Available online at www.sciencedirect.com
SciVerse ScienceDirect
Procedia IUTAM 7 (2013) 135 – 140
Procedia IUTAM
www.elsevier.com/locate/procedia

Topological Fluid Dynamics: Theory and Applications

The vortex filament equation in any dimension

Boris Khesin*

Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada

Abstract

We present the vortex filament (or localized induction approximation) equation in any dimension. For an arbitrary $n \geq 3$ the evolution of vorticity supported on vortex membranes of codimension 2 in $\mathbb{R}^n$ is described by the skew (or binormal) mean-curvature flow, which generalises to any dimension the classical binormal equation in $\mathbb{R}^3$. This paper is a brief summary of the results in Khesin (2012) and Shashikanth (2012) [4, 6].

Selection and/or peer-review under responsibility of the Isaac Newton Institute for Mathematical Sciences, University of Cambridge

Keywords: Vortex filament equation; LIA, vorticity Euler equation, mean curvature.

1. The vortex filaments, membranes, and skew-mean-curvature flow

The *vortex filament (or binormal) equation* is the evolution equation

$$\partial_t \gamma = \gamma' \times \gamma'' , \tag{1}$$

of an arc-length parametrized space curve $\gamma(\cdot, t) \subset \mathbb{R}^3$, where $\gamma' := \partial\gamma/\partial\theta$. For an arbitrary parametrisation the filament equation reads $\partial_t \gamma = k \cdot \mathbf{b}$, where k and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ stand, respectively, for the curvature value and binormal unit vector of the curve γ at the corresponding point.

This binormal equation is known to be Hamiltonian with the Hamiltonian function given by the length functional $H(\gamma) = \text{length}(\gamma) = \int_\gamma \|\gamma'(\theta)\| \, d\theta$ and relative to the *Marsden-Weinstein symplectic structure* on non-parametrized oriented space curves in $\mathbb{R}^3$, see e.g. [2, 5]. At a curve γ this symplectic structure is

$$\omega_\gamma^{MW}(V, W) := \int_\gamma i_V i_W \mu = \int_\gamma \mu(V, W, \gamma') \, d\theta \tag{2}$$

where V and W are two vector fields attached to the curve γ and regarded as variations of this curve, while the volume form μ is evaluated on the three vectors V, W. Equivalently, the Marsden-Weinstein symplectic structure can be defined by means of the operator J of almost complex structure on curves: any variation V is rotated by the

* *E-mail address:* khesin@math.toronto.edu

2210-9838 © 2013 The Authors. Published by Elsevier B.V.
Selection and/or peer-review under responsibility of the Isaac Newton Institute for Mathematical Sciences, University of Cambridge
doi:10.1016/j.piutam.2013.03.016

operator J in the planes orthogonal to γ by $\pi/2$ in the positive direction (which makes a skew-gradient from a gradient field).

Furthermore, the Hasimoto transformation at any time t sends a curve $\gamma(\theta)$ with curvature $k(\theta)$ and torsion $\tau(\theta)$ to the wave function $\psi(\theta) = k(\theta)\exp\{i\int^\theta \tau(\zeta)\,d\zeta\}$ satisfying the 1-dimensional focusing nonlinear Schrödinger equation (NLS):

$$i\partial_t\psi + \psi'' + \frac{1}{2}|\psi|^2\psi = 0\,.$$

In particular, the binormal equation is an infinite-dimensional integrable system.

A natural extension of the binormal equation to higher dimensions is as follows. Consider a closed oriented embedded submanifold (membrane) P of codimension 2 in $\mathbb{R}^n$ (or more generally, in a Riemannian manifold M^n) with $n \geq 3$. The Marsden-Weinstein symplectic structure ω^{MW} on membranes of codimension 2 in $\mathbb{R}^n$ (or in any n-dimensional manifold) with a volume form μ is defined similar to the 3-dimensional case: two variations of a membrane P are regarded as a pair of normal vector fields attached to the membrane P and the value of the symplectic structure on them is

$$\omega_P^{MW}(V,W) := \int_P i_V i_W \mu\,.$$

Here $i_V i_W \mu$ is an $(n-2)$-form integrated over P. Note that this symplectic structure can be thought of as the 'total' averaging of the symplectic structures in each normal space N_pP to P. (The Marsden-Weinstein structure in higher dimensions was studied in [2, 3].)

