Closed Problems Session

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A Mathematical Trivium

V. I. Arnold

The standard of mathematical culture is falling; both undergraduate and graduate students educated at our colleges, including the Mechanics and Mathematics Department of Moscow State University, are becoming no less ignorant than the professors and teachers. What is the reason for this abnormal phenomenon? Following the general principle of propagation of knowledge, under normal conditions students usually know their subject better than their professors: new knowledge prevails not because old men learn it, but because new a generation enters the field already knowing it.

Russian Mathematical Surveys, 46:1 (1991), 271-278.

Comments: ARNOLD: Swimming against the tide, AMS, 2014

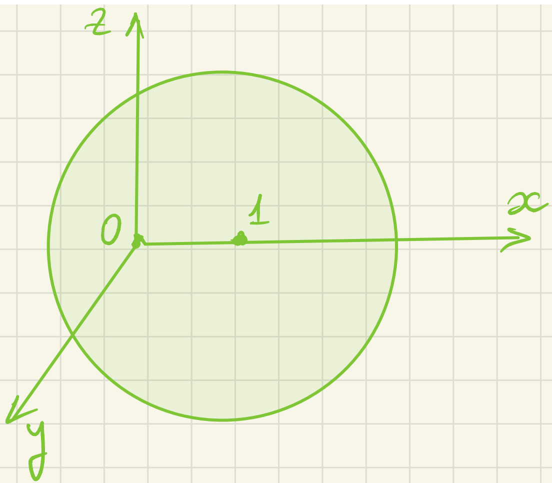
Thus, *to put an end to this spurious enhancement of the results, we must specify not a list of theorems, but a collection of problems which students should be able to solve. These lists of problems must be published annually (I think there should be ten problems for each one-semester course).* Then we shall see what we really teach our students and how successful we are. And in order for students to learn how to use their knowledge, *all examinations must be written examinations.*

The compilation of model problems is a laborious job, but I think it must be done. As an attempt I give below a list of one hundred problems forming a mathematical minimum for a physics student. Model problems (unlike syllabi) are not uniquely defined, and many will probably not agree with me. Nonetheless I believe that it is necessary to start defining mathematical standards using written examinations and model problems. One wants to hope that in the future students will receive model problems for each course at the beginning of each semester, while oral examinations with cramming of rote learning will become a thing of the past.

(12) Find the flux of the vector field $\vec{r}/r^3$ through the surface

$$(x-1)^2 + y^2 + z^2 = 2.$$

12. Note that this vector field is $\nabla(-1/r)$, the gradient of the potential function $-1/r$. By the divergence theorem the given integral over the sphere equals the integral of the divergence of this field, $\operatorname{div}(\nabla(-1/r)) = \Delta(-1/r)$, over the ball bounded by the sphere. In turn, the function $1/r$ is a multiple of the Green function in $\mathbb{R}^3$, so its integral over a domain depends on whether the source point is inside the sphere or not.



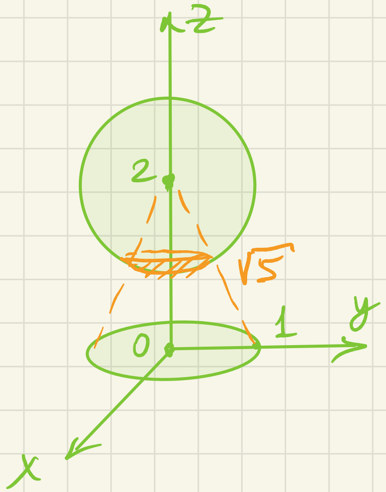
$$\text{Flux } v\Big|_{S^2} = \int_{B^3} (\operatorname{div} v)\, \text{vol} = \int_{B^3} \Delta\left(-\frac{1}{r}\right) \text{vol}$$

$$= 4\pi \int_{B^3} G(0,\bar{x})\, d^3x = 4\pi \int_{B^3} \delta(\bar{x})\, d^3x = 4\pi$$

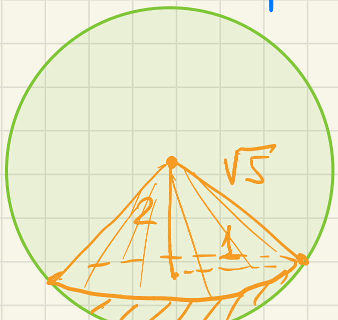
(69) Prove that the solid angle based on a given closed contour is a function of the vertex of the angle that is harmonic outside of the contour.

(70) Calculate the mean value of the solid angle by which the disc $x^2 + y^2 \leq 1$ lying in the plane $z = 0$ is seen from points of the sphere $x^2 + y^2 + (z-2)^2 = 1$.

