

Characterization of Steady Solutions to the 2D Euler Equation

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Steady fluid flows have very special topology. In this paper we describe necessary and sufficient conditions on the vorticity function of a 2D ideal flow on a surface with or without boundary, for which there exists a steady flow among isovorticed fields. (Here the necessary condition is for any metric, while the sufficient condition is for an appropriate one.) For this we introduce the notion of an antiderivative (or circulation function) on a measured graph, the Reeb graph associated with the vorticity function on the surface, while the criterion is related to the total negativity of this antiderivative. It turns out that given topology of the vorticity function, the set of coadjoint orbits of the symplectomorphism group admitting steady flows with this topology (and a certain metric) forms a convex polytope. As a byproduct of the proposed construction, we also describe a complete list of Casimirs for the 2D Euler hydrodynamics: we define generalized enstrophies which, along with circulations, form a complete set of invariants for coadjoint orbits of area-preserving diffeomorphisms on a surface.

1 Introduction

The study of topology of steady fluid flows is one of central topics of ideal hydrodynamics. Steady flows are key objects in a variety of questions related to hydrodynamical stability, existence of attractors, in dynamo constructions and magnetohydrodynamics,

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etc. Such flows satisfy very special constraints and are difficult to construct explicitly (beyond certain very symmetric settings).

In this paper we present necessary conditions for the existence of steady flows with a given vorticity function on an arbitrary 2D surface, possibly with boundary. Furthermore, we show that these conditions are also sufficient in the sense that for any vorticity satisfying these conditions there is a steady flow for an appropriate choice of metric. This solves one of Arnold's problems raised in [4] about existence of smooth minimizers in Dirichlet problems for initial functions of various topology.

One is usually looking for Euler stationary solutions among fields with prescribed vorticity since vorticity of the fluid is frozen into the flow. Our main tool in the study of isovorticed fields is a description of fine properties of the measured Reeb graphs of vorticity functions and the notion of a graph antiderivative. In this hydrodynamical application we call this antiderivative a circulation function. It turns out that the criterion for the existence of a steady flow is that the corresponding circulation function is balanced, that is, circulations in the vicinity of every singular level have the same signs, as we explain below. Furthermore, given the geometry of the vorticity function, it turns out that the set of coadjoint orbits admitting a stationary Euler flow forms a convex polytope. This opens a variety of questions related to deeper connections between steady flows and convex geometry.

Example 1.1. The following example illustrates the results discussed below in a nutshell. Consider a generic vorticity function on a surface M , see Figure 1 where vorticity is the height function on a pretzel. (For a surface with boundary we consider a vorticity function constant on the boundary ∂M .)

Associate with this function its Reeb graph Γ_F (also called Kronrod graph, see [1, 14]), which is the graph representing the set of connected components of levels of the height vorticity function F . Critical points of F correspond to the vertices of the graph Γ_F . This graph comes with a natural parametrization by the values of F . Furthermore, the surface M is equipped with an area form, preserved by the fluid motion. This area form on the surface gives rise to a log-smooth measure on the Reeb graph Γ_F (see Definition 3.5). It turns out that such a measured Reeb graph is the only invariant of a simple Morse function (i.e., a Morse function with distinct critical values) on a surface modulo symplectomorphisms. \square

Theorem A (=Theorem 3.8). *The mapping assigning the measured Reeb graph Γ_F to a simple Morse function F provides a one-to-one correspondence between simple Morse*

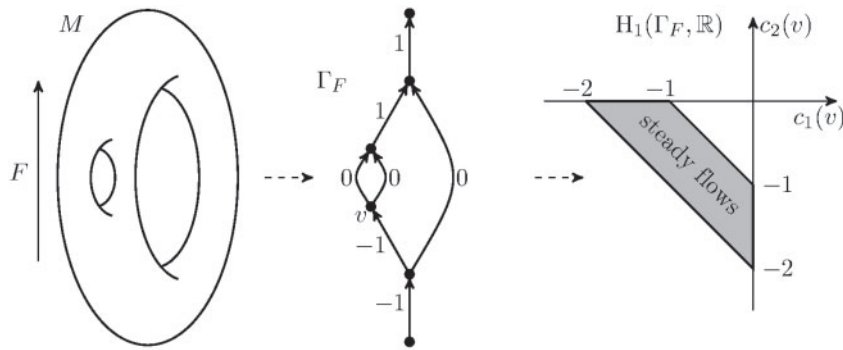


Fig. 1. A vorticity F on a pretzel, its measured Reeb graph, and the corresponding polytope of totally negative circulation functions.

functions on M up to symplectomorphisms and measured Reeb graphs compatible with M .