Furthermore, define the Hamiltonian function on those membranes by taking their $(n-2)$-volume:

$$H(P) = \text{volume}(P) = \int_P \mu_P\,,$$

where μ_P is the volume form of the metric induced from $\mathbb{R}^n$ to P. (For a closed curve γ in $\mathbb{R}^3$ this Hamiltonian is the length functional discussed above.)

Theorem 1.1. *In any dimension $n \geq 3$ the Hamiltonian vector field for the Hamiltonian H and the Marsden-Weinstein symplectic structure on codimension 2 membranes $P \subset \mathbb{R}^n$ is*

$$v_H(p) = C_n \cdot J(\mathbf{MC}(p))\,,$$

where C_n is a constant, J is the operator of positive $\pi/2$ rotation in every normal space N_pP to P, and $\mathbf{MC}(p)$ is the mean curvature vector to P at the point p.

This statement holds for any Riemannian manifold M. The expression of v_H via the trace of the second fundamental form without reference to the mean curvature appeared in [3], Proposition 3. For 4D this theorem was obtained in [6] and for higher dimensions in [4], where we refer to for the proof. Here and below we use the notation C_n for *some* constant depending on the dimension in the case of $\mathbb{R}^n$ (above $C_n = 4 - 2n$), or on the geometry of M^n in the general case.

Recall the definition of the mean curvature vector field for a smooth submanifold of any dimension.

Definition 1.2. a) Let P be a smooth submanifold of dimension l in the Euclidean space $\mathbb{R}^n$. Its second fundamental quadratic form at a point $p \in P$ is a map from the tangent space T_pP to the normal space N_pP. The *mean curvature vector* $\mathbf{MC}(p) \in N_pP$ is the normalized trace of the second fundamental form at p, i.e. the trace divided by l.

b) Equivalently, the *mean curvature vector* $\mathbf{MC}(p) \in N_pP$ is the mean value of the curvature vectors of geodesics in P passing through the point p when we average over the sphere S^{l-1} of all possible unit tangent vectors in T_pP for these geodesics.

Definition 1.3. The *higher vortex filament equation* on submanifolds of codimension 2 in $\mathbb{R}^n$ is given by the *binormal (or skew) mean-curvature flow*:

$$\partial_t P(p) = -J(\mathbf{MC}(p))\,. \tag{3}$$

Note that the skew mean-curvature flow differs by the $\pi/2$-rotation from the mean-curvature one. Respectively, it does not stretch the submanifold while moving its points orthogonally to the mean curvatures. In particular, the volume of the submanifold P is preserved under this evolution, as it should, being the Hamiltonian function of the corresponding dynamics.

For dimension $n = 3$ the mean curvature vector is the curvature vector $k \cdot \mathbf{n}$ of a curve γ: $\mathbf{MC} = k \cdot \mathbf{n}$, while the skew mean-curvature flow becomes the binormal equation: $\partial_t \gamma = -J(k \cdot \mathbf{n}) = k \cdot \mathbf{b}$. Unlike the case $n = 3$, for larger $n \geq 4$ the skew mean-curvature flow is apparently non-integrable.

It would be very interesting to find an analogue of the Hasimoto transformation for any n relating the higher vortex filament equation with the higher-dimensional (and already non-integrable) nonlinear Schrödinger equation; (for $n = 4$ this question was posed in [6]).

2. The vorticity Euler equation

To describe the relation of the skew mean-curvature to hydrodynamics we start by recalling the vorticity form of the Euler equation. Consider an inviscid incompressible fluid filling a Riemannian manifold M. The fluid motion is described as an evolution of its velocity field v in M governed by the classical Euler equation:

$$\partial_t v + (v, \nabla) v = -\nabla p \,. \tag{4}$$

Here the field v is assumed to be divergence-free ($\operatorname{div} v = 0$) with respect to the Riemannian volume form μ and tangent to the boundary of M. The pressure function p is defined uniquely modulo an additive constant by these restrictions on the velocity v. The term $(v, \nabla)v$ stands for the Riemannian covariant derivative $\nabla_v v$ of the field v in the direction of itself.

