- Today: Integration of trigonometric functions.

- Homework before Thursday's class: watch videos 9.10, as well as 9.11, 9.12.

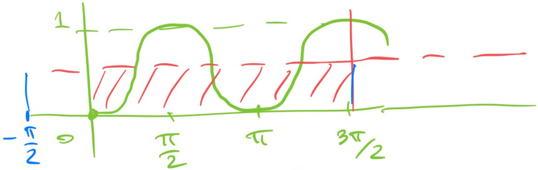
Warm up: Symmetry

We want to compute the number

$$A = \int_{-\pi/2}^{3\pi} \sin^2 x \, dx.$$

Sketch the graph of $y = \sin^2 x$. Look at it and search for symmetries. **You can get the answer without performing any integration.**

Check it using $\sin^2 x = \frac{1-\cos(2x)}{2}$



$$\int_{-\frac{\pi}{2}}^{3\pi} \sin^2 x \, dx = \int_{-\frac{\pi}{2}}^{3\pi} \frac{1}{2} \, dx = \frac{1}{2} \cdot \left(3\pi + \frac{\pi}{2}\right) = \dots$$

Practice: Integrals with trigonometric functions

Compute the following antiderivatives. (Once you get them to a form from where it is easy to finish, you may stop.)

1. $\displaystyle\int \sin^{10} x \cos x \; dx$
2. $\displaystyle\int \sin^{10} x \cos^3 x \, dx$
3. $\displaystyle\int e^{\cos x} \cos x \sin^5 x \, dx$
4. $\displaystyle\int \cos^2 x \, dx$
5. $\displaystyle\int \sin^4 x \, dx$
6. $\displaystyle\int \csc x \, dx$

Useful trig identities

$$\sin^2 x + \cos^2 x = 1 \qquad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\tan^2 x + 1 = \sec^2 x \qquad \underline{\cos^2 x} = \frac{1 + \cos(2x)}{2}$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \underbrace{\cos^2 x - \sin^2 x}_{\displaystyle \parallel} = \cos^2 x - (1-\cos^2 x)$$

$$= 2\cos^2 x - 1$$

$$(1-\sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$$

1. $\int \sin^{10}x \cdot (\cos x \cdot dx) = \int \sin^{10}x \, d(\sin x) = \int u^{10} du$

$df = f'(x)dx$, $d\sin x = \cos x \cdot dx$, $u = \sin x$

$$= \frac{u^{11}}{11} + C = \frac{\sin^{11}x}{11} + C$$

2. $\int \sin^{10}x \cdot \cos^3 x \, dx = \int \sin^{10}x (1 - \sin^2 x) d\sin x$

$\cos^3 x = \underbrace{\cos^2 x}_{= 1-\sin^2 x} \cdot \cos x$

$= \int u^{10}(1-u^2)du = \ldots$

3. $\int e^{\cos x} \cdot \cos x \cdot \sin^5 x \, dx =$

$\sin^5 x = \sin^4 x \cdot \sin x$

$\cos x = u$, $du = -(\sin x)dx$

$$= -\int e^{\cos x} \cdot \cos x \cdot (\sin^2 x)^2 \cdot d\cos x$$

$$= -\int e^{u} \cdot u \cdot (1-u^2)^2 \, du = \dots$$

$\int u^5 e^u \, du$ — integrate by parts 5 times

4. $\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2} dx$

$= \int \frac{1}{2} dx + \frac{1}{2}\int \cos 2x\, dx = \frac{x}{2} + \frac{1}{4}\int \cos(\overset{u}{\overset{\shortparallel}{2x}})\, d(2x)$

$= \frac{x}{2} + \frac{1}{4}\int \cos u\, du = \frac{x}{2} + \frac{\sin u}{4}\Big|_{u=2x} + C$

$= \frac{x}{2} + \frac{\sin(2x)}{4} + C$

5. $\int \underset{(\sin^2 x)^2}{\underset{\shortparallel}{\sin^4 x}}\, dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx$

$= \ldots$

$$6.\ \int \csc x\,dx = \int \frac{dx}{\sin x} = \int \frac{\sin x\,dx}{\sin^2 x} \qquad u = \cos x$$

$$= -\int \frac{d\cos x}{1-\cos^2 x} = -\int \frac{du}{1-u^2} = \int \frac{du}{u^2-1}$$