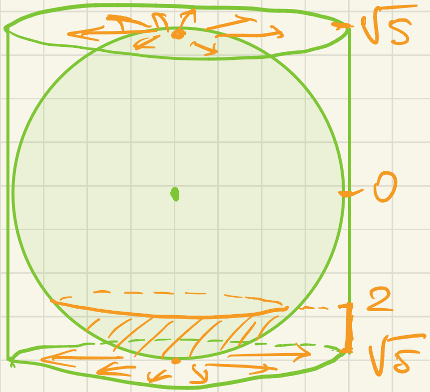
70. Use problem 69: such a solid angle is a harmonic function in $\mathbb{R}^3$ outside the circle. Hence its mean value on the sphere is equal to its value at the sphere center.



Now we need to find the solid angle at the point $(0,0,2)$



Use Archemedes' thm: the projection sphere $\to$ cylinder is area-preserving!



Then

$$\frac{S_{cap}}{S_{sphere}} = \frac{S_{\text{lower part}}}{S_{cylinder}} = \frac{\sqrt{5}-2}{2\sqrt{5}}$$

and the solid angle $= 4\pi \cdot \dfrac{\sqrt{5}-2}{2\sqrt{5}}$

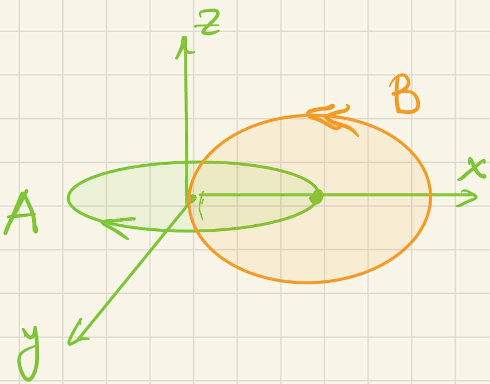
(39) Calculate the Gauss integral

$$\iint \frac{(d\vec{A}, d\vec{B}, \vec{A}-\vec{B})}{|\vec{A}-\vec{B}|^3}, = 4\pi \cdot lk(A, B)$$

where $\vec{A}$ runs along the curve $x = \cos\alpha$, $y = \sin\alpha$, $z = 0$, and $\vec{B}$ along the curve $x = 2\cos^2\beta$, $y = \frac{1}{2}\sin\beta$, $z = \sin 2\beta$.

Note! $x = 2\cos^2\beta = 1 + \cos 2\beta$

$z = \sin 2\beta$



$lk(A, B) = 2$

(33) Find the linking coefficient of the phase trajectories of the equation of small oscillations $\ddot{x} = -4x, \ddot{y} = -9y$ on a level surface of the total energy.

33. These phase trajectories give the Hopf fibration of the 3-dimensional sphere, the level surface of the total energy. Different Hooke coefficients imply that the periods of oscillations in the $(x, \dot{x})$- and the $(y, \dot{y})$-planes differ. For the common period, one trajectory traverses the circle twice, and the other trice.

$$E = \underbrace{\frac{\dot{x}^2}{2} + 2x^2}_{= c_1} + \underbrace{\frac{\dot{y}^2}{2} + \frac{9}{2}y^2}_{= c_2},$$

level $E = \text{const} = c_1 + c_2$ is an ellipsoid, fibered by tori $c_1 = \text{const}$, $c_2 = \text{const}$

frequencies of oscillators are 2 & 3

$\left(\ddot{x} = -\omega^2 x \Rightarrow \text{frequency is } \omega \text{ and } T = \frac{2\pi}{\omega}\right)$