Definition 2.1. The *vorticity (or Helmholtz) form of the Euler equation* is

$$\partial_t \xi + L_v \xi = 0 \,, \tag{5}$$

where L_v is the Lie derivative along the field v and which means that the vorticity field $\xi := \operatorname{curl} v$ is transported by (or 'frozen into') the fluid flow. In any dimension the vorticity of a fluid motion geometrically is the 2-form defined by $\xi := dv^\flat$, where $v^\flat$ is the 1-form obtained from the vector field v by the metric lifting of indices. (In 3D the vorticity field ξ can be thought of as a vector field, while in 2D it is a scalar vorticity function.)

The Euler equation has a Hamiltonian formulation on the dual space to the Lie algebra $\mathfrak{g} = \text{Vect}_\mu(M)$, which consists of smooth divergence-free vector fields in M tangent to the boundary ∂M, see e.g. [1, 2, 5]. The natural 'regular dual' space for this Lie algebra is the space of vorticities ξ, i.e. exact 2-forms on M. As the dual space to a Lie algebra, this space of vorticities $\text{Vect}_\mu(M)^* = \{\xi\}$ has the natural Lie-Poisson structure. Its symplectic leaves are co-adjoint orbits of the corresponding group $\text{Diff}_\mu(M)$ of volume-preserving diffeomorphisms of M, which are sets of fields with diffeomorphic vorticities. There exists the corresponding (Kirillov-Kostant) symplectic structure ω^{KK} on these orbits in $\text{Vect}_\mu(M)^*$. The Euler equation is Hamiltonian with respect to this structure ω^{KK} and the Hamiltonian function given by the energy $E(v) = \frac{1}{2}\int_M (v, v)\mu$ for $v = \text{curl}^{-1}\,\xi$.

Regular vorticities ξ have support of full dimension, i.e. of codimension 0 in M, while singular ones can have support of codimension 1 or 2. Vortex sheets are singular vorticities with support of codim $= 1$. Below we discuss vortex membranes, which are singular vorticities with support of codim $= 2$. The main types of singular vorticities, as well as related to them symplectic structures and Hamiltonian equations studied below, are summarized in the following table, see more details in [4]. (Note that the Marsden-Weinstein symplectic structure on membranes coincides with the Kirillov-Kostant symplectic structure on coadjoint orbits of singular vorticities ξ_P supported on membranes P.)

The Euler equation (5) is nonlocal in terms of vorticities, since so is the operation of finding $v = \text{curl}^{-1}\xi$. The localized induction approximation (LIA) of the vorticity motion is a procedure which allows one to keep only the local terms in the Euler equation for singular vorticity, as we discuss below.

support codim	vorticity types	symplectic structure	evolution equation	Hamiltonian
0	smooth vorticities ξ	$\omega^{KK}_{\xi}(V,W) = \int_M \xi \wedge i_V i_W \mu$	vorticity Euler equation $\partial_t \xi = -L_v \xi$	energy $H = \frac{1}{2}\int_M (v,v)\,\mu$
1	vortex sheets $\partial_\Gamma \wedge \alpha$	$\omega_{\partial_\Gamma \wedge \alpha}(V,W) = \int_\Gamma \alpha \wedge i_V i_W \mu$	Euler ⇒ Birkhoff-Rott LIA – ?	$H = ?$
2	2D: point vortices $\sum \kappa_j \delta_{z_j}$	$\omega_{(\kappa_j, z_j)} = \sum \kappa_j \, dx_j \wedge dy_j$	Euler ⇒ Kirchhoff LIA=0	$H =$ Kirchhoff Hamiltonian $\mathcal{H}$
	3D: filaments $C \cdot \delta_\gamma$	$\omega^{MW}_{\gamma}(V,W) = \int_\gamma i_V i_W \mu$	LIA: binormal eqn $\partial_t \gamma = \gamma' \times \gamma''$	$H = \mathrm{length}(\gamma)$
	any D: membranes (higher filaments) $C \cdot \delta_P$	$\omega^{MW}_{P}(V,W) = \int_P i_V i_W \mu$	LIA: skew mean curvature flow $\partial_t P = J(\mathbf{MC}(P))$	$H = \mathrm{volume}(P)$