Steady flows are known to be critical points of the energy functional restricted to the sets of isovorticed fields, that is, to coadjoint orbits of the group of volume- or area-preserving diffeomorphisms of the manifold. To describe coadjoint orbits of the group of symplectomorphisms of a surface M (see Theorem 3.19) we introduce the notion of an antiderivative on a graph. Unlike integration on a segment, for a measured oriented graph the antiderivatives form a space of dimension equal to the first Betti number of the graph itself, see Section 3.2. (In the case with boundary, the space of antiderivatives has dimension of the corresponding relative homology group.) For a pretzel on Figure 1 this space is two-dimensional, parametrized by circulations over two independent cycles. The space of such antiderivatives, called circulation functions, on a graph is in one-to-one correspondence with the space of coadjoint orbits for a given vorticity. Roughly speaking, an antiderivative of a measure is a function which is a classical antiderivative at any smooth point of the graph, vanishes at the points of maximum and minimum, while at vertices it can be discontinuous but satisfies the Kirchhoff rule. In particular, at any trivalent point in the Reeb graph the total value of “incoming” circulations must be equal to the total values of “outgoing” ones. One of the main results of the paper fully characterizes which circulation functions correspond to steady flows.

Theorem B (=Theorem 4.4). *Let (M, ω) be a compact symplectic surface, possibly with boundary. Then Morse-type coadjoint orbit corresponding to a given vorticity function admits a steady flow if and only if the corresponding circulation function has the same signs near every singular level of the vorticity.*

For closed surfaces one can sharpen this description:

Theorem C (=Theorems 4.25 and 4.27). *For a symplectic surface without boundary a Morse-type coadjoint orbit for a given vorticity function F admits a steady flow if and only if the corresponding circulation function is totally negative. Moreover, the set of such totally negative circulation functions is an open convex polytope in the affine space of all circulation functions. In particular, the set of totally negative circulation functions on the Reeb graph Γ_F is bounded.*

The latter implies that “most orbits” do not admit any smooth steady flow. For instance, consider again the vorticity represented by the height function F of a pretzel, see Figure 1. For each edge e of the Reeb graph Γ_F consider the domain $M_e \subset M$, the preimage of this edge bounded by the corresponding critical levels of F . Consider the mean values $\int_{M_e} F \omega$ of the vorticity function in these regions and mark them on the edges of the graph. Assume these mean values of F are as marked on the graph edges in Figure 1. In order to describe the set of circulation functions admitting a steady flow we note that the set of all circulation functions on Γ_F for given F is parametrized, for example, by limits of circulations $c_1(v), c_2(v)$ at branches of the vertex v . (The numbers $c_i(v)$ can be thought of as circulations of the fluid velocity along two cycles across two left handles of the pretzel.) It turns out that for such a vorticity the set of orbits admitting Euler steady solutions is given only by the pairs of circulations lying inside the shaded trapezoid, see Section 4.3.

One should note that Theorems B and C provide a necessary condition for the existence of a steady flow for any metric on the surface (in particular, for any given metric most orbits do not contain a steady solution). This condition is also sufficient, but only for an appropriate metric, that is, for each orbit satisfying this condition we construct a (generally speaking, dependent on the orbit) metric providing a steady flow. Our construction is explicit in terms of the measured Reeb graph with a circulation function and is rather rigid at hyperbolic points. Therefore, it is not clear if the metrics so constructed form a dense and/or open set, or if for a given orbit there exists a transversal submanifold consisting of steady flows (cf. [8]). The question whether a coadjoint orbit admits a steady solution for a fixed metric is still open. This problem is of particular interest for flat and constant curvature metrics in view of hydrodynamical applications.

To summarize, it turns out that there are very restrictive conditions on the sets of isovorticed fields, so that among them there existed a steady flow. It is interesting to compare these results with topological (rather than symplectic) necessary conditions on vorticity functions to admit steady flows obtained in [10]. In the 2D case they follow from

results of this paper, see Example 4.6. In a sense, in [10] the restrictions are obtained by using only the information on the Morse indices of the vorticity, that is, on its Reeb graph, while in this paper we show that the use of a measured Reeb graph allows one to sharpen the necessary conditions and, in addition, to obtain sufficient conditions on the existence of steady flows. On the other hand, in the case of a simple laminar flow in an annulus, any vorticity without critical points and constant on the boundary produces a steady flow on the corresponding orbit (Corollary 4.5), in a perfect matching with results of [8]. It would be also interesting to find the relation of steady flows with infinite-dimensional convex geometry related to the group of symplectomorphisms of an annulus [6] and of arbitrary toric manifolds [7].