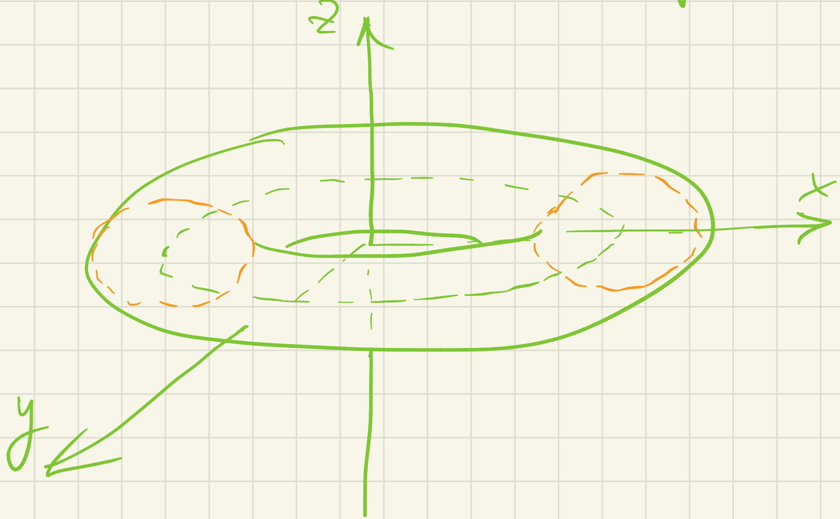
linking for the common period is

$$\mathrm{lk} = 2 \cdot 3 = 6$$

(35) Sketch the geodesics on the surface

$$(x^2 + y^2 - 2)^2 + z^2 = 1.$$

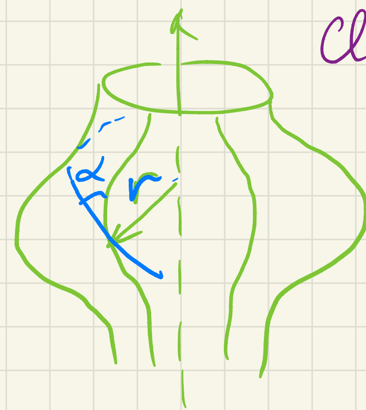
The surface is the torus of revolution around the z-axis



for geodesics on a surface of revolution

$r \cdot \sin\alpha = \text{const} = r_0$

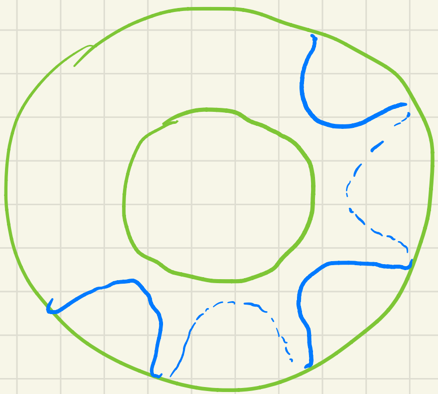
Clairaut's thm

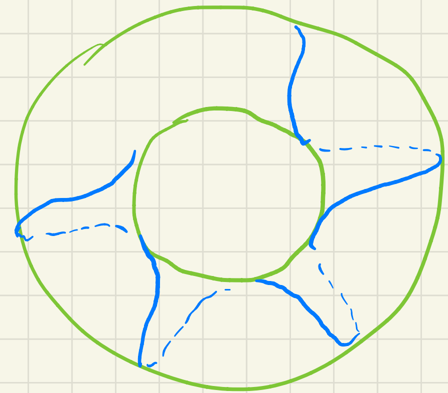


Note: since $|\sin\alpha| \leq 1$

$r \geq r_0$

For the torus there can be e.g. such geodesics
(view from above)



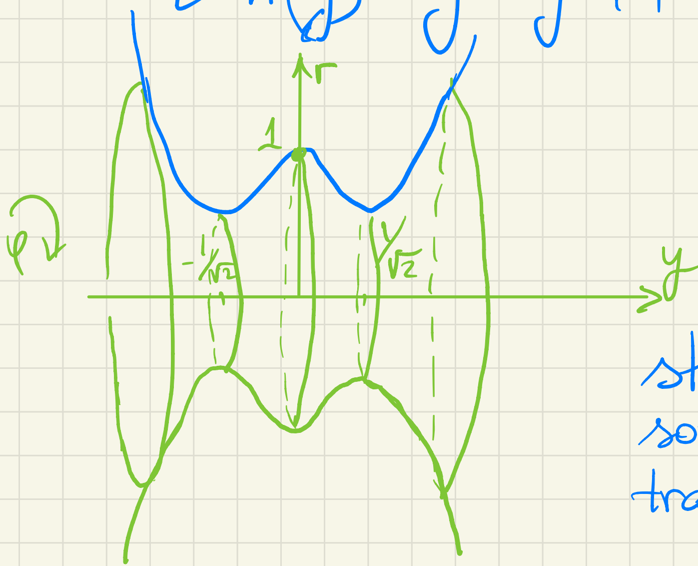


(19) Investigate the path of a light ray in a plane medium with refractive index $n(y) = y^4 - y^2 + 1$, using Snell's law $n(y) \sin \alpha = \text{const}$, where α is the angle made by the ray with the y-axis.

Hint: Note similarity with Clairaut's thm

Consider the surface of revolution

$r = n(y) = y^4 - y^2 + 1$ in cylindrical coord's (r, φ, y)



Then geodesics in (φ, y)

$\rightsquigarrow$ light rays in (x, y)

This picture allows one to study behaviour of geodesics: some go through, some are trapped in the middle section, etc.

