- Today: Integrable functions.
- Homework before Tuesday's class: watch videos 7.7, 7.8, 7.11.

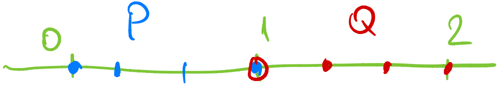
A partition of $[a,b]$ is $P=\{x_0, x_1, \dots, x_n\}$, $x_i \in [a,b]$, $x_0 = a$, $x_n = b$

Which ones are partitions of $[0,2]$?

1. $[0,2]$ ✗
2. $(0,2)$ ✗
3. $\{0,2\}$ ✓
4. $\{1,2\}$ ✗
5. $\{0, 1.5, 1.6, 1.7, 1.8, 1.9, 2\}$ ✓

$\{0, e, 2\}$ ✗ $e \approx 2.7 \notin [0,2]$

6. $\{0, \frac{n}{n+1}, n \in \mathbb{Z}_+; 1, 2\}$ ✗

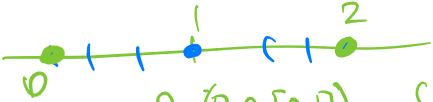


1. Let P be a partition of $[0,1]$. $R = P \cup Q$
   Let Q be a partition of $[1,2]$.
   How do we construct a partition R of $[0,2]$ from them?

2. Let R be a partition of $[0,2]$. $\overline{R} = R \cup \{1\}$
   How do we construct partitions of $[0,1]$ and $[1,2]$ from it?

   $P = \overline{R} \cap [0,1]$

   $Q = \overline{R} \cap [1,2]$



or $P = (R \cap [0,1]) \cup \{1\}$

Warm up: lower and upper sums

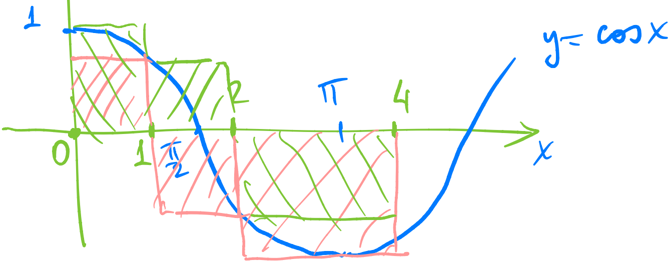
Reminder $L_P(f) = \sum_{i=1}^{n} m_i \Delta x_i$

for a contin f

$\Delta x_i = x_i - x_{i-1}$, $m_i = \inf_{\Delta x_i} f \left(= \min_{\Delta x_i} f(x) \right)$

Let $f(x) = \cos x$.
Consider the partition $P = \{0, 1, 2, 4\}$ of the interval $[0, 4]$.
Calculate $L_P(f)$ and $U_P(f)$.

$$P = \{ \underset{x_0}{\overset{\parallel}{0}}, \underset{x_1}{\overset{\parallel}{1}}, \underset{x_2}{\overset{\parallel}{2}}, \underset{x_3}{\overset{\parallel}{4}} \}$$

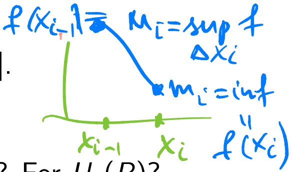


$$L_p(\cos x) = \underset{f(x_1)}{\underset{\parallel}{(\cos 1)}}\underset{x_1 - x_0}{\underset{\parallel}{(1-0)}} + (\cos 2)(2-1) + \overset{-1}{\overset{\parallel}{(\cos \pi)}}(4-2)$$

$$= \cos 1 + \cos 2 - 1 \cdot 2$$

$$U_p(\cos x) = \underset{1}{\underset{\parallel}{(\cos 0)}}(1-0) + (\cos 1)(2-1) + (\cos 2)(4-2)$$

$$= 1 + \cos 1 + 2\cos 2$$



Let f be a **decreasing**, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \dots, x_N\}$ be a partition of $[a, b]$

Which one (or ones) is a valid equation for $L_P(f)$? For $U_f(P)$?

✗ 1. $\sum_{i=0}^{N} \Delta x_i \, f(x_i)$ $\nexists x_{-1}$

✗ 3. $\sum_{i=0}^{N-1} \Delta x_i \, f(x_i)$ $\nexists x_{-1}$

✓ 5. $\sum_{i=1}^{N} \Delta x_i \, f(x_{i-1})$ $U_P(f)$

2. $\sum_{i=1}^{N} \Delta x_i \, f(x_i)$ $L_P(f) = \sum \Delta x_i \cdot m_i$

✗ 4. $\sum_{i=1}^{N} \Delta x_i \, f(x_{i+1})$ $\nexists x_{N+1}$

✓ 6. $\sum_{i=0}^{N-1} \Delta x_{i+1} \, f(x_i)$ $U_P(f)$

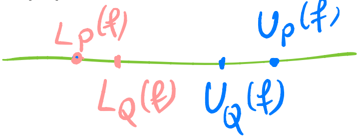
$i+1=j$

$= \sum_{j=1}^{N} \Delta x_j \, f(x_{j-1})$

Recall: $\Delta x_i = x_i - x_{i-1}$.

Assume

$$L_P(f) = 2, \quad U_P(f) = 6$$
$$L_Q(f) = 3, \quad U_Q(f) = 8$$



1. Is $P \subseteq Q$? If $P \leq Q$
2. Is $Q \subseteq P$?
3. What can you say about $L_{P\cup Q}(f)$ and $U_{P\cup Q}(f)$?

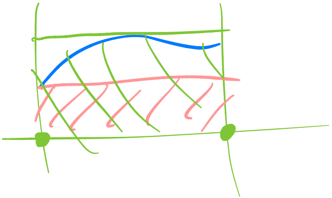
$3 \leq L_{P\cup Q}(f)$, $\quad U_{P\cup Q}(f) \leq 6$

$$\begin{matrix} L_P(f) \leq \\ L_Q(f) \leq \end{matrix} \; L_{P\cup Q}(f) \leq U_{P\cup Q}(f) \; \begin{matrix} \leq U_P(f) \\ \leq U_Q(f) \end{matrix}$$

1. Let f be a bounded function on $[0,1]$.
   Assume f is not constant.
   Prove that there exists a partition P of $[0,1]$ such that

$$L_P(f) \neq U_P(f).$$

   Hint: This is easier than it looks.



1. Let f be a bounded function on $[0, 1]$.
   Assume f is not constant.
   Prove that there exists a partition P of $[0, 1]$ such that

$$L_P(f) \neq U_P(f).$$

   Hint: This is easier than it looks.

2. For which functions f
   is there a partition P of $[0, 1]$ such that
   $L_P(f) = U_P(f)$?

$c^2 = a^2 + b$