The Mystery of Pentagram Maps

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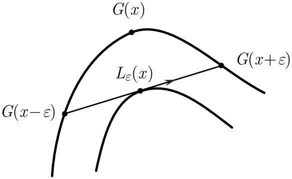
Mystery Itinerary

1. Boussinesq and higher KdV equations
2. Pentagram map in 2D
3. Pentagram maps in any dimension
4. Duality
5. Numerical integrability and non-integrability

Geometry of the Boussinesq equation

Let $G : \mathbb{R} \to \mathbb{RP}^2$ be a *nondegenerate curve*, i.e. G' and G'' are not collinear $\forall x \in \mathbb{R}$.



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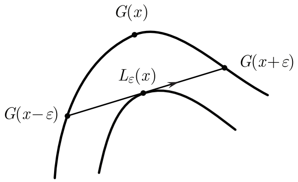

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Given $\epsilon > 0$ take the *envelope* L_ϵ of chords $[G(x-\epsilon), G(x+\epsilon)]$.

Expand the envelope L_ϵ in ϵ:

$$L_\epsilon(x) = G(x) + \epsilon^2 B_G(x) + O(\epsilon^4)$$

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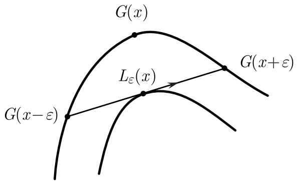

Reminder on envelopes:

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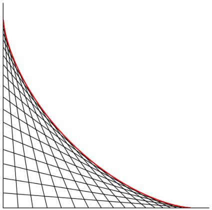

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Theorem (Ovsienko-Schwartz-Tabachnikov 2010)

The evolution equation $\partial_t G(x,t) = B_G(x,t)$ is equivalent to the Boussinesq equation $u_{tt} + 2(u^2)_{xx} + u_{xxxx} = 0$.

Remark. The Boussinesq equation is the $(2,3)$-equation of the Korteweg-de Vries hierarchy.

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Mystery 1: Where is shallow water in curve envelopes?

Reminder on the KdV hierarchy

- Consider the space of vector-functions $\{(u_0, ..., u_{m-2})(x)\}$ and view them as linear DOs

$$R = \partial^m + u_{m-2}(x)\partial^{m-2} + u_{m-3}(x)\partial^{m-3} + ... + u_1(x)\partial + u_0(x),$$

where $\partial^j := d^j/dx^j$. Define its mth root $Q = R^{1/m}$ as a formal pseudo-differential operator

$$Q = \partial + a_1(x)\partial^{-1} + a_2(x)\partial^{-2} + ...,$$

such that $Q^m = R$. (Use the Leibniz rule $\partial f = f\partial + f'$.)

- Define its fractional power $R^{k/m} = \partial^k + ...$ for any $k = 1, 2, ...$ and take its purely differential part $Q_k := (R^{k/m})_+$.

Example

for $k = 1$ one has $Q_1 = \partial$,
for $k = 2$ one has $Q_2 = \partial^2 + (2/m)u_{m-2}(x)$.

The **(k, m)-KdV equation** is

$$\frac{d}{dt} R = [Q_k, R] .$$

Given order m, these evolution equations on $R = \partial^m + ... + u_0(x)$ commute for different $k = 1, 2, \ldots$ and form integrable hierarchies.

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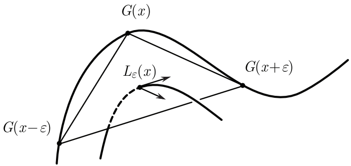
Example

- the **Korteweg-de Vries equation** $u_t + uu_x + u_{xxx} = 0$ is the "(3,2)-KdV equation". It is the 3*rd* evolution equation on Hill's operator $R = \partial^2 + u(x)$ of order $m = 2$.
- the **Boussinesq equation** $u_{tt} + 2(u^2)_{xx} + u_{xxxx} = 0$ is the "(2,3)-KdV equation". It is the 2*nd* evolution equation on operator $R = \partial^3 + u(x)\partial + v(x)$ of order $m = 3$, after exclusion of v.

In higher dimensions...

