- Today: Sequences.

- Homework before Tuesday's class: watch videos 11.3, 11.4.

Sequences

A sequence is just an infinite list of numbers. Slightly more precise it is a function defined on the natural numbers. Some sequences:

$$\{1, -1, 1, -1, ...\} \qquad a_n = (-1)^n$$

$$\{1, \frac{1}{2}, \frac{1}{3}, ...\} \qquad a_n = \frac{1}{n}$$

$$\{1, 1, 2, 3, 5, 8, ...\} \qquad a_0 = 1;\ a_1 = 1;\ a_n = a_{n-1} + a_{n-2}$$

Warm up

Write a formula for the general term of these sequences

1. $\{a_n\}_{n=0}^{\infty} = \{\,1,\ 4,\ 9,\ 16,\ 25,\ \ldots\,\}$
2. $\{b_n\}_{n=1}^{\infty} = \{\,1,\ -2,\ 4,\ -8,\ 16,\ -32,\ \ldots\,\}$
3. $\{c_n\}_{n=1}^{\infty} = \left\{\,\frac{2}{1!},\ \frac{3}{2!},\ \frac{4}{3!},\ \frac{5}{4!},\ \ldots\,\right\}$
4. $\{d_n\}_{n=1}^{\infty} = \{\,1,\ 4,\ 7,\ 10,\ 13,\ \ldots\,\}$

True or False?

Let f be a function with domain at least $[1, \infty)$.
We define a sequence as $a_n = f(n)$.
Let $L \in \mathbb{R}$.

1. IF $\lim_{x\to\infty} f(x) = L$, THEN $\lim_{n\to\infty} a_n = L$.
2. IF $\lim_{n\to\infty} a_n = L$, THEN $\lim_{x\to\infty} f(x) = L$.
3. IF $\lim_{n\to\infty} a_n = L$, THEN $\lim_{n\to\infty} a_{n+1} = L$.

Definition of limit of a sequence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
2. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \quad n > n_0 \implies |L - a_n| < \varepsilon.$
3. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
4. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{R}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
5. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| \leq \varepsilon.$
6. $\forall \varepsilon \in (0,1),\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
7. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \frac{1}{\varepsilon}.$
8. $\forall k \in \mathbb{Z}^+ > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < k.$
9. $\forall k \in \mathbb{Z}^+ > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \frac{1}{k}.$

Which of the following sequences converge?

1. $a_n = \dfrac{1}{n}$
2. $\{1, -1, 1, -1, ...\}$
3. $\{0.3, 0.33, 0.333, ...\}$
4. $a_n = \dfrac{n}{n+1}$
5. $a_n = (1 + \dfrac{1}{n})^n$

Definition of limit of a sequence (continued)

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

8. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except the first few.
9. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except finitely many.
10. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains cofinitely many of the terms of the sequence.
11. $\forall \varepsilon > 0$, the interval $[L - \varepsilon, L + \varepsilon]$ contains cofinitely many of the terms of the sequence.
12. Every interval that contains L must contain cofinitely many of the terms of the sequence.
13. Every open interval that contains L must contain cofinitely many of the terms of the sequence.

Notation: "cofinitely many" = "all but finitely many"

Convergence and divergence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.
Write the formal definition of the following concepts:

1. $\{a_n\}_{n=0}^{\infty}$ is convergent.
2. $\{a_n\}_{n=0}^{\infty}$ is divergent.
3. $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞.