Geometric Fluid Dynamics

Henan University, Sept - Oct 2021

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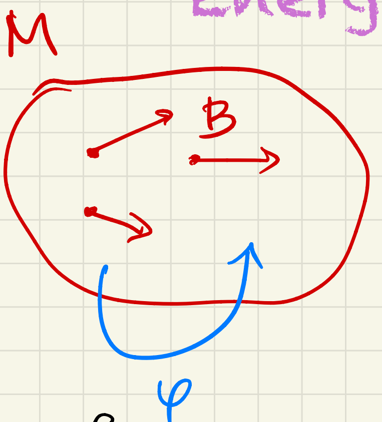
Lecture 5

Energy and helicity



Let M be a simply-conn cpt. Riem. mfd B - a (magnetic) vect. field on M (e.g. $M \subset \mathbb{R}^3$) $\operatorname{div} B = 0$ w.r.t. $\mu = d^3x$ - volume form on M

The energy of B is $E(B) = \|B\|^2_{L^2(M)} = \int_M (B, B)\,\mu$

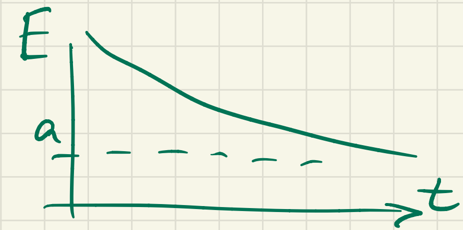
Consider a volume-preserving diffeo'm $\varphi : M \circlearrowleft$

Question Given a field B find $a := \inf_\varphi E(\varphi_* B)$

Is $a = 0$ or $a > 0$?

Motivation: B – magnetic field of a star (the sun) "frozen into" the media (plazma), i.e.

$$\partial_t B = -L_v B = -\operatorname{curl}(v \times B)$$

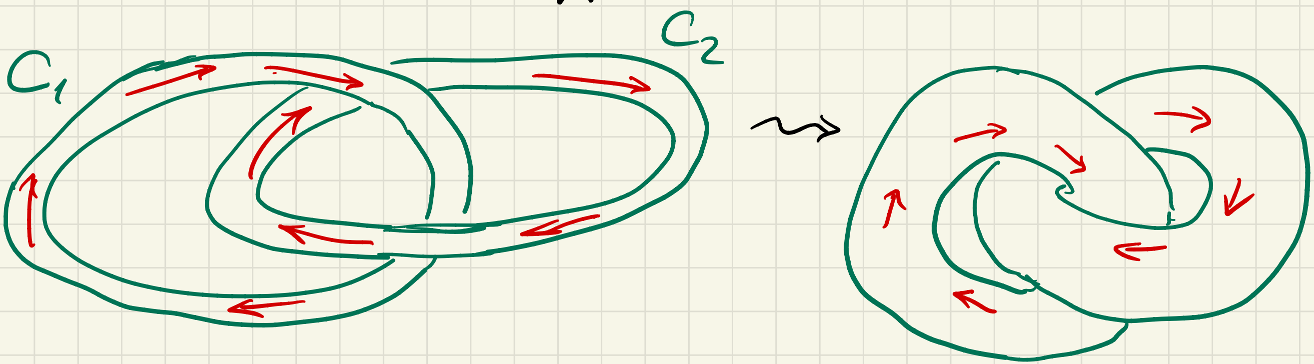


The star radiates its energy

Question: Will the star extinguish completely? (will a be positive or $=0$?)

Topological obstruction:

Assume that $\operatorname{supp} B = C_1 \cup C_2$, two solid tori



$E(B) \searrow$ $\approx$ most orbits shrink

Indeed, length $\rightsquigarrow$ length$/\lambda$, time remains the same

$\Rightarrow$ $B \rightsquigarrow B/\lambda$ $\Rightarrow$ $E \rightsquigarrow E/\lambda^2$.

But linking prevents tori from infinite flattening, since transformations are volume-preserving.

Prop (Arnold 1973) $E(B) \geq c\,|\mathrm{Hel}(B)|$,

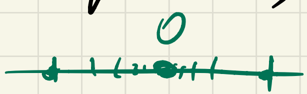
where $\mathrm{Hel}(B) := \int_M (B, \operatorname{curl}^{-1} B)\,\mu$ – helicity of B,

and $c = c(M)$. (Note: "geometry $\geq$ topology")

Rm. Here $A = \operatorname{curl}^{-1} B$ is a div-free vector-potential of B, i.e. $\nabla \times A = B$, $\operatorname{div} A = 0$. The integral is independent of the choice of A.