It is worth mentioning one more byproduct of the use of measured Reeb graphs in the description of coadjoint orbits of symplectomorphism groups: it allows one to give a complete list of Casimirs in two-dimensional hydrodynamics, which was a folklore open question since Arnold's 1974 paper [3] on the Hopf invariant. It is known that the 2D Euler equation admits an infinite number of conserved quantities called enstrophies, which are moments of the vorticity function. They are called Casimirs, or first integrals of the motion for any Riemannian metric fixed on the surface. However, the set of all enstrophies is known to be incomplete for flows with generic vorticities: there are non-diffeomorphic vorticities with the same values of enstrophies. In Appendix 3, we show how measured Reeb graphs can be used to define generalized enstrophies, and prove that they together with the set of circulations form a complete list of Casimir invariants for flows of an ideal 2D fluid with simple Morse vorticity functions (Corollary A.5).

To the best of our knowledge, a complete description of Casimirs in 2D fluid dynamics has not previously appeared in the literature, while various partial results could be found in [5, 15–17]. Steady flows are conditional extrema of the energy functional on the sets of isovorticed fields, so Casimirs allow one to single out such sets in order to introduce appropriate Lagrange multipliers. Furthermore, Casimirs in fluid dynamics are a cornerstone of the energy-Casimir method for the study of hydrodynamical stability (see, e.g., [2]).

2 Euler Equations and Steady Flows

2.1 Geodesic and Hamiltonian frameworks of the Euler equation

Consider an inviscid incompressible fluid filling an n -dimensional Riemannian manifold M with the Riemannian volume form μ and, possibly, with boundary ∂M . The motion of

an ideal fluid is governed by the hydrodynamical Euler equations

$$\begin{aligned}\partial_t u + (u, \nabla)u &= -\nabla p, \\ \operatorname{div} u &= 0 \quad \text{and} \quad u \parallel \partial M,\end{aligned}\tag{1}$$

describing an evolution of the velocity field u of a fluid flow in M , supplemented by the divergence-free condition and tangency to the boundary. The pressure function p entering the Euler equation is defined uniquely modulo an additive constant by the above constraints on the velocity u . The notation $(u, \nabla)u$ stands for the Riemannian covariant derivative $\nabla_u u$ of the field u along itself.

Arnold [2] showed that the Euler equations in any dimension can be regarded as an equation of the geodesic flow on the group $\operatorname{SDiff}(M) := \{\phi \in \operatorname{Diff}(M) \mid \phi^* \mu = \mu\}$ of volume-preserving diffeomorphisms of M with respect to the right-invariant L^2 -metric on the group given by the energy of the fluid's velocity field: $E(u) = \frac{1}{2} \int_M (u, u) \mu$. In this approach an evolution of the velocity field $u(t)$ according to the Euler equations is understood as an evolution of the vector in the Lie algebra $\mathfrak{vect}(M) = \{u \in \mathfrak{vect}(M) \mid L_u \mu = 0\}$, tracing the geodesic on the group $\operatorname{SDiff}(M)$ defined by the given initial condition $u(0) = u_0$ (here L_u stands for the Lie derivative along the field u).

The geodesic description implies the following Hamiltonian framework for the Euler equation. Consider the (smooth) dual space $\mathfrak{g}^* = \mathfrak{vect}^*(M)$ to the space $\mathfrak{g} = \mathfrak{vect}(M)$ of divergence-free vector fields on M . This dual space has a natural description as the space of cosets $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$: for a one-form α on M its coset of one-forms is

$$[\alpha] = \{\alpha + df \mid \text{for all } f \in C^\infty(M)\} \in \Omega^1(M)/d\Omega^0(M).$$

The pairing between cosets and divergence-free vector fields is straightforward: $\langle [\alpha], W \rangle := \int_M \alpha(W) \omega$ for any field $W \in \mathfrak{vect}(M)$. The coadjoint action of the group $\operatorname{SDiff}(M)$ on the dual \mathfrak{g}^* is given by the change of coordinates in (cosets of) one-forms on M by means of volume-preserving diffeomorphisms.