Let $G : \mathbb{R} \to \mathbb{RP}^d$ be a nondegenerate curve, i.e. $(G', G'', \ldots, G^{(d)})$ are linearly independent $\forall x \in \mathbb{R}$.



Given $\epsilon > 0$ and reals $\varkappa_1 < \varkappa_2 < \ldots < \varkappa_d$ such that $\sum_j \varkappa_j = 0$ define hyperplanes $P_\epsilon(x) = [G(x + \varkappa_1 \epsilon), \ldots, G(x + \varkappa_d \epsilon)]$.

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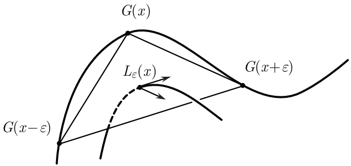

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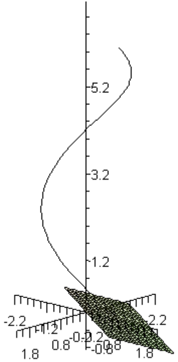
Example

In $\mathbb{RP}^3$ above $\varkappa_j = -1, 0, 1$

Let $L_\epsilon(x)$ be the envelope curve for the family of hyperplanes $P_\epsilon(x)$ for a fixed ϵ.

Reminder on envelopes of plane families

The envelope condition means that for each x the point $L_\epsilon(x)$ and the derivative vectors $L'_\epsilon(x), \ldots, L_\epsilon^{(d-1)}(x)$ belong to the plane $P_\epsilon(x)$.



Expand the envelope in ϵ:

$$L_\epsilon(x) = G(x) + \epsilon^2 B_G(x) + O(\epsilon^3)$$

Theorem (K.-Soloviev 2016)

The evolution equation $\partial_t G(x,t) = B_G(x,t)$ is equivalent to the $(2, d+1)$-KdV equation for any choice of $\varkappa_1, ..., \varkappa_d$. In particular, it is an integrable infinite-dimensional system.

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Remark. If $\sum_j \varkappa_j \neq 0$, then the expansion is

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Mystery 2: Are there higher $(k, d+1)$-KdV equations for all $k \geq 3$ hidden in the geometry of envelopes of space curves?

Relation of curves and differential operators

How to associate a curve $G \subset \mathbb{RP}^d$ to a differential operator

$$R = \partial^{d+1} + u_{d-1}(x)\partial^{d-1} + \dots + u_0(x) \quad ?$$

- Consider the linear differential equation $R\,\psi = 0$.

  Take any fundamental system of solutions

  $$\Psi(x) := (\psi_1(x), \psi_2(x), \dots, \psi_{d+1}(x)).$$

  Regard it as a map $\Psi : \mathbb{R} \to \mathbb{R}^{d+1}$.

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- Pass to the corresponding homogeneous coordinates:

$$G(x) := (\psi_1(x) : \psi_2(x) : \ldots : \psi_{d+1}(x)) \in \mathbb{RP}^d, \quad \text{i.e.} \quad G : \mathbb{R} \to \mathbb{RP}^d.$$

We associated a curve $G \subset \mathbb{RP}^d$ to a linear differential operator R. Moreover,

- *Wronskian* $\Psi(x) \neq 0 \ \forall x \iff G(x)$ *is nondegenerate* $\forall x$, i.e. $(G', G'', ..., G^{(d)})$ are linearly independent for all x.

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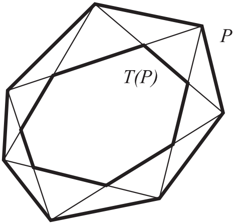
  The curve $G \subset \mathbb{RP}^d$ is defined modulo PSL_{d+1} (i.e. projective) transformations.
- For differential operator R with periodic coefficients, solutions Ψ of $R\Psi = 0$ (and hence, the curve G) are quasiperiodic: there is a monodromy $M \in SL_{d+1}$ such that $\Psi(x + 2\pi) = M\Psi(x)$ for all $x \in \mathbb{R}$.

