- Today: Integration of rational functions.
- Homework before Tuesday's class: watch video 10.1.

Rational integrals

1. Calculate $\int \frac{1}{x+a}\,dx$ $= \ln|x+a| + C$

2. Reduce to common denominator $\frac{2}{x} - \frac{3}{x+3}$

3. Calculate $\int \frac{-x+6}{x^2+3x}\,dx$ $= \int \left(\frac{2}{x} - \frac{3}{x+3}\right) dx$

   $= 2\ln|x| - 3\ln|x+3| + C$

   $= \ln\left|\frac{x^2}{(x+3)^3}\right| + C$

4. Calculate $\int \frac{1}{x^2+3x}\,dx$

5. Calculate $\int \frac{1}{x^3-x}\,dx$

$$\frac{1}{x^3-x} = \frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

II plug in:

$x=0 \quad 1 = A(0-1)(0+1) \Rightarrow A = -1$

$x=1 \quad 1 = B\cdot 1(1+1) \Rightarrow B = \frac{1}{2}$

$x=-1 \quad 1 = C\cdot(-1)(-1-1) \Rightarrow C = 1/2$

$$\int \frac{dx}{x^3-x} = -\int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$= -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

$$= \ln \frac{\sqrt{|(x-1)(x+1)|}}{|x|} + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\frac{2}{x}-\frac{3}{x+3}=\frac{2(x+3)-3x}{x(x+3)}=\frac{-x+6}{x(x+3)}=\frac{-x+6}{x^2+3x}$$

$$\frac{1}{x^2+3x}=\frac{1}{x(x+3)}=\frac{A}{x}+\frac{B}{x+3}=\frac{A(x+3)+Bx}{x^2+3x}$$

$A=\frac{1}{3}$, $B=-\frac{1}{3}$

$$1=A(x+3)+Bx$$

I

$$0x+1=(A+B)x+3A$$

$$\begin{cases} A+B=0 \\ 3A=1 \end{cases} \Rightarrow A=1/3 \qquad B=-1/3$$

II plug in

$x=0$: $1=A(0+3)+\cancel{B\cdot 0}$ $\Rightarrow A=\frac{1}{3}$

$x=-3$: $1=A(\cancel{-3+3})+B(-3)$ $\Rightarrow B=-1/3$

$$\int\frac{dx}{x^2+3x}=\int\frac{1/3\,dx}{x}-\int\frac{1/3\,dx}{x+3}=\frac{1}{3}\ln|x|-\frac{1}{3}\ln|x+3|+C$$

$$=\frac{1}{3}\ln\left|\frac{x}{x+3}\right|+C=\ln\sqrt[3]{\left|\frac{x}{x+3}\right|}+C$$

Compute

$$\int \sec x \, dx$$

using the substitution $u = \sin x$.

$$\int \sec x \, dx = \int \frac{dx}{\cos x} = \int \frac{\cos x \, dx}{\underbrace{\cos^2 x}_{1-\sin^2 x}} = \int \frac{d \sin x}{1 - \sin^2 x}$$

$$= \int \frac{du}{1-u^2} \text{ for } u = \sin x$$

$$\frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u} = \frac{1/2}{1-u} + \frac{1/2}{1+u}. \text{ Then}$$

$$\int \frac{1}{1-u^2}\,du = \frac{1}{2}\underbrace{\int \frac{du}{1-u}}_{= -\frac{1}{2}\int \frac{du}{u-1}} + \frac{1}{2}\int \frac{du}{1+u} = -\frac{1}{2}\ln|u-1| + \frac{1}{2}\ln|u+1| + C$$

$\cdot(\sin x+1)$

$$= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C = \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(\sin x+1)^2}{\underbrace{\sin^2 x-1}_{= -\cos^2 x}}\right| = \cancel{\frac{1}{2}}\ln\left|\frac{\sin x+1}{\cos x}\right|^{\cancel{2}} + C$$

$$= \ln\left|\frac{\sin x+1}{\cos x}\right| + C = \ln|\tan x + \sec x| + C$$

Repeated factors

1. Calculate $\int \frac{1}{(x+1)^n}\,dx \quad$ for $n > 1$ =
2. Calculate $\int \frac{(x+1)-1}{(x+1)^2}\,dx = \int \frac{x\,dx}{(x+1)^2}$
3. Calculate $\int \frac{2x+6}{(x+1)^2}\,dx$
4. Calculate $\int \frac{x^2}{(x+1)^3}\,dx$
5. How would you calculate $\int \frac{\text{polynomial}}{(x+1)^3}\,dx$?

