Today: Sequences.

Homework before Monday’s class: watch videos 11.3, 11.4.
Sequences

A sequence is just an infinite list of numbers. Slightly more precise it is a function defined on the natural numbers. Some sequences:

\[
\{1, -1, 1, -1, ...\} \quad a_n = (-1)^n
\]

\[
\left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\} \quad a_n = \frac{1}{n}
\]

\[
\{1, 1, 2, 3, 5, 8, ...\} \quad a_0 = 1; \quad a_1 = 1; \quad a_n = a_{n-1} + a_{n-2}
\]
Warm up

Write a formula for the general term of these sequences

1. \( \{a_n\}_{n=0}^\infty = \{1, 4, 9, 16, 25, \ldots\} \)

2. \( \{b_n\}_{n=1}^\infty = \{1, -2, 4, -8, 16, -32, \ldots\} \)

3. \( \{c_n\}_{n=1}^\infty = \left\{ \frac{2}{1!}, \frac{3}{2!}, \frac{4}{3!}, \frac{5}{4!}, \ldots \right\} \)

4. \( \{d_n\}_{n=1}^\infty = \{1, 4, 7, 10, 13, \ldots\} \)
True or False?

Let $f$ be a function with domain at least $[1, \infty)$. We define a sequence as $a_n = f(n)$. Let $L \in \mathbb{R}$.

1. IF $\lim_{x \to \infty} f(x) = L$, THEN $\lim_{n \to \infty} a_n = L$.

2. IF $\lim_{n \to \infty} a_n = L$, THEN $\lim_{x \to \infty} f(x) = L$.

3. IF $\lim_{n \to \infty} a_n = L$, THEN $\lim_{n \to \infty} a_{n+1} = L$. 
Definition of limit of a sequence

Let \( \{a_n\}_{n=0}^\infty \) be a sequence. Let \( L \in \mathbb{R} \).

Which statements are equivalent to “\( \{a_n\}_{n=0}^\infty \to L \)”?

1. \( \forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| < \epsilon \).
2. \( \forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \ n > n_0 \implies |L - a_n| < \epsilon \).
3. \( \forall \epsilon > 0, \exists n_0 \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| < \epsilon \).
4. \( \forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{R}, \ n \geq n_0 \implies |L - a_n| < \epsilon \).
5. \( \forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| \leq \epsilon \).
6. \( \forall \epsilon \in (0, 1), \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| < \epsilon \).
7. \( \forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| < \frac{1}{\epsilon} \).
8. \( \forall k \in \mathbb{Z}^+ > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| < k \).
9. \( \forall k \in \mathbb{Z}^+ > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| < \frac{1}{k} \).
Which of the following sequences converge?

1. \( a_n = \frac{1}{n} \)

2. \( \{1, -1, 1, -1, \ldots\} \)

3. \( \{0.3, 0.33, 0.333, \ldots\} \)

4. \( a_n = \frac{n}{n + 1} \)

5. \( a_n = \left(1 + \frac{1}{n}\right)^n \)
Definition of limit of a sequence (continued)

Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence. Let \( L \in \mathbb{R} \).
Which statements are equivalent to \( \{a_n\}_{n=0}^{\infty} \to L \)?

8. \( \forall \varepsilon > 0 \), the interval \((L - \varepsilon, L + \varepsilon)\) contains all the elements of the sequence, except the first few.

9. \( \forall \varepsilon > 0 \), the interval \((L - \varepsilon, L + \varepsilon)\) contains all the elements of the sequence, except finitely many.

10. \( \forall \varepsilon > 0 \), the interval \((L - \varepsilon, L + \varepsilon)\) contains cofinitely many of the terms of the sequence.

11. \( \forall \varepsilon > 0 \), the interval \([L - \varepsilon, L + \varepsilon]\) contains cofinitely many of the terms of the sequence.

12. Every interval that contains \( L \) must contain cofinitely many of the terms of the sequence.

13. Every open interval that contains \( L \) must contain cofinitely many of the terms of the sequence.

Notation: “cofinitely many” = “all but finitely many”
Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence.

Write the formal definition of the following concepts:

1. \( \{a_n\}_{n=0}^{\infty} \) is convergent.

2. \( \{a_n\}_{n=0}^{\infty} \) is divergent.

3. \( \{a_n\}_{n=0}^{\infty} \) is divergent to \( \infty \).