Defining the pentagram map (R.Schwartz 1992)

The **pentagram map** takes a (convex) n-gon $P \subset \mathbb{RP}^2$ into a new polygon $T(P)$ spanned by the "shortest" diagonals of P (modulo projective equivalence):



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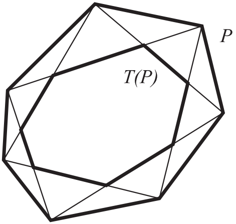
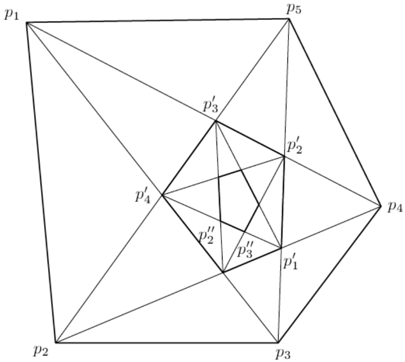


- $T = id$ for $n = 5$
- $T^2 = id$ for $n = 6$
- T is quasiperiodic for $n \geq 7$

Hidden integrability?

Why $T = id$ for a pentagon?



A pentagon in $\mathbb{RP}^2$ is fully defined by the cross-ratios at its vertices.

$$CR(z_1, z_2, z_3, z_4) := \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

Cross-ratio CR is a projective invariant of 4 points on $\mathbb{RP}^1$ or 4 lines through one vertex in $\mathbb{RP}^2$.

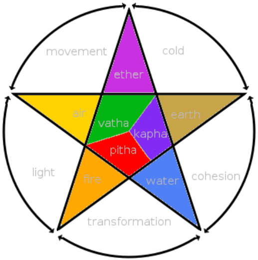
Now note:

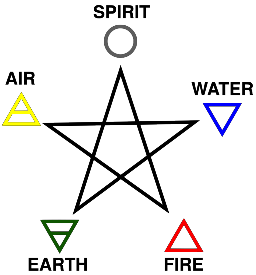
$$CR(4 \text{ lines at } p_1)$$

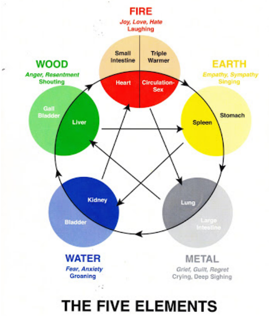
$$= CR(p_2, p_4', p_3', p_5)$$

$$= CR(4 \text{ lines at } p_1')$$

A Pentagram map?

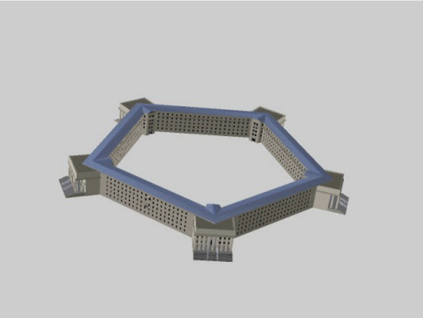






The Pentagram dates back to Pythagoras, the Greek mathematician and philosopher., He held that the number five was the number of Man because of the five-fold division of the body. He used the Pentagram to symbolize the five Elements that made up man, Earth, Air, Fire, Water and Spirit.

Or, rather, a Pentagon map?



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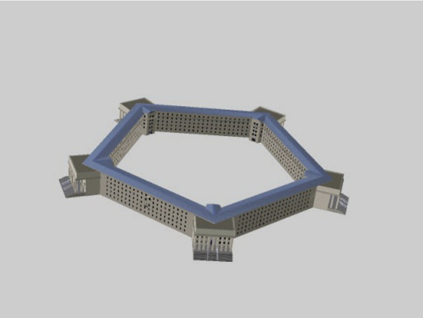

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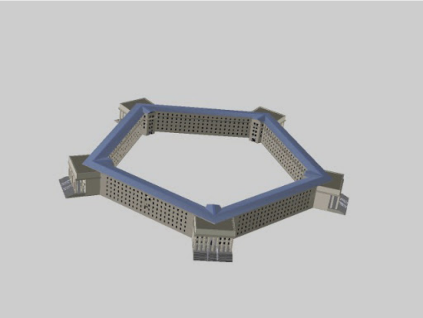

Properties of the pentagram map

- The integrability was proved for closed and twisted (i.e. with fixed monodromy) polygons in 2D (Ovsienko-Schwartz-Tabachnikov 2010, Soloviev 2012).
- There are first integrals, an invariant Poisson structure, and a Lax form.
- Pentagram map is related to cluster algebras, frieze patterns, etc.
- Its continuous limit is the Boussinesq equation.
- Extension to corrugated polygons, polygonal spirals, etc.