$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{1/2}{u-1} + \frac{-1/2}{u+1}$$

$$= \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u+1} = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + C$$

$$= \frac{1}{2} \ln \left| \frac{(\cos x - 1)^2}{(\cos x + 1)\underbrace{(\cos x - 1)}_{\cos^2 x - 1 = -\sin^2 x}} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\sin x} \right|^2 + C = \ln \left| \frac{\cos x - 1}{\sin x} \right| + C$$

$$= \ln \left| \cot x - \csc x \right| + C$$

A reduction formula

Let $I_n = \int_0^{2\pi} \sin^n x \, dx$.

1. Compute I_0 and I_1.

$$I_0 = 2\pi \qquad I_1 = 0$$

Let $I_n = \int_0^{2\pi} \sin^n x \, dx$.

1. Compute I_0 and I_1.
2. Starting with I_n, use integration by parts.
   Then use the main trig identity to obtain an equation involving I_n and I_{n-2}.

$$I_n = \int_0^{2\pi} \underbrace{\sin^n x}_{\sin^{n-1} x \cdot \sin x} dx = -\int_0^{2\pi} \sin^{n-1} x \, d \cos x \overset{\text{by parts}}{=} - \cancel{\sin^{n-1} x \cos x \Big|_0^{2\pi}}^{\,0}$$

$$+ \int_0^{2\pi} \cos x \, d \sin^{n-1} x = (n-1) \int_0^{2\pi} \underbrace{\cos x \cdot \sin^{n-2} x \cdot \cos x}_{} dx$$

$$= (n-1)\int_0^{2\pi} (1-\sin^2 x)\sin^{n-2} x\, dx$$

$$\boxed{I_n = (n-1)\, I_{n-2} - (n-1)\, I_n} \Rightarrow n I_n = (n-1) I_{n-2}$$

$$\Rightarrow \boxed{I_n = \frac{n-1}{n} I_{n-2}} = \frac{n-1}{n}\cdot\frac{n-3}{n-2} I_{n-4} = \dots$$

For odd n $\quad I_n = \dots I_{n-2} = \dots I_{n-4} = \dots I_1 = 0$

$n = 2m+1$

For even n $\quad I_8 = \frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2} I_0 = \frac{35}{64}\pi$

$$I_{2m} = \frac{(2m-1)(2m-3)\dots 1}{(2m)(2m-2)\dots(2)} I_0 = \frac{(2m-1)!!}{(2m)!!} I_0$$

A reduction formula

Let $I_n = \int_0^{2\pi} \sin^n x\, dx$.

1. Compute I_0 and I_1.
2. Starting with I_n, use integration by parts.
   Then use the main trig identity to obtain an equation involving I_n and I_{n-2}.
3. Use the previous answers to get a formula for I_n for every positive integer I_n.
4. Compute I_8. (The answer should be $\frac{35}{64}\pi$).

We want to compute

$\sin^2 x \cdot \sin x$

$$I = \int \overset{\shortparallel}{\sin^3 x} \cos^2 x \, dx$$

$d(\cos x) \; \shortparallel \; u$

1. Attempt the substitution $u = \sin x$
2. Attempt the substitution $u = \cos x$
3. One worked better than the other. Which one? Why? Finish the problem.

We want to compute

$$I = \int \sin^3 x \cos^2 x \, dx$$

1. Attempt the substitution $u = \sin x$
2. Attempt the substitution $u = \cos x$
3. One worked better than the other. Which one? Why? Finish the problem.
4. Assume we want to compute

$$\int \sin^n x \cos^m x \, dx$$

When will the substitution $u = \sin x$ be helpful?

When will the substitution $u = \cos x$ be helpful?

Integral of products of secant and tangent

To integrate

$$\int \sec^n x \tan^m x \, dx$$

- If $\boxed{???}$, then try the substitution $u = \tan x$.
- If $\boxed{???}$, then try the substitution $u = \sec x$.

Hint: You will need

- $\dfrac{d}{dx}[\tan x] = \ldots$
- $\dfrac{d}{dx}[\sec x] = \ldots$

- The trig identity involving sec and tan