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Graduate course MAT 1126HS
Spring 2013
"Lie Groups and Hamiltonian PDEs"

Syllabus:

I. Introduction and main notions.

1. Lie groups and Lie algebras.
2. Adjoint and coadjoint orbits.
3. Central extensions.
4. The Lie–Poisson (or Euler) equations for Lie groups.
5. Bihamiltonian systems.
6. Symplectic reduction.

II. Geometry of infinite-dimensional Lie groups and their orbits.

1. Affine Kac–Moody Lie algebras and groups.
   - 1.1. Definition of the affine Kac–Moody Lie algebras.
   - 1.2. Affine Lie groups.
   - 1.3. Their coadjoint orbits.
   - 1.4. The quotient (WZW) construction of the affine groups.
2. The Virasoro algebra and group. The KdV equation.
   - 2.1. Definitions.
   - 2.2. The group of circle diffeomorphisms.
   - 2.3. The Virasoro coadjoint action.
   - 2.4. Virasoro coadjoint orbits.
   - 2.5. The Virasoro group and Korteweg-de Vries equation.
   - 2.6. Bihamiltonian structure of the KdV.
3. Groups of diffeomorphisms. The hydrodynamical Euler equation.
   - 3.1. The Lie group of volume-preserving diffeomorphisms and its Lie algebra.
   - 3.2. Coadjoint action and Casimirs.
   - 3.3. Other diffeomorphism groups.
4. Groups of (pseudo)differential operators. Integrable KP-KdV hierarchies.
   - 4.1. Pseudodifferential operators and cocycles on them.
   - 4.2. The Lie group of pseudodifferential operators of complex degree.
   - 4.3. Integrable KP-KdV hierarchies.

5. The double loop (or elliptic) Lie groups and Lie algebras.
    5.1. Definitions.
    5.2. Classification of coadjoint orbits.
    5.3. Monodromy and holomorphic loop algebras.

III. Poisson structures on moduli spaces.

1. Definition and integrability of holomorphic bundles.
2. Moduli spaces of flat connections.
3. The Poincaré residue and Cauchy–Stokes formula.
4. Moduli spaces of holomorphic bundles.

IV. Around the Chern–Simons functional.

1. A reminder on the Lagrangian formalism.
2. Main example: the Chern–Simons action functional.
3. The holomorphic Chern–Simons functional.
4. The Chern–Simons functional and linking numbers.

References:

1. B. Khesin and R. Wendt "The geometry of infinite-dimensional groups," Ergebnisse der Mathematik und Grenzgebiete 3.Folge, 51, Springer-Verlag (2008), xviii+304pp, see *http://www.math.toronto.edu/khesin/papers/Lecture_notes.pdf*

2. A. Pressley and G. Segal: "Loop Groups," Clarendon Press, Oxford (1986)

Prerequisites:

A basic course (or familiarity with main notions) of symplectic geometry would be helpful.