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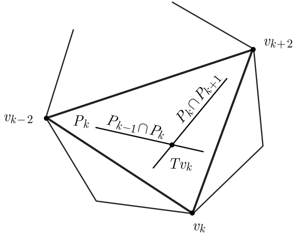
- No generalizations to polyhedra: many choices of hyperplanes passing through neighbours of any given vertex.
- Diagonal chords of a space polygon may be skew and do not intersect in general.

Example and integrability of a pentagram map in 3D

For a generic n-gon $\{v_k\} \subset \mathbb{RP}^3$, for each k consider the two-dimensional **"short-diagonal plane"** $P_k := [v_{k-2}, v_k, v_{k+2}]$.

Then the **space pentagram map** T_{sh} is the intersection point

$$T_{sh} v_k := P_{k-1} \cap P_k \cap P_{k+1}.$$



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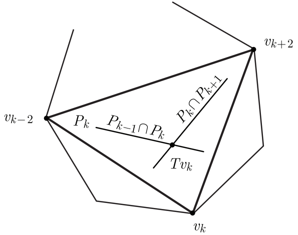
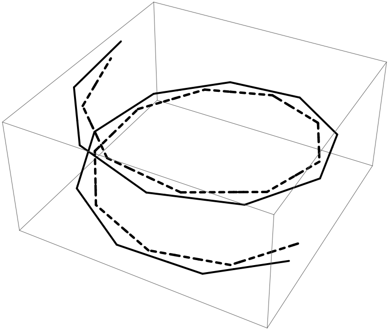


Theorem (K.-Soloviev 2012)

The 3D short-diagonal pentagram map is a discrete integrable system (on a Zariski open subset of the complexified space of closed n-gons in 3D modulo projective transformations PSL_4). It has a Lax representation with a spectral parameter. Invariant tori have dimensions $3\lfloor n/2 \rfloor - 6$ for odd n and $3(n/2) - 9$ for even n.

Twisted polygons

Remark. There is a version for twisted space n-gons, where vertices are related by a fixed monodromy $M \in PSL_4$: $v_{k+n} = Mv_k$ for any $k \in \mathbb{Z}$. The dimension of the space of *closed* n-gons modulo projective equivalence is $3n - \dim PSL_4 = 3n - 15$. This dimension for *twisted* n-gons is $3n - 15 + 15 = 3n$.



Analogy and coordinates on the spaces of polygons

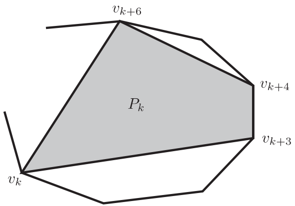
A *differential operator* $R = \partial^{d+1} + \ldots + u_0(x)$ defines a "solution curve" $\Psi : \mathbb{R} \to \mathbb{R}^{d+1}$ such that $\Psi^{(d+1)} + u_{d-1}(x)\Psi^{(d-1)} + \ldots + u_0(x)\Psi = 0$ $\forall x \in \mathbb{R}$, which defines a nondegenerate curve $G : \mathbb{R} \to \mathbb{RP}^d$ (mod projective equivalence).

A *difference operator* defines a (twisted) "polygonal curve" $V : \mathbb{Z} \to \mathbb{R}^{d+1}$ such that $V_{i+d+1} + a_{i,d}V_{i+d} + \ldots + a_{i,1}V_{i+1} \pm V_i = 0$ $\forall i \in \mathbb{Z}$, which defines a generic twisted polygon $v : \mathbb{Z} \to \mathbb{RP}^d$ (mod projective equivalence).