Pf is an appl. of the Schwarz & Poincaré inequalities to $A = \operatorname{curl}^{-1} B$

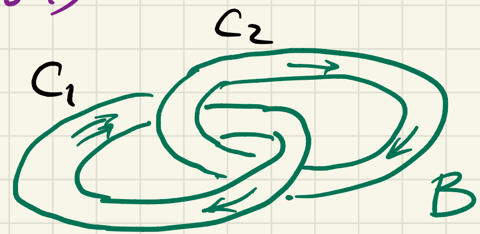
Recall, that "$\operatorname{curl}\operatorname{curl} = -\Delta$", i.e. "$\operatorname{curl} \approx \sqrt{-\Delta}$".

curl^{-1} on a cpt mfd is a bounded symmetric operator, whose spectrum accumulates to 0 on both sides: 

Then $c(M)$ is the max abs. value of its eigenvalues, depends on M.

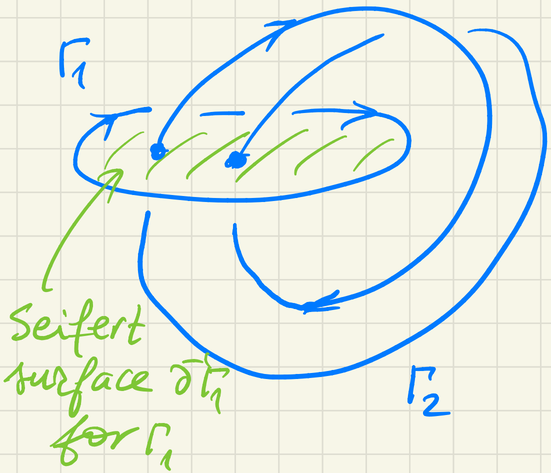
An important example (Moffatt 1969)

For a vect. field B as above, without net twist inside C_1, C_2



$$Hel(B) = 2\, lk(C_1, C_2) \cdot Flux_1 \cdot Flux_2 \neq 0$$

Rm Recall: the linking number of two oriented curves Γ_1 and Γ_2 in M^3 is



$$lk(\Gamma_1, \Gamma_2) := \#(\partial^{-1}\Gamma_1) \cap \Gamma_2$$

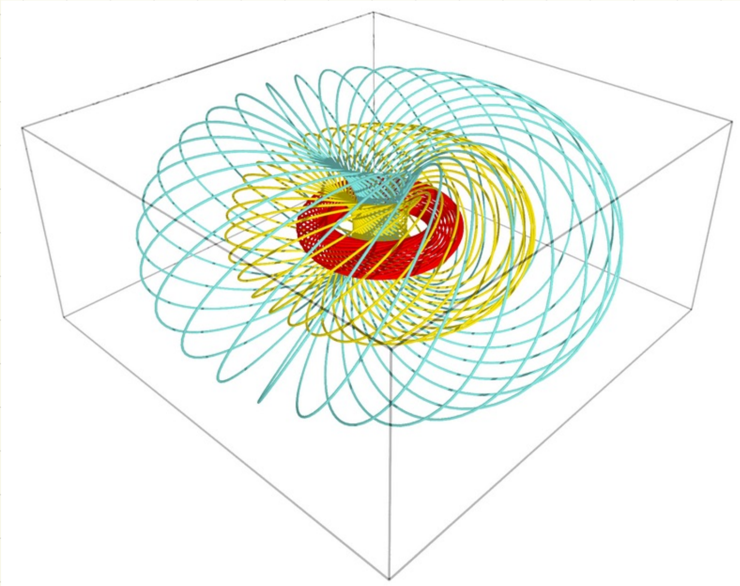
signed #

Note! lk
- is symmetric
- does not depend on the choice of $\partial^{-1}\Gamma_1$
- has a higher-dim generalization

An example: the Hopf field in S^3

For $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^{4} x_i^2 = 1\}$

define $v(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$



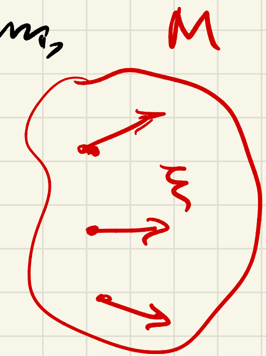
Exer v corresponds to the max eigenvalue $= \frac{1}{2}$ of curl^{-1} on S^3.