The Riemannian metric (\cdot, \cdot) on the manifold M allows one to identify (the smooth part of) the Lie algebra and its dual by means of the so-called inertia operator: given a vector field u on M one defines the one-form $\alpha = u^\flat$ as the pointwise inner product with the velocity field u : $u^\flat(W) := (u, W)$ for all $W \in T_x M$, see details in [5]. Note also that divergence-free fields u correspond to co-closed one-forms u^\flat . The Euler equation (1) rewritten on one-forms $\alpha = u^\flat$ is

$$\partial_t \alpha + L_u \alpha = -dP$$

for an appropriate function P on M . In terms of the cosets of one-forms $[\alpha]$, the Euler equation on the dual space \mathfrak{g}^* looks as follows:

$$\partial_t[\alpha] + L_u[\alpha] = 0. \quad (2)$$

The Euler equation (2) on $\mathfrak{g}^* = \mathfrak{vect}^*(M)$ turns out to be a Hamiltonian equation with the Hamiltonian functional \mathcal{H} given by the fluid's kinetic energy,

$$\mathcal{H}([\alpha]) = E(u) = \frac{1}{2} \int_M (u, u) \mu$$

for $\alpha = u^\flat$. The corresponding Poisson structure is given by the natural linear Lie–Poisson bracket on the dual space \mathfrak{g}^* of the Lie algebra \mathfrak{g} , see details in [2, 5]. The corresponding Hamiltonian operator is given by the Lie algebra coadjoint action ad_u^* , which in the case of the diffeomorphism group corresponds to the Lie derivative: $\text{ad}_u^* = L_u$. Its symplectic leaves are coadjoint orbits of the corresponding group $\text{SDiff}(M)$. All invariants of the coadjoint action, also called Casimirs, are first integrals of the Euler equation for *any choice* of Riemannian metric. Stationary (or steady) solutions of the Euler equation satisfying $\partial_t u = 0$ (or, equivalently, $\partial_t[\alpha] = 0$) are critical points of the restriction of the Hamiltonian functional \mathcal{H} to coadjoint orbits. The main result of this paper is a complete characterization of those coadjoint orbits that admit steady solutions.

Remark 2.1. Recall that according to the Euler equation (2) in any dimension the coset of one-forms $[\alpha]$ evolves by a volume-preserving change of coordinates, that is, during the Euler evolution it remains in the same coadjoint orbit in \mathfrak{g}^* . Introduce the *vorticity two-form* $\xi := du^\flat$ as the differential of the one-form $\alpha = u^\flat$ and note that the vorticity exact two-form is well-defined for cosets $[\alpha]$: one-forms α in the same coset have equal vorticities $\xi = d\alpha$. The corresponding Euler equation assumes the vorticity (or Helmholtz) form

$$\partial_t \xi + L_u \xi = 0, \quad (3)$$

which means that the vorticity form is transported by (or “frozen into”) the fluid flow (Kelvin’s theorem). The definition of vorticity ξ as an exact two-form $\xi = du^\flat$ makes sense for a manifold M of any dimension. \square

Remark 2.2. In the case of two-dimensional oriented surfaces M the group $\text{SDiff}(M)$ of volume-preserving diffeomorphisms of M coincides with the group $\text{Symp}(M)$ of symplectomorphisms of M with the area form $\mu = \omega$ given by the symplectic structure.

Furthermore, in 2D we identify the vorticity two-form ξ with a function F satisfying $\xi = F\omega$. Since locally any symplectic vector field u is Hamiltonian, it satisfies $i_u\omega = -dH$ for a (locally defined) Hamiltonian function H . Then vorticity function F and the Hamiltonian H are related via $\Delta H = -F$, by means of the Laplace–Beltrami operator for the Riemannian metric on the surface M . \square

2.2 Steady flows and Dirichlet variational problem

As we mentioned above, steady flows on a manifold M (of any dimension) are defined by the Euler stationary equation $\partial_t[\alpha] = 0$, which means

$$L_u[\alpha] = 0,$$

where $[\alpha]$ is the coset of the one-form $\alpha = u^\flat$ obtained from a divergence-free vector field u on M tangent to the boundary. (The Euler stationary equation is often written in the form $(u, \nabla)u = -\nabla p$ for a field u with $\operatorname{div} u = 0$, $u \parallel \partial M$, and an appropriate pressure function p .)