As $n \to \infty$ a generic n-gon $v_i, i \in \mathbb{Z}$ in $\mathbb{RP}^d$ "becomes" a nondegenerate curve $G(x), x \in \mathbb{R}$ in $\mathbb{RP}^d$.

Pentagram maps in any dimension

For a generic n-gon $\{v_k\} \subset \mathbb{RP}^d$ and
any fixed $(d-1)$-tuple $I = (i_1, ..., i_{d-1})$ of "jumps" $i_\ell \in \mathbb{N}$ define
an I-**diagonal hyperplane** P_k^I by
$P_k^I := [v_k, v_{k+i_1}, v_{k+i_1+i_2}, ..., v_{k+i_1+...+i_{d-1}}]$.



Example

The diagonal hyperplane P_k^I for the jump tuple $I = (3, 1, 2)$ in $\mathbb{RP}^4$.

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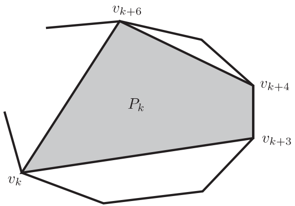

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Pentagram map T^I in $\mathbb{RP}^d$ is

$$T^I v_k := P_k^I \cap P_{k+1}^I \cap ... \cap P_{k+d-1}^I .$$

Integrability in higher dimensions

Theorem

The pentagram map T^I on (projective equivalence classes of) n-gons in $\mathbb{RP}^d$ is an integrable system, i.e. it admits a Lax representation with a spectral parameter, for

- *the "short-diagonal case", $I = (2, 2, ..., 2)$;* ($d = 2$ O.-S.-T., $d = 3$ K.-S., $d \geq 4$ K.-S.+G.Mari-Beffa)
- *the "deep-dented case", $I = (1, ..., 1, p, 1, ..., 1)$ for any $p \in \mathbb{N}$.* (any d K.-S.)

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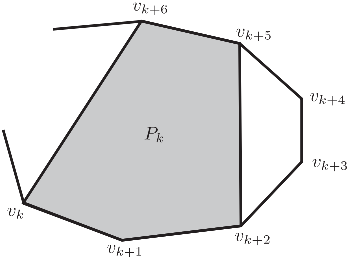
Remark. In all cases, the continuous limit $n \to \infty$ is the integrable $(2, d + 1)$-KdV equation!

Example

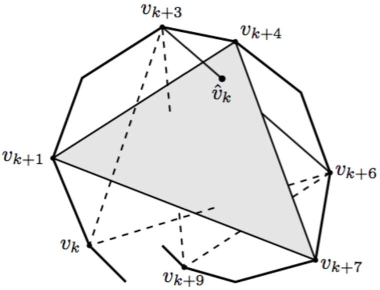
A "deep-dented" diagonal hyperplane for $I = (1, 1, 3, 1)$ in $\mathbb{RP}^5$:



The corresponding pentagram map T^I is defined by intersecting 5 consecutive hyperplanes P_k for each vertex, and it is integrable!

Theorem (Izosimov 2018)

The pentagram map in $\mathbb{R}P^d$ defined by $Tv_k := P_k^+ \cap P_k^-$ intersecting two planes $P^\pm = (v_{k+j}, j \in R^\pm)$ of complementary dimensions in $\mathbb{R}P^d$, where $R^\pm$ are two m-arithmetic sequences, is a discrete integrable system.
These pentagram maps can be obtained as refactorizations of difference operators, have natural Lax forms and invariant Poisson structures.



Corollary on long-diagonal maps

Corollary (Izosimov-K. 2020)

1) The (dual) long-diagonal pentagram map T_m^I in $\mathbb{R}P^d$ defined by the jump tuple $I = (m, ..., m, p, m, ..., m)$ (or, more generally, for the union of two m-arithmetic sequences), and $T_m^I := P_k \cap P_{k+m} \cap ... \cap P_{k+m(d-1)}$, is a completely integrable system, i.e. it admits a Lax representation with a spectral parameter and invariant Poisson structure.

2) The continuous limit is the $(2, d+1)$-KdV equation for all those cases.

Remark. Such long-diagonal pentagram maps T_m^I, along with their duals, include all known integrable cases!