$\Rightarrow$ The Hopf field has the minimal energy among diffeomophic ones (by volume-pres. diffeo's)

A metric-free definition of helicity

Let M be a simply-conn mfd, μ - volume form, and ξ a div-free vect. field on M, i.e. $L_\xi \mu = 0$.

$\Leftrightarrow$ $\omega_\xi := i_\xi \mu$ is a closed ($\Rightarrow$ exact) 2-form on M



Hence $\omega_\xi = d\alpha$ for some 1-form α.

Def The helicity $Hel(\xi)$ of ξ on M is

$$Hel(\xi) = \int_M \alpha \wedge d\alpha = \int_M d\alpha \wedge \alpha = \int_M \omega_\xi \wedge d^{-1}\omega_\xi,$$

for $d\alpha = \omega_\xi$.

Ex $\xi = \text{curl}\, v$. Then

$$Hel(\xi) = \int_M i_\xi \mu \wedge d^{-1}(i_\xi \mu) = \int_M du \wedge u = \int_M (\text{curl}\, v, v)\, \mu.$$

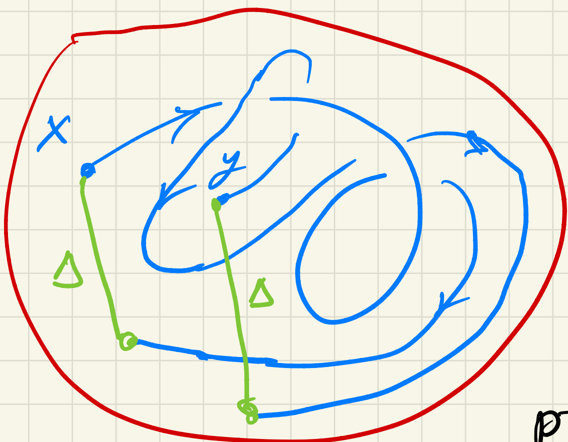
Cor (of coord-free def'n of helicity)
The helicity $Hel(\xi)$ is preserved under the action on ξ of volume-pres. diffeo's of M (i.e. it is a topological invariant).

Rm. This was an "integral def'n" of helicity. What is its topological meaning?

Helicity as an asymptotic linking

Let M be a simply-connected closed 3D mfd with a volume form μ (we do not fix metric now).

Def For a div-free vector field ξ on M (i.e. s.t. $L_\xi \mu = 0$) introduce the following function $\lambda_\xi(x,y)$, $x, y \in M$



Let $g^t(x)$, $g^s(y)$ be pieces of ξ-trajectories for times T, S resp. Assume that Δ is a "system of short paths" (joining any pair of pts on M and chosen a priori, e.g. geodesics)

Close up the trajectory pieces by Δ.

Then define $\lambda_\xi(x,y) := \lim\limits_{T,S\to\infty} \frac{1}{T\cdot S}\, lk(g^t(x), g^s(y), \Delta)$

Rm The limit exists for almost all $x, y \in M$ and doesn't depend on Δ under some cond's (Arnold)

Better: Δ - a system of geodesics, the limit exists in $L^1(M \times M)$ (T. Vogel)

Pf is based on the Birkhoff or L^1 - ergodic theorems.

Thm (V. Arnold 1973) Helicity $Hel(\xi) := \int (i_\xi \mu) \wedge d^{-1}(i_\xi \mu)$ is equal to the averaged linking:

$$Hel(\xi) = \iint_{M \times M} \lambda_\xi(x, y)\, \mu_x \mu_y$$

It is natural to call it the asymptotic Hopf invariant.