1. $\int \frac{dx}{(x+1)^n} = \int (x+1)^{-n} dx = \frac{(x+1)^{-n+1}}{-n+1} + C$

$n > 1$

$$\int u^{-n} du = \frac{u^{-n+1}}{-n+1} + C$$

$x+1 = u$

$dx = du$

2. $\int \frac{x\,dx}{(x+1)^2} = \int \frac{\cancel{(x+1)}dx}{(x+1)^{\cancel{2}}} - \int \frac{dx}{(x+1)^2}$

$= \ln|x+1| + \frac{1}{x+1} + C$

3-5. $u = x+1$, $x = u-1$, $x^2 = (u-1)^2 = u^2 - 2u + 1$

$x^2 = (x+1)^2 - 2(x+1) + 1$ $\qquad dx = du$

Then $\int \frac{x^2\,dx}{(x+1)^3} = \int \frac{(u^2-2u+1)\,du}{u^3}$

$u = x+1$

$= \int (u^{-1} - 2u^{-2} + u^{-3})\,du = \dots$

$$\int \frac{\mathrm{pol}(u)}{u^3}\,du = \dots$$

1. Calculate $\int \frac{1}{x^2+1}\,dx$ and $\int \frac{x}{x^2+1}\,dx$.
   Hint: These two are very short.

2. Calculate $\int \frac{2x+3}{x^2+1}\,dx = \int \frac{d(x^2+1)}{x^2+1} + 3\int \frac{dx}{x^2+1}$

   $= \ln(x^2+1) + 3\arctan x + C$

3. Calculate $\int \frac{x^2+1-1}{x^2+1}\,dx$

4. Calculate $\int \frac{x}{x^2+x+1}\,dx$

   Hint: Transform it into one like the previous ones

$$1.\ \int \frac{dx}{x^2+1} = \arctan x + C$$

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C$$

$$dx^2 = 2x\,dx$$

$$3.\ \int \frac{x^2+1}{x^2+1}\,dx - \int \frac{dx}{x^2+1} = \overbrace{\int dx}^{=x} - \arctan x + C$$

4. $\displaystyle\int \frac{x}{x^2+x+1}\,dx = \int \frac{x+\frac{1}{2}-\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \int \frac{u-\frac{1}{2}}{u^2+\frac{3}{4}}\,du$

complete square:

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\left(x+\frac{1}{2}\right)^2 = x^2+x+\frac{1}{4}$$

$u = x+\frac{1}{2}$

$du = dx$

Note: $\displaystyle\int \frac{u\,du}{u^2+5} = \int \frac{\cancel{\sqrt{5}}\,t\,d(\cancel{\sqrt{5}}\,t)}{\cancel{5}t^2+\cancel{5}} = \int \frac{t\,dt}{t^2+1}$

$u = \sqrt{5}\,t$

$du = \sqrt{5}\,dt$

$u^2 = 5t^2$

$$= \frac{1}{2}\ln\left|t^2+1\right|+C = \frac{1}{2}\ln\left|\left(\frac{u}{\sqrt{5}}\right)^2+1\right|+C$$

$$= \frac{1}{2}\ln\left|\frac{u^2}{5}+1\right|+C = \frac{1}{2}\ln\left|u^2+5\right|+C_1$$

$$\int \frac{du}{u^2+3} = \int \frac{\sqrt{3}\,dt}{3t^2+3} = \frac{1}{\sqrt{3}} \int \frac{dt}{t^2+1} = \frac{1}{\sqrt{3}} \arctan t + C$$

$u = \sqrt{3}\,t$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) + C$$

1. How could we compute an integral of the form

$$\int \frac{\text{polynomial}^{\deg < 4}}{(x+1)^3(x+2)}dx \; ?$$

$$= \int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x+2} \, dx$$

Messier rational functions

1. How could we compute an integral of the form
$$\int \frac{\text{polynomial}}{(x+1)^3(x+2)}dx\ ?$$

2. How could we compute an integral of the form
$$\int \frac{\text{polynomial}}{(x+1)^3(x+2)x^4(x^2+1)(x^2+4x+7)}dx\ ?$$

A harder antiderivative

1. Calculate

$$\frac{d}{dx}\left[\arctan x\right], \qquad \frac{d}{dx}\left[\frac{x}{1+x^2}\right].$$

2. Use the previous answer to calculate

$$\int \frac{1}{\left(1+x^2\right)^2}\,dx$$

3. Calculate

$$\int \frac{1}{\left(1+x^2\right)^3}\,dx$$