(53) Investigate the singular points of the differential form $dt = dx/y$ on the compact Riemann surface $y^2/2 + U(x) = E$, where U is a polynomial and E is not a critical value.

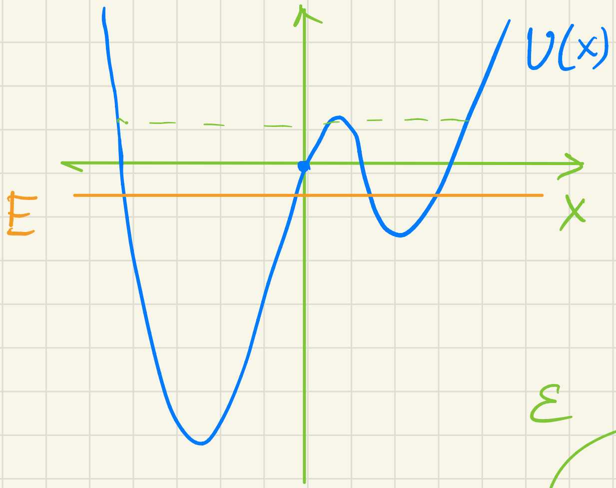
(54) Let $\ddot{x} = 3x - x^3 - 1$. In which of the potential wells is the period of oscillation greater (in the more shallow one or in the deeper one) for equal values of the total energy?

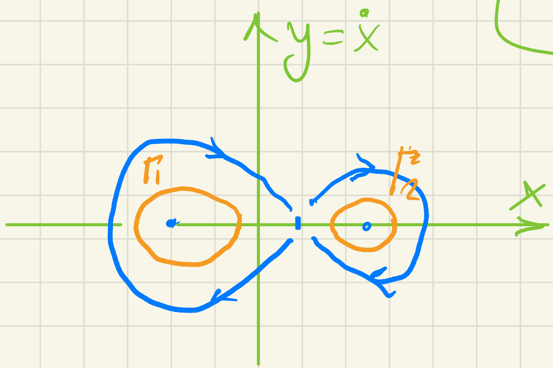
- For a noncritical value of E the 1-form $\frac{dx}{y}$ has no singularities*. (Indeed, differentiate to get $y\,dy + U'(x)\,dx = 0$. Then $\frac{dx}{y} = -\frac{dy}{U'(x)}$, which is holomorphic for $x \neq \infty$. At $x = \infty$ do a variable change.)

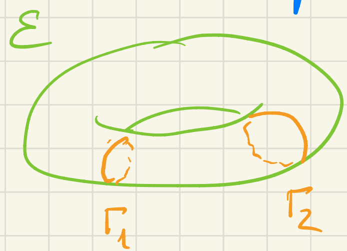
  *) for $\deg U \geq 3$. The cases of $\deg U = 1$ and 2 are special.

- For Newton's eq'n $\ddot{x} = 3x - x^3 - 1$ $\left(= -\frac{\partial U}{\partial x}\right)$ the potential is a polynomial of $\deg = 4$:

$$U(x) = \frac{x^4}{4} - \frac{3x^2}{2} + x$$







The level

$$\frac{x^4}{4} - \frac{3x^2}{2} + x = E$$

is an elliptic curve $\mathcal{E}$ in $\mathbb{C}^2 = \langle x, y \rangle$

The period $T_1 = \int_{\Gamma_1} dt = \int_{\Gamma_1 \subset \mathcal{E}} \frac{dx}{y}$

since Γ_1 is homologous to Γ_2 in $\mathcal{E}$ we obtain

$$= \int_{\Gamma_2 \subset \mathcal{E}} \frac{dx}{y} = T_2 ,$$

i.e the periods are the same!

(61) What is the largest value of t for which the solution of the problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \sin x, \quad u|_{t=0} = 0,$$

can be extended to the interval $[0, t)$?

(62) Find all solutions of the equation $y\,\partial u/\partial x - \sin x\,\partial u/\partial y = u^2$ in a neighbourhood of the point $(0, 0)$.

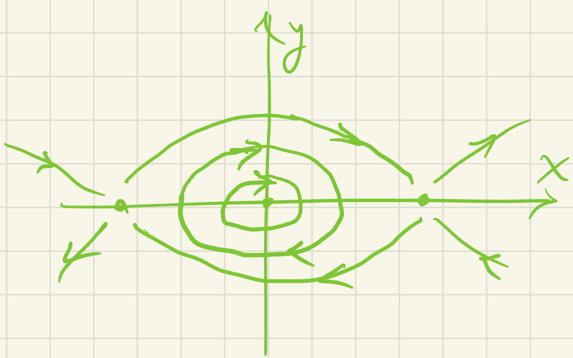
(63) Is there a solution of the Cauchy problem $y\,\partial u/\partial x + \sin x\partial u/\partial y = y$, $u|_{x=0} = y^4$ on the whole (x, y) plane? Is it unique?