3. The localized induction approximation (LIA) in any dimension

Let $P^{n-2} \subset \mathbb{R}^n$, $n \geq 3$ be a closed oriented submanifold of codimension 2. Consider the vorticity 2-form ξ_P supported on this submanifold: $\xi_P = C \cdot \delta_P$. We will call P a higher(-dimensional) vortex filament or membrane. Note that the exactness of the 2-form ξ_P implies that the membrane strength C is constant.

We would like to find the divergence-free vector field v which has a prescribed vorticity 2-form ξ, i.e. $\xi_P = dv^\flat \in \Omega^2(\mathbb{R}^n)$. In dimension 3, where vorticity can be regarded as a vector field, the corresponding vector potential v in $\mathbb{R}^3$ is reconstructed by means of the Biot-Savart formula, and now we are looking for its analogue in any dimension $n \geq 3$. Denote by $G(q,p)$ the Green function of the Laplace operator in $\mathbb{R}^n$, i.e. given a point $q \in \mathbb{R}^n$ one has $\Delta_p G(q,p) = \delta_q(p)$, the delta-function supported at q.

Theorem 3.1. (see [6] for 4D and [4] for any n) *For any dimension $n \geq 3$ the divergence-free vector field v in $\mathbb{R}^n$ satisfying $\operatorname{curl} v = \xi_P$ (i.e. $\xi_P = dv^\flat$) in the distributional sense is given by the following generalized Biot-Savart formula: for any point $q \notin P$ one has*

$$v(q) := C_n \cdot \int_P J\left(\mathrm{Proj}_N \nabla_p G(q,p)\right)\, \mu_P(p)\,,$$

where $\mathrm{Proj}_N \nabla_p G(\cdot,p)$ is the orthogonal projection of the gradient $\nabla_p G(\cdot,p)$ of the Green function $G(\cdot,p)$ to the fiber N_pP of the normal bundle to P at $p \in P$, the operator J is the positive rotation around p by $\pi/2$ in this 2-dimensional space N_pP, and μ_P is the induced Riemannian $(n-2)$-volume form on the submanifold $P \subset \mathbb{R}^n$.

Note that as the point q approaches the membrane P the vector field $v(q)$ may go to infinity. Consider the following truncation of the integral above. For $q \in P$ and given $\epsilon > 0$ take the above integral not over P but over all points $p \in P$ also satisfying $\|q-p\| \geq \epsilon$, i.e. at the distance at least ϵ from q:

$$v_\epsilon(q) := C_n \cdot \int_{p \in P,\ \|q-p\| \geq \epsilon} J\left(\mathrm{Proj}_N \nabla_p G(q,p)\right)\, \mu_P(p)\,.$$

Furthermore, consider the energy Hamiltonian $E(v) = \frac{1}{2}\int_{\mathbb{R}^n}(v,v)\,\mu$ on fast decaying divergence-free velocity vector fields v. As before, let ξ be the vorticity 2-form of the field v, i.e. $\xi = dv^\flat$. If the vorticity $\xi = \xi_P$ is supported on a membrane $P \subset \mathbb{R}^n$ of codimension 2, the corresponding energy $E(v)$ for the velocity v defined by $\xi_P = dv^\flat$ is divergent and requires a regularisation. Let the *regularised energy* be

$$E_\epsilon(v) := \frac{1}{2}\int_{\mathbb{R}^n}(v, v_\epsilon)\,\mu\,,$$

where v_ϵ is here understood as a smooth extension of the above field v_ϵ from P to $\mathbb{R}^n$. The regularised energy E_ϵ depends on this extension, but its principal term does not, as the following theorem shows.