Pf sketch for $M \subset \mathbb{R}^3$

Consider the Biot-Savart integral for vector-potential $A = \operatorname{curl}^{-1}\xi$:

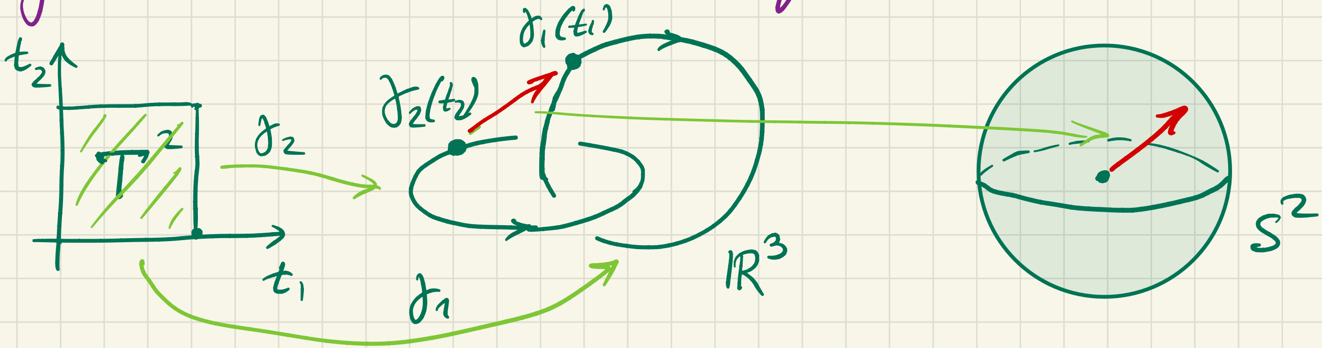
$$A(y) = -\frac{1}{4\pi}\int_M \frac{\xi(x) \times (x-y)}{\|x-y\|^3}\,\mu_x$$

Then the helicity is

$$\operatorname{Hel}(\xi) = \int(\xi, A) = \frac{1}{4\pi}\int_M\int_M \frac{(\xi(x), \xi(y), x-y)}{\|x-y\|^3}\,\mu_x\,\mu_y$$

On the other hand, recall the explicit formula for the linking number.

Digression on the Gauss formula



Gauss Thm: The linking number of closed curves $\gamma_1(S^1), \gamma_2(S^1) \subset \mathbb{R}^3$ is given by

$$lk(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_0^{T_1}\int_0^{T_2} \frac{(\dot\gamma_1, \dot\gamma_2, \gamma_1 - \gamma_2)}{\|\gamma_1 - \gamma_2\|^3} \, dt_1 \, dt_2$$

Note: $lk(\gamma_1, \gamma_2) = \deg(f : T^2 \to S^2)$, where $f = F/\|F\|$ for the map $F(t_1, t_2) := \gamma_1(t_1) - \gamma_2(t_2)$

Hence for 2 pieces of ξ-trajectories their lk is

$$\lambda_\xi(x,y) = \lim_{T,S\to\infty} \frac{1}{4\pi T\cdot S} \int_0^T \int_0^S \frac{(\dot{x}(t), \dot{y}(s), x(t)-y(s))}{\|x(t)-y(s)\|^3}\, dt\, ds$$

where $x(t) = g^t(x)$, and we neglect the integrals
$y(s) = g^s(y)$ over short paths Δ.

Now the result follows from the Birkhoff ergodic thm: the time average of $\lambda_\xi(x,y)$ along the measure-pres. flow of ξ coincides with the space average, given by the integral expression of $\mathrm{Hel}(\xi)$.

QED.

Return to energy estimates

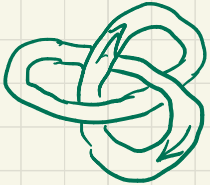
Cor. If a div. free field has nonzero helicity, its energy cannot be made arbitrarily small.

But what if $Hel(\xi) = 0$?

For instance two pairs of solid tori linked in opposite directions?

Thm (Freedman-He 1991) Suppose a vect. field ξ in $\mathbb{R}^3$ has an invariant torus T forming a nontrivial knot of type K. Then

$$E(B) \geq \left(\frac{16}{\pi \cdot Vol(T)}\right)^{1/3} \cdot |Flux\, B|^2 (2\, genus(K) - 1)$$



Cor For a nontrivial K, $E(\varphi_* B) > 0$

Cor If a field B has at least one closed linked trajectory of an elliptic type $\Rightarrow E(\varphi_* B) > 0$.