#62: Let $v = y\frac{\partial}{\partial x} - \sin x \frac{\partial}{\partial y}$. Then the eq'n is

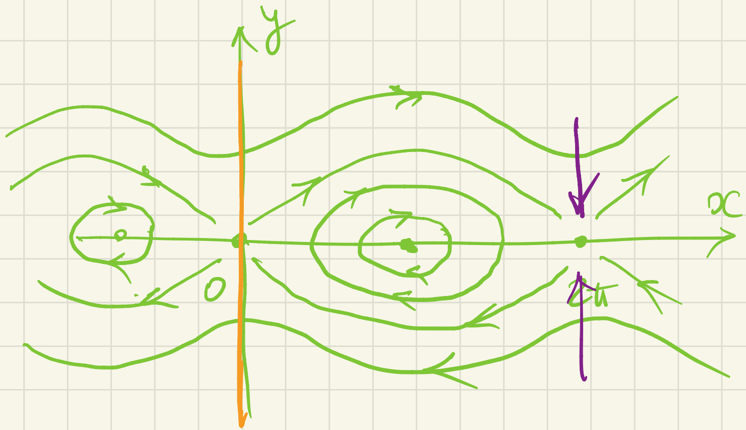
$$L_v u = u^2 \text{ near } (0,0),$$

i.e. solution u always increases along periodic orbits!

Thus $u \equiv 0$



#63: Now $v = y\frac{\partial}{\partial x} + \sin x \frac{\partial}{\partial y}$ and $L_v u = y$, $u\big|_{x=0} = y^4$



The initial cond'n on the y-axis defines solution u along characteristics intersecting the y-axis, i.e. everywhere on $\mathbb{R}^2$ outside "the eyes". For instance,

$$u = 4(y^2 + \cos x - 1)^2 + x \text{ is a solution on } \mathbb{R}^2.$$

$\Rightarrow$ existence

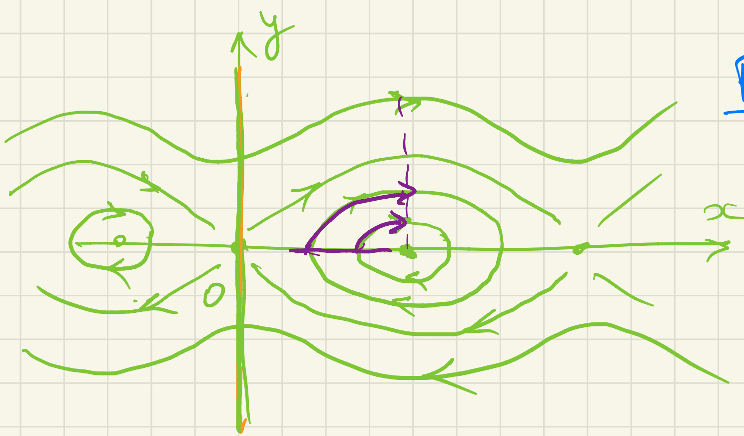
Any choice inside the eyes $\Rightarrow$ nonuniqueness

\# 61: The eq'n of characteristics (pendulum eq'n)

$$\begin{cases} \dot{t} = 1 \\ \dot{x} = u \\ \dot{u} = \sin x \end{cases}$$

The surface defined by $u\big|_{t=0} = 0$ is not a graph $u = u(x,t)$ where the pendulum trajectories start "passing by" each other, i.e. for $t \geq \frac{\pi}{2}$.

Thus the max interval is $[0, \frac{\pi}{2})$



<u>Rm</u> This is the Burgers eq'n w/ forcing:

for $x = \varphi(t)$, $\dot{\varphi} = u(x,t)$

$\ddot{\varphi} = u_t + u_x \cdot \dot{\varphi} = u_t + u u_x$, i.e.

$\ddot{x} = \sin x$ – pendulum

(75) On account of the annual fluctuation of temperature the ground at a given town freezes to a depth of 2 metres. To what depth would it freeze on account of the daily fluctuation of the same amplitude?

75. Heat propagation is described by the heat equation

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

where $T(x,t)$ is the temperature at depth x at time t. This equation is invariant under the one-parameter group $(x,t) \mapsto (cx, c^2 t)$. Therefore if the time is changed by a factor of 365 then the depth is changed by a factor of $\sqrt{365} \approx 19$. Hence the ground would freeze at depth of about 10 cm.