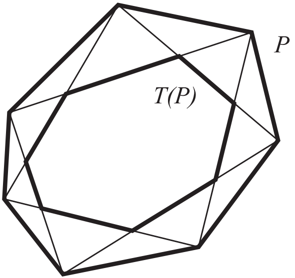
Duality

Example

The 2D pentagram map is $P \to T(P)$, i.e. T uses "step-2" diagonals and intersects them consecutively, $T := T^{(2)(1)}$.

The inverse map $T(P) \to P$ uses sides, or "step-1" diagonals, and intersects them through one, $T^{-1} = T^{(1)(2)}$.

$$(T^{(2)(1)})^{-1} = T^{(1)(2)}.$$



More generally ...

Duality

- for $(d-1)$-tuple of jumps $I = (i_1, ..., i_{d-1})$, the I-**diagonal plane** is $P_k^I := [v_k, v_{k+i_1}, v_{k+i_1+i_2}, ..., v_{k+i_1+...+i_{d-1}}]$.
- for $(d-1)$-tuple of intersections $J = (j_1, ..., j_{d-1})$, the **pentagram map** $T^{I,J}$ in $\mathbb{RP}^d$ is

$$T^{I,J} v_k := P_k^I \cap P_{k+j_1}^I \cap ... \cap P_{k+j_{d-1}}^I .$$

Example

- The 2D pentagram map is $T^{(2)(1)}$ for $I = (2)$ and $J = (1)$.
- The short-diagonal map in 3D has $I = (2, 2)$ and $J = (1, 1)$.
- Pentagram map T^I has $I = (i_1, ..., i_{d-1})$ and $J = (1, ..., 1)$.

Integrability of $T^{I,J}$ for general I and J is unknown!

Duality

Let $I^* = (i_{d-1}, ..., i_1)$ be the $(d-1)$-tuple I in the opposite order.

Theorem (K.-Soloviev 2015)

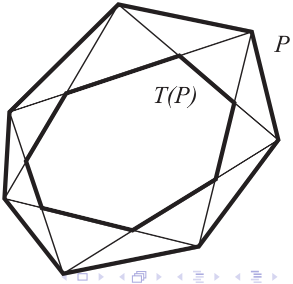
There is the following duality for the pentagram maps $T^{I,J}$:

$$(T^{I,J})^{-1} = T^{J^*,I^*}.$$

Example

For the 2D pentagram map:

$$(T^{(1)(2)})^{-1} = T^{(2)(1)}.$$



Height of a rational number and polygon

The **height of a rational number** $a/b \in \mathbb{Q}$ in the lowest terms is $ht(a/b) = \max(|a|, |b|)$. Use projectively-invariant (cross-ratio) coordinates (x_i, y_i, z_i) on the space of n-gons in $\mathbb{QP}^3$ (i.e., having only rational values of coordinates).

The **height of an n-gon** P is

$$H(P) := \max_{0 \le i \le n-1} \max(ht(x_i), ht(y_i), ht(z_i)).$$

Numerical experiments in 3D

We trace how fast the height of an initial 11-gon grows with the number of iterates of different pentagram maps in 3D.

Fix a twisted 11-gon in $\mathbb{QP}^3$ by specifying vectors in $\mathbb{Q}^4$.
Their coordinates are randomly distributed in $[1, 10]$.

Numerical experiments in 3D

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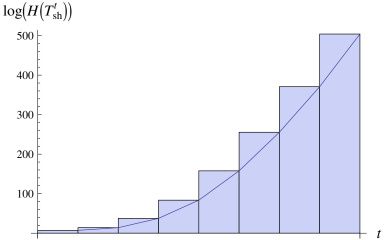
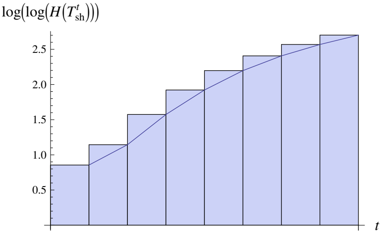
Fix a twisted 11-gon in $\mathbb{QP}^3$ by specifying vectors in $\mathbb{Q}^4$.
Their coordinates are randomly distributed in $[1, 10]$.