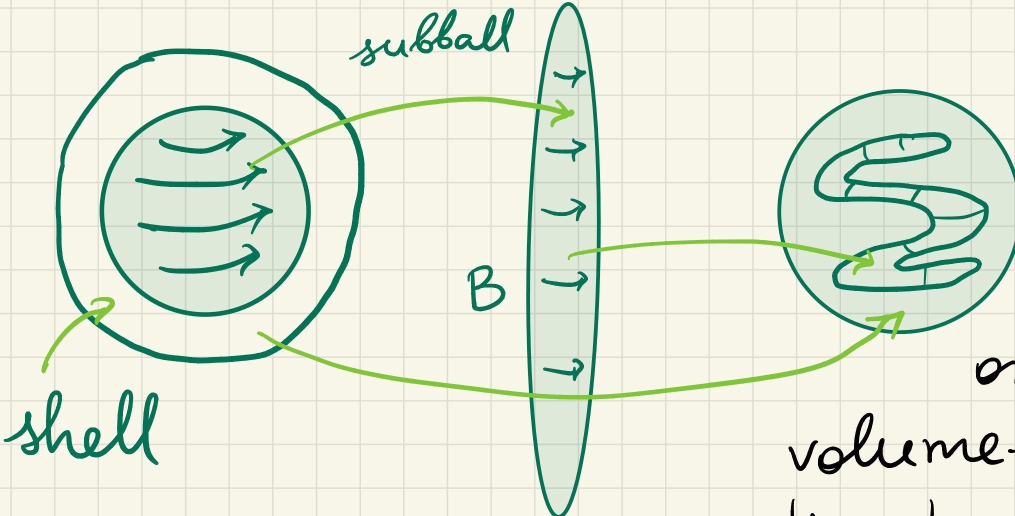
The Sakharov–Zeldovich problem

Let B be the rotation field of a ball in $\mathbb{R}^3$.

Problem: Is $\inf_{\varphi} E(\varphi_* B) = 0$?

Thm (Freedman 1990) There exists a sequence of volume-pres. diffeo's $\varphi^{(n)} : M \circlearrowleft$ such that $E(\varphi_*^{(n)} B) \to 0$ as $n \to \infty$

Pf idea:



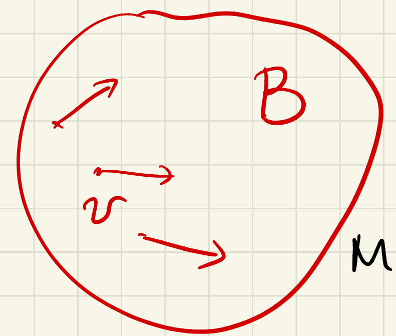
Stretch a subball to shorten trajectory and put the snake obtained inside the sphere.

Use Moser's lemma on existence of volume-pres. diffeo'm to estimate the energy in the shell image.

Another direction: Fast dynamo problem

Def. The kinematic dynamo equation is

$$\begin{cases} \partial_t B = -L_v B + \eta \Delta B \\ \operatorname{div} B = 0 \end{cases}$$



The unknown magnetic field $B(t)$ is stretched by the fluid flow with velocity v, while a low diffusion dissipates the magnetic energy $E(B)$.

Problem Does there exist a div-free vect. field v in M s.t. $E(B(t))$ grows exponentially in time (for some initial $B(0)$) as $\eta \to 0$ or $\eta = 0$?

Look for solutions $B = e^{\lambda(\eta)t} B(0)$ such that $\mathrm{Re}\,\lambda(\eta) \geq \lambda_0 > 0$ as $\eta \to 0$ or $\eta = 0$.

A non-dissipative dynamo ($\eta = 0$) corresponds to a frozen magnetic field.

Rm There are many works constructing explicit dynamos and proving a necessity of chaotic behavior of v for $\eta \neq 0$. One of popular examples is the ABC-flow.

Ex The ABC flows (Arnold-Beltrami-Childress) are

$$\mathbf{v} = (A\sin z + C\cos y)\frac{\partial}{\partial x} + (B\sin x + A\cos z)\frac{\partial}{\partial y} + (C\sin y + B\cos x)\frac{\partial}{\partial z}$$

on 3D torus $T^3 = \{(x, y, z) \mid \mathrm{mod}\ 2\pi\}$.

