Today: Suprema and infima.

Homework before Thursday’s class: watch videos 7.5, 7.6.
Warm up: suprema and infima

Find the supremum, infimum, maximum, and minimum of the following sets (if they exist):

1. \( A = [-1, 5) \)
   \( \inf A = \min A = -1 \)
   \( \sup A = 5 \notin A \)
   \( \max A \)

2. \( B = (-\infty, 0) \cup (3, \infty) \)

3. \( C = \{2, 3, 4\} \)

4. \( D = \left\{ \frac{1}{n} : n \in \mathbb{Z}, n > 0 \right\} \)

5. \( E = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{Z}, n > 0 \right\} \)

6. \( F = \{2^n : n \in \mathbb{Z}\} \)
1) 

2) 

3) \( \inf = \min = 2, \quad \sup = \max = 4 \)

4) \( \max = \sup = 1, \quad \inf = 0 \)

5) 

6) \( 1 = \min, \quad 0 = \inf, \quad 4 = \max, \quad \sup \)
Empty set

1. Does \( \emptyset \) have an upper bound?
2. Does \( \emptyset \) have a supremum?
3. Does \( \emptyset \) have a maximum?
4. Is \( \emptyset \) bounded above?
1. Does $\emptyset$ have an upper bound? Yes, e.g. 10

2. Does $\emptyset$ have a supremum? No, $-10$, $-\infty$, $-1000$

3. Does $\emptyset$ have a maximum? No

4. Is $\emptyset$ bounded above? No

Recall:

Let $A \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$.

- $a$ is an **upper bound** of $A$ means $\forall x \in A, x \leq a$.
- $a$ is the **least upper bound** (lub) or **supremum** (sup) of $A$ means
  - $a$ is an upper bound of $A$, and
  - there are no smaller upper bounds.
**Equivalent definitions of supremum**

**Assume** \( S \) **is an upper bound of the set** \( A \). Which one (or ones) of the following statements is equivalent to “\( S \) is the supremum of \( A \)”?

1. If \( R \) is an upper bound of \( A \), then \( S \leq R \).
2. \( \forall R \geq S, \ R \text{ is an upper bound of } A \).
3. \( \forall R \leq S, \ R \text{ is not an upper bound of } A \).
4. \( \forall R < S, \ R \text{ is not an upper bound of } A \).
5. \( \forall R < S, \ \exists x \in A \text{ such that } R < x \).
6. \( \forall R < S, \ \exists x \in A \text{ such that } R \leq x \).
7. \( \forall R < S, \ \exists x \in A \text{ such that } R < x \leq S \).
8. \( \forall R < S, \ \exists x \in A \text{ such that } R < x < S \).
9. \( \forall \varepsilon > 0, \ \exists x \in A \text{ such that } S - \varepsilon < x \leq S \).
Infima and suprema exercises

Let $A, B \subseteq \mathbb{R}$. Which of the following are true or false? If false, find a counterexample.

1. If $B \subseteq A$ and $A$ is bounded above, then $B$ is bounded above.
2. If $B \subseteq A$ and $B$ is bounded above, then $A$ is bounded above.
3. If $B \subseteq A$ and $A$ is bounded above, then $\sup B \leq \sup A$.
4. If $B \subseteq A$ and $A$ is bounded above, then $\inf B \leq \inf A$.
5. If $A$ and $B$ are bounded above and $\sup A \leq \sup B$, then $A \subseteq B$.
6. If $A$ and $B$ are bounded above, then $\sup(A \cup B) = \max\{\sup A, \sup B\}$. Yes
7. If $A$ and $B$ are bounded above, then $\sup(A \cap B) = \min\{\sup A, \sup B\}$. No
\[ A \cap B = \{2\} \]
\[ \sup (A \cap B) = \sup \{2, 3\} = 2 \]
\[ \min (\sup A, \sup B) = \min \{4, 3\} = 3 \]
Maximum and supremum

Let \( f(x) = x^3 - 3x \).

Find four open intervals \( I_1, I_2, I_3, I_4 \) such that

1. \( f \) has a maximum and a minimum on \( I_1 \).
2. \( f \) has a supremum and no maximum on \( I_2 \).
3. \( f \) has a supremum and no infimum on \( I_3 \).
4. \( f \) does not have a supremum or an infimum on \( I_4 \).
\[ y = x^3 - 3x \]

\[ I_4 = \mathbb{R} \]
1. Let $f$ and $g$ be bounded functions on $[a, b]$. Then

$$\sup_{[a, b]} (f + g) \leq \sup_{[a, b]} f + \sup_{[a, b]} g$$

2. Let $a < b < c$. Let $f$ be a bounded function on $[a, c]$. Then

$$\sup_{[a, c]} f = \max \left( \sup_{[a, b]} f, \sup_{[b, c]} f \right)$$

3. Let $f$ be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\sup_{[a, b]} (cf) = |c| \left( \sup_{[a, b]} |f| \right)$$
$f = \ln x$
on $[-\pi, 0]$
c $c = -1$