Instructor: B. Khesin

Graduate course MAT 1126HS
Spring 2018
"Lie Groups and Hamiltonian PDEs"

Syllabus:

I. Introduction and main notions.

1. Lie groups and Lie algebras.
2. Adjoint and coadjoint orbits.
3. Central extensions.
4. The Lie–Poisson (or Euler) equations for Lie groups.
5. Bihamiltonian systems.

II. Geometry of infinite-dimensional Lie groups and their orbits.

1. Affine Kac–Moody Lie algebras and groups.
   - 1.1. Definition of the affine Kac–Moody Lie algebras.
   - 1.2. Affine Lie groups.
   - 1.3. Their coadjoint orbits.
   - 1.4. The quotient (WZW) construction of the affine groups.
2. The Virasoro algebra and group. The KdV equation.
   - 2.1. The group of circle diffeomorphisms.
   - 2.2. The Virasoro group and coadjoint action.
   - 2.3. Virasoro coadjoint orbits.
   - 2.4. The Virasoro group and Korteweg-de Vries equation.
   - 2.5. Bihamiltonian structure of the KdV.
3. Groups of (pseudo)differential operators. Integrable KP-KdV hierarchies.
   - 3.1. Pseudodifferential operators and cocycles on them.
   - 3.2. The Lie group of pseudodifferential operators of complex degree.
   - 3.3. Integrable KP-KdV hierarchies.
4. Groups of diffeomorphisms. The hydrodynamical Euler equation.
   - 4.1. The Lie group of volume-preserving diffeomorphisms and its Lie algebra.
   - 4.2. Coadjoint action and Casimirs.

III. Applications to PDEs in geometric fluid dynamics (time permitted).

1. Ideal hydrodynamics and optimal mass trasport. Otto's calculus.
2. Compressible fluid dynamics.
3. Properties of the Madelung transform.
4. Structures on and dynamics of vortex sheets.
5. H^1 geometry on diffeomorphism groups and Fisher-Rao metric.

References:

1. B. Khesin and R. Wendt "The geometry of infinite-dimensional groups," Ergebnisse der Mathematik und Grenzgebiete 3.Folge, 51, Springer-Verlag (2008), xviii+304pp, see *http://www.math.toronto.edu/khesin/papers/Lecture_notes.pdf*

2. A. Pressley and G. Segal: "Loop Groups," Clarendon Press, Oxford (1986)

Prerequisites:

A basic course (or familiarity with main notions) of symplectic geometry would be helpful.