Observed: a sharp contrast in the height growth for different maps.
However, the borderline between numerically integrable and non-integrable cases is difficult to describe.
We group those cases separately.

Numerically integrable cases

First study the short-diagonal map $T_{\rm sh}$ in 3D, which is known to be integrable.
After $t = 8$ iterations, the height of the twisted 11-gon in $\mathbb{QP}^3$ becomes of the order of 10^{500}:

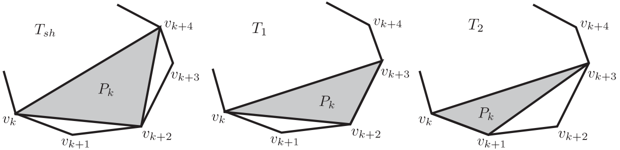




"Polynomial growth" of $\log H$ for the integrable pentagram map $T_{\rm sh}$ in 3D as a function of t.

Numerically integrable cases (cont'd)

The height also grows moderately fast for the integrable dented maps $T^{(2,1)}$ and $T^{(1,2)}$, reaching the value of the order of 10^{800}.



Similar moderate growth is observed for the (integrable) deep-dented map $T^{(1,3)}$ in 3D: the height remains around 10^{1000}.

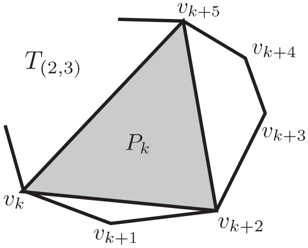
More numerically integrable cases:

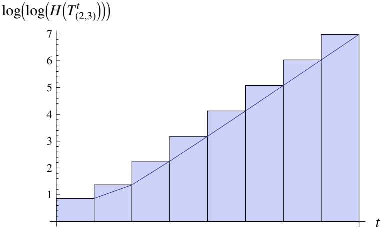
Map $T^{I,J}$	Name of pent. map	Height at $t=8$
$T_{\rm sh} := T^{(2,2),(1,1)}$	short-diagonal	10^{500}
$T^{(2,1)} := T^{(2,1),(1,1)}$	dented	10^{800}
$T^{(3,1)} := T^{(3,1),(1,1)}$	deep-dented	10^{1000}
$T^{(2,2),(1,2)}$	long-diagonal	10^{1000}
$T^{(1,2),(1,2)}$		10^{2000}
$T^{(1,3),(1,3)}$		10^{3000}
$T^{(2,3),(2,3)}$		10^{3000}

Observation - Conjecture. The pentagram maps $T^{I,I}$, i.e. those with $I=J$, are integrable.

Numerically non-integrable cases

The first case not covered by theorems: for the map $T^{(2,3)}$ after 8 iterations the height is already of order 10^{10^7}.





Linear growth of $\log\log H$ for the map $T^{(2,3)}$ in 3D indicates super-fast growth of its height and apparent non-integrability.

More numerically non-integrable cases:

Map $T^{I,J}$	**Height at** $t = 8$
$T^{(1,2),(3,1)}$	$10^{3\cdot 10^7}$
$T^{(1,2),(1,3)}$	$10^{3\cdot 10^7}$
$T^{(2,3)} := T^{(2,3),(1,1)}$	10^{10^7}
$T^{(2,4)} := T^{(2,4),(1,1)}$	10^{10^7}
$T^{(3,3)} := T^{(3,3),(1,1)}$	10^{10^7}

More numerically non-integrable cases:

Map $T^{I,J}$	Height at $t = 8$
$T^{(1,2),(3,1)}$	$10^{3 \cdot 10^7}$
$T^{(1,2),(1,3)}$	$10^{3 \cdot 10^7}$
$T^{(2,3)} := T^{(2,3),(1,1)}$	10^{10^7}
$T^{(2,4)} := T^{(2,4),(1,1)}$	10^{10^7}
$T^{(3,3)} := T^{(3,3),(1,1)}$	10^{10^7}

More Mysteries / Open Problems:

- Prove non-integrability in those cases.
- Describe the border of integrability and non-integrability.

Intricate border of integrable and non-integrable cases:

Several references

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THANK YOU!