Remark 2.3. For a two-dimensional surface M steady flows can be described by means of the following data. Recall that in the 2D case the vorticity two-form can be thought of as a function $F = d\alpha/\omega$, and the corresponding stationary Euler equation assumes the vorticity form $L_u F = 0$. Since an area-preserving field on a surface M is locally Hamiltonian, one can rewrite the stationary condition in terms of the corresponding stream (or Hamiltonian) function H of the vector field $u = X_H$. Namely, the 2D stationary Euler equation $L_u F = 0$ assumes the form $\{F, H\} = 0$ of vanishing the Poisson bracket between the vorticity F and (local) stream function H . The latter in 2D means that F and H have common level curves. In particular, if the Hamiltonian $H = H(F)$ is a smooth function of vorticity F (or vice versa), the corresponding Hamiltonian field defines a steady flow.

Also note that for steady flows the vorticity F is necessarily constant on connected components of the boundary of M , since so is the stream function H (the latter must be constant on boundary components in view of the condition $u \parallel \partial M$). \square

In the above description of steady flows one can separate the role of the metric and the area form. First note that on a surface with a symplectic structure ω and a Riemannian metric $g(\cdot, \cdot)$ there is a natural almost complex structure J relating them: $\omega(V, JW) = g(V, W)$ for any pair of vector fields V and W on M . (On surfaces an almost

complex structure is always a complex structure for dimensional reasons.) Given only a symplectic structure ω we say that an almost complex structure J is compatible with ω if the formula above produces a positive-definite metric g .

Definition 2.4. Let (M, ω) be a symplectic surface, and let $F: M \rightarrow \mathbb{R}$ be a smooth function. Then a triple (α, J, H) , where α is a one-form, J is an almost complex structure compatible with ω , and $H = H(F)$ is a smooth function of F , such that $J^*\alpha = -dH$ and $d\alpha = F\omega$, is called an *F-steady triple*. \square

Proposition 2.5. A steady triple (α, J, H) on a symplectic surface (M, ω) gives rise to a steady flow with vorticity F and stream function H . \square

Proof. Indeed, let us show that such a one-form α defines a steady flow $u = \alpha^\flat$, that is, that the condition $L_u[\alpha] = 0$ is fulfilled. Since $L_u = i_u d + di_u$ we need to show that the one-form $i_u d\alpha$ is exact. The latter holds, since $i_u d\alpha = i_u F\omega = Fi_u\omega = -FdH(F) = d\Phi(F)$ for an appropriate function Φ . Furthermore, the relation $J^*\alpha = -dH$ along with the condition of compatibility of J and ω for the Hamiltonian field, which satisfies $i_u\omega = -dH$, gives $\alpha(\cdot) = g(u, \cdot)$, that is, precisely $u = \alpha^\flat$ in the metric g . Thus the Hamiltonian field u for the Hamiltonian function H is a steady solution of the Euler equation. \blacksquare

It is worth mentioning that, in addition to the direct definition of steady flows by means of the stationary Euler equation, there are two variational problems leading to the same characterization, both of which have hydrodynamical origin.

Proposition 2.6. (a) Steady flows of an ideal fluid correspond to critical points of the L^2 -energy Hamiltonian $H([\alpha])$ on the sets of isovorticed fields.

(b) Steady flows of an ideal fluid correspond to critical points of the L^2 -energy $E(u)$ on the sets of fields related by the action of volume-preserving diffeomorphisms. \square

Indeed, the first description is a direct consequence of the aforementioned Hamiltonian framework in *ideal hydrodynamics*: being Hamiltonian on the dual space \mathfrak{g}^* , steady flows are critical points of the Hamiltonian function, which is the L^2 -energy, on *coadjoint orbits* of volume-preserving diffeomorphisms. Such orbits consist of (cosets of) one-forms α which are diffeomorphic by means of volume-preserving diffeomorphisms, and the corresponding vector fields are called isovorticed, since they correspond to diffeomorphic vorticity two-forms $d\alpha$.

In the second description, originated in *magnetohydrodynamics*, one is looking for critical points of the same energy functional but on the sets of diffeomorphic fields, which are *adjoint orbits* in the Lie algebra \mathfrak{g} of the group of volume-preserving diffeomorphisms, and the sets of conditional extrema in the two problems coincide, see [5, 13].

In terms of the stream function H , the second variational problem is closely related to the following minimization problem, posed by Arnold in [3]: given a simple Morse function \bar{H} on a surface M , find among the functions $H = \phi^* \bar{H}$ obtained from it by volume-preserving coordinate change $\phi \in \text{SDiff}(M)$ those realizing the minimum for the Dirichlet functional: $DF(