Theorem 3.2. (see [6] for 4D and [4] for any n) *For any dimension $n \geq 3$ and a membrane $P \subset \mathbb{R}^n$*

i) the velocity field v defined in Theorem 3.1 and satisfying $\xi_P = dv^\flat$ has the following asymptotics of the truncation v_ϵ: for $q \in P \subset \mathbb{R}^n$ one has

$$\lim_{\epsilon\to 0}\frac{v_\epsilon(q)}{\ln\epsilon} = C_n\cdot J\left(\mathbf{MC}(q)\right)\,;$$

ii) the regularised energy $E_\epsilon(v)$ for the velocity of P has the following asymptotics:

$$\lim_{\epsilon\to 0}\frac{E_\epsilon(v)}{\ln\epsilon} = C_n\cdot\int_P \mu_P = C_n\cdot \mathrm{volume}\,(P)\,.$$

By reparametrising the time variable $t \to -(C_n\cdot\ln\epsilon)t$ to absorb the logarithmic singularity we come to the following LIA equation for a membrane $P \subset \mathbb{R}^n$.

Corollary 3.3. *The LIA approximation for a vortex membrane (or higher filament) P in $\mathbb{R}^n$ coincides with the skew mean-curvature flow:*

$$\partial_t P(q) = -J\left(\mathbf{MC}(q)\right)\,,$$

where $\mathbf{MC}(q)$ is the mean curvature vector at $q \in P$. In particular, the LIA equation is Hamiltonian with respect to the Marsden-Weinstein symplectic structure ω^{MW} and Hamiltonian function $H(P)$ given by the volume of the membrane P.

We refer to [6, 4] for proofs and more detail.

Remark 3.4. The hydrodynamical Euler equation remains Hamiltonian under the local induction approximation. Indeed, the LIA takes the Hamiltonian Euler equation on $\mathrm{Vect}_\mu(M)^*$ into the Hamiltonian skew mean-curvature equation on the space of membranes $\{P\}$ by ‘keeping only the logarithmic divergences’ given by the local terms.

Note that we are looking for a vector potential v *induced* by the vorticity ξ_P. Since $v_\epsilon(q) = C_n\cdot J\left(\mathbf{MC}(q)\right)\ln\epsilon + \mathcal{O}(1)$, the points remote from q contribute to $\mathcal{O}(1)$, while the leading term in the expansion is determined by points in P that are ϵ-close to q. Thus keeping only the leading term corresponds to *local* contribution in this approximation, and hence the term of the *localized induction approximation* or *LIA*.

The LIA evolution is close to the actual Euler evolution of a vortex filament only for a short time (when the local term in dominant). For large times the LIA filament may, e.g., self-intersect, while the incompressible Euler dynamics has a frozen-in vorticity and it does not allow topology changes of the filaments.

Acknowledgements

I am grateful to the Isaac Newton Institute in Cambridge, UK, for its kind hospitality. This work was partially supported by an NSERC research grant.

References

[1] Arnold, V.I. (1969) The Hamiltonian nature of the Euler equation in the dynamics of rigid body and of an ideal fluid. *Uspekhi Matem. Nauk* **24:3**, 225–226.

[2] Arnold, V.I., Khesin, B.A. (1998) Topological Methods in Hydrodynamics. *Applied Mathematical Sciences*, vol. 125, Springer-Verlag, New York, 374pp.

[3] Haller, S., Vizman, C. (2003) Nonlinear Grassmannians as coadjoint orbits. *Preprint arXiv: math.DG/0305089*, 13pp, which is a longer version of *Math. Ann.* **329:4**, 771–785.

[4] Khesin, B. (2012) Symplectic structures and dynamics on vortex membranes. *Moscow Math. Journal*, **12:2**, 413–434; (arXiv:1201.5914)

[5] Marsden, J., Weinstein, A. (1983) Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. *Physica D*, **7:1-3**, 305–323.

[6] Shashikanth, B.N. (2012) Vortex dynamics in $\mathbb{R}^4$. *J. Math. Phys.*, **53:1**, 013103 (22pp); (arXiv:1110.2717)