Today: Volumes (cont’d)

Tomorrow, on Thursday we will start sequences. Please watch videos 11.1, 11.2. I strongly recommend that you watch these videos!
An equation for volumes by “cylindrical shells”

Let \( a < b \). 

Let \( f \) be a continuous, positive function defined on \([a, b]\). 
Let \( R \) be the region in the first quadrant bounded between the graph of \( f \) and the \( x \)-axis. 
Find a formula for the volume of the solid of revolution obtained by rotation the region \( R \) around the \( y \)-axis.
approx: \[ \sum_{i=0}^{n} 2\pi x_i \cdot f(x_i) \Delta x_i \]

volume of the shell

\[ V = \int_{a}^{b} x \cdot f(x) \, dx \]

volume of the solid of revolu

circumference = \(2\pi x_i\)

height \(f(x_i)\)

thickness \(\Delta x_i\)
Last time we found the volume of a sphere by the cross-section method, by rotating a circle around the $x$-axis.

Recall how you did this. Recall the equation of a circle (or half-circle) of radius $R$ centered at $(0, 0)$.

Now reorient your half-circle, rotate it about the $y$-axis and find the sphere volume using the shell method.
The volume of the ball of radius $R$ is

$$V(R) = 2\pi \int_0^R x \left(\sqrt{R^2 - x^2} - \left(-\sqrt{R^2 - x^2}\right)\right) \, dx$$

height of the shell

$$d^2(x) = f(x) \, dx$$

$$d^2(x) = 2x \, dx$$

$$d^2(x) = -du$$

$$dx^2 = -du$$

$$dx^2 = 2x \, dx$$

$$\int_0^R \sqrt{u} \, du = 2\pi \left[ \frac{u^{3/2}}{3/2} \right]_0^R = 2\pi \cdot \frac{2}{3} \cdot (R^2) = \frac{4}{3} \pi R^3$$
Volume of a circus tent

A circus tent is made by taking the graph of $y = (x - 1)^4 + 1$, $0 \leq x \leq 1$ and rotating it around the $y$-axis.

Find the volume by cutting it into cylindrical shells.

Now try to find the volume by the cross-section method, aka the “washer method.”

Imagine that we had a more complicated function. Which method would work better?
$f(x) = y = (x-1)^4 + 1$, $0 \leq x \leq 1$

- The shell method:

$$V = 2\pi \int_0^1 x (x-1)^4 + 1 \, dx$$

The integral becomes:

$$V = 2\pi \int_0^1 (x (x-1)^4 + x) \, dx = 2\pi \int_0^1 ((x-1)(x-1)^4 + (x-1)^4 + x) \, dx$$

Simplifying the integrand:

$$V = 2\pi \int_0^1 ((x-1)^5 + (x-1)^4 + x) \, dx = 2\pi \left( \frac{(x-1)^6}{6} + \frac{(x-1)^5}{5} + \frac{x^2}{2} \right) \bigg|_0^1$$

Evaluating the limits:

$$= 2\pi \left( \left( \frac{1-1}{6} + \frac{1-1}{5} + \frac{1^2}{2} \right) - \left( \frac{-\frac{1}{6} + \frac{1}{5} + \frac{1}{2}}{2} \right) \right) = 2\pi \left( \frac{-5+6+15}{30} \right) = \frac{16}{15} \pi$$
washer method for \( y \)-axis

\[ V = V_1 + V_2 \]

\[ V_2 = \pi \cdot \pi R^2 = \pi \cdot 1^2 = \pi \]

cylinder

\[ V = \pi \int_1^2 x(y)^2 dy \]

\[ V_1 : \quad y = (x-1)^4 + 1, \quad \text{express } x = x(y) = -\sqrt[4]{y-1} + 1 \]

Then

\[ V_1 = \pi \int (\sqrt[4]{y-1} + 1)^2 dy = \pi \int (\sqrt[4]{y-1} - 2\sqrt[4]{y-1} + 1)^2 dy \]

\[ = \pi \left( \left. \left( \frac{y-1}{3/2} - 2 \left( \frac{y-1}{5/4} \right) + y \right) \right|^2 \right|_1 = \pi \left( \frac{2}{3} - 2 \frac{4}{5} + 1 \right) = \pi \frac{10 - 24 + 15}{15} = \pi \frac{15}{15} = \pi \]

Finally,

\[ V = V_1 + V_2 = \frac{\pi}{15} + \pi = \frac{16}{15} \pi \]
Challenge

Two cylinders have the same radius $R$ and their axes meet at a right angle. Find the volume of their intersection.

Hint: You can slice the resulting solid by parallel cuts in three different directions. One of the three makes the problem much, much simpler than the other two.
At any height \( z \)

\[ x^2 + z^2 = R^2 \]

for a y-cylinder

\[ x = \pm \sqrt{R^2 - z^2} \]

area \( \int \) of \( z \)-section

\[ = (2x)^2 = (2 \sqrt{R^2 - z^2})^2 = 4(R^2 - z^2) \]

Then the volume

\[ V = \int_{-R}^{R} \text{area}(z) \, dz = \int_{-R}^{R} 4(R^2 - z^2) \, dz \]

\[ = \ldots \]