They are eigen for curl: $\mathrm{curl}\,\mathbf{v} = \mathbf{v}$

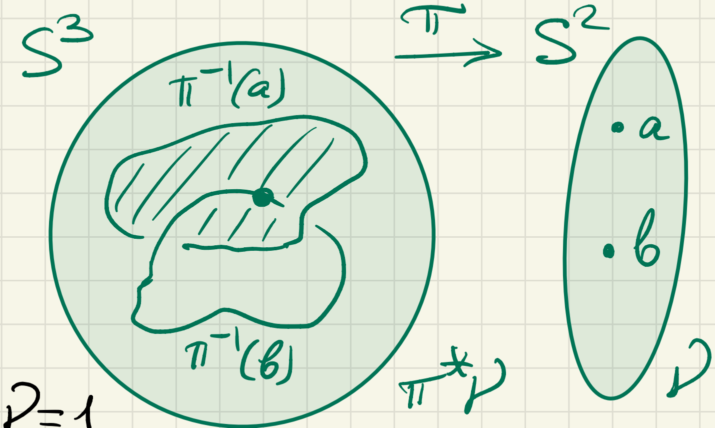
Bonus: Why Hopf?

Recall: The Hopf invariant of a map $\pi: S^3 \to S^2$ has 2 def's:

a) geometric/topological:

$$\mathrm{Hopf}_1(\pi) = \mathrm{lk}(\pi^{-1}(a), \pi^{-1}(b))$$

It doesn't depend on $a, b \in S^2$



b) integral: take $\nu \in \Omega^2(S^2)$, $\int \nu = 1$, then $\pi^* \nu \in \Omega^2(S^3)$ is closed $\Rightarrow$ exact on S^3. Then

$$\mathrm{Hopf}_2(\pi) = \int_{S^3} \pi^* \nu \wedge d^{-1}(\pi^* \nu)$$

Why Hopf_2 is an integer?

Note: in the formula for Hopf_2 the closed 2-form ν can be replaced by a cohomological one, $\tilde{\nu}$ on S^2, since their difference is exact, $\nu - \tilde{\nu} = d\alpha$, $\alpha \in \Omega^1(S^2)$, and

$$\int_{S^3} \pi^*\nu \wedge d^{-1}(\pi^*\nu) - \int_{S^3} \pi^*\nu \wedge d^{-1}(\pi^*\tilde{\nu}) = \int_{S^3} \pi^*\nu \wedge d^{-1}(\pi^* d\alpha)$$

$$= \int_{S^3 = \pi^{-1}(S^2)} \pi^*\nu \wedge \pi^*\alpha = \int_{S^2} \nu \wedge \alpha = 0 .$$

Now take $\nu = \delta(a)$, $\tilde{\nu} = \delta(b)$ – the δ-type 2-forms in S^2, supported at pts a and b. Then $\pi^*\nu = \delta(\pi^{-1}(a))$ and $\pi^*\tilde{\nu} = \delta(\pi^{-1}(b))$, δ-type 2-forms in S^3 supported on $\pi^{-1}(a)$, $\pi^{-1}(b)$.

Then $d^{-1}(\pi^*\nu) = \delta(\partial^{-1}\pi^{-1}(a))$ – 1-form supported on a Seifert surface $\partial^{-1}\pi^{-1}(a)$ in S^3.

Hence $\mathrm{Hopf}_2(\pi) = \int_{S^3} \pi^*\nu \wedge d^{-1}(\pi^*\nu) = \int_{S^3} \pi^*\tilde{\nu} \wedge d^{-1}(\pi^*\nu)$

$= \int_{S^3} \delta(\pi^{-1}(b)) \wedge \delta(\partial^{-1}\pi^{-1}(a)) = \# \, \partial^{-1}\pi^{-1}(a) \cap \pi^{-1}(b)$

intersections of a Seifert surface with another curve

$= \mathrm{lk}(\pi^{-1}(a), \pi^{-1}(b)) = \mathrm{Hopf}_1(\pi)$

This proves the equivalence of 2 definitions.

Arnold's thm is an asymptotic version of this equivalence. Namely instead of a map $\pi: S^3 \to S^2$, where the fibers are closed (which corresponds to a field ξ whose all trajectories are closed) consider a div-free v.f. ξ with arbitrary trajectories.