Today: Properties of sequences.

Homework before Wednesday’s class: watch videos 11.5, 11.6.
Review: the limit of a sequence

Recall: $\lim_{n \to \infty} a_n = L$ means that

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$$  

Find the limit and prove it directly from the definition for

$$a_n = \frac{n}{3n + 5}$$
True or False – Monotonic sequences vs functions

Let $f$ be a function with domain $[0, \infty)$. We define a sequence as $a_n = f(n)$.

1. IF $f$ is increasing, THEN $\{a_n\}_{n=0}^{\infty}$ is increasing.

2. IF $\{a_n\}_{n=0}^{\infty}$ is increasing, THEN $f$ is increasing.

If you think any of them is false, prove it with a counterexample.
Examples

Construct 8 examples of sequences. If any of them is impossible, cite a theorem to justify it.

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<th>convergent</th>
<th>divergent</th>
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<td><strong>monotonic</strong></td>
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<td><strong>not monotonic</strong></td>
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A suspicious calculation – What is wrong?

The sequence \( \{ a_n \}_{n=0}^{\infty} \) defined by

\[
\begin{align*}
& a_0 = 1 \\
& \forall n \in \mathbb{N}, \quad a_{n+1} = 1 - a_n
\end{align*}
\]

has limit 1/2.

Proof.

- Let \( L = \lim_{n \to \infty} a_n \).
- \( a_{n+1} = 1 - a_n \)
- \( \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} [1 - a_n] \)
- \( L = 1 - L \)
- \( L = 1/2 \).
Consider the sequence \( \{ R_n \}_{n=0}^{\infty} \) defined by

\[
\begin{align*}
R_0 &= 1 \\
\forall n \in \mathbb{N}, \quad R_{n+1} &= \frac{R_n + 2}{R_n + 3}
\end{align*}
\]

Compute \( R_1, R_2, R_3 \).
Is this proof correct?

Let \( \{R_n\}_{n=0}^{\infty} \) be the sequence in the previous slide.

**Claim:**
\[
\{R_n\}_{n=0}^{\infty} \rightarrow -1 + \sqrt{3}.
\]

**Proof.**

- Let \( L = \lim_{n \to \infty} R_n \).
- \( R_{n+1} = \frac{R_n + 2}{R_n + 3} \)
- \( \lim_{n \to \infty} R_{n+1} = \lim_{n \to \infty} \frac{R_n + 2}{R_n + 3} \)
- \( L = \frac{L + 2}{L + 3} \)
- \( L(L + 3) = L + 2 \)
- \( L^2 + 2L - 2 = 0 \)
- \( L = -1 \pm \sqrt{3} \)
- \( L \) must be positive, so \( L = -1 + \sqrt{3} \)
Consider the sequence \( \{ R_n \}_{n=0}^{\infty} \) defined by
\[
\begin{align*}
R_0 &= 1 \\
\forall n \in \mathbb{N}, \quad R_{n+1} &= \frac{R_n + 2}{R_n + 3}
\end{align*}
\]

1. Prove \( \{ R_n \}_{n=0}^{\infty} \) is bounded below by 0.
2. Prove \( \{ R_n \}_{n=0}^{\infty} \) is decreasing (use induction)
3. Prove \( \{ R_n \}_{n=0}^{\infty} \) is convergent (use a theorem)
4. Now the calculation in the previous slide is correct, and we can get the value of the limit.