MAT137

- Today: Suprema and infima.

- Homework before Thursday's class: watch videos 7.5, 7.6.

Warm up: suprema and infima

Find the supremum, infimum, maximum, and minimum of the following sets (if they exist):

1. $A = [-1, 5)$
2. $B = (-\infty, 0) \cup (3, \infty)$
3. $C = \{2, 3, 4\}$
4. $D = \left\{ \frac{1}{n} \ : \ n \in \mathbb{Z}, n > 0 \right\}$
5. $E = \left\{ \frac{(-1)^n}{n} \ : \ n \in \mathbb{Z}, n > 0 \right\}$
6. $F = \{2^n \ : \ n \in \mathbb{Z}\}$

Empty set

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Recall:

Let $A \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$.

- a is an **upper bound** of A means $\forall x \in A, x \leq a$.
- a is the **least upper bound** (lub) or **supremum** (sup) of A means
  - a is an upper bound of A, and
  - there are no smaller upper bounds.

Equivalent definitions of supremum

Assume S is an upper bound of the set A.
Which one (or ones) of the following statements is equivalent to "S is the supremum of A"?

1. If R is an upper bound of A, then $S \leq R$.
2. $\forall R \geq S$, R is an upper bound of A.
3. $\forall R \leq S$, R is not an upper bound of A.
4. $\forall R < S$, R is not an upper bound of A.
5. $\forall R < S$, $\exists x \in A$ such that $R < x$.
6. $\forall R < S$, $\exists x \in A$ such that $R \leq x$.
7. $\forall R < S$, $\exists x \in A$ such that $R < x \leq S$.
8. $\forall R < S$, $\exists x \in A$ such that $R < x < S$.
9. $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x \leq S$.

Infima and suprema exercises

Let $A, B \subseteq \mathbb{R}$. Which of the following are true or false?
If false, find a counterexample.

1. If $B \subseteq A$ and A is bounded above, then B is bounded above.
2. If $B \subseteq A$ and B is bounded above, then A is bounded above.
3. If $B \subseteq A$ and A is bounded above, then $\sup B \leq \sup A$.
4. If $B \subseteq A$ and A is bounded above, then $\inf B \leq \inf A$.
5. If A and B are bounded above and $\sup A \leq \sup B$, then $A \subseteq B$.
6. If A and B are bounded above,
   then $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
7. If A and B are bounded above,
   then $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

Maximum and supremum

Let $f(x) = x^3 - 3x$.
Find four open intervals I_1, I_2, I_3, I_4 such that

1. f has a maximum and a minimum on I_1.
2. f has a supremum and no maximum on I_2.
3. f has a supremum and no infimum on I_3.
4. f does not have a supremum or an infimum on I_4.

Fix these FALSE statements

1. Let f and g be bounded functions on $[a, b]$. Then

$$\begin{array}{c}\text{sup of } (f+g) \\ \text{on } [a,b]\end{array} = \begin{array}{c}\text{sup of } f \\ \text{on } [a,b]\end{array} + \begin{array}{c}\text{sup of } g \\ \text{on } [a,b]\end{array}$$

2. Let $a < b < c$. Let f be a bounded function on $[a, c]$. Then

$$\begin{array}{c}\text{sup of } f \\ \text{on } [a,c]\end{array} = \begin{array}{c}\text{sup of } f \\ \text{on } [a,b]\end{array} + \begin{array}{c}\text{sup of } f \\ \text{on } [b,c]\end{array}$$

3. Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\begin{array}{c}\text{sup of } (cf) \\ \text{on } [a,b]\end{array} = c \left(\begin{array}{c}\text{sup of } f \\ \text{on } [a,b]\end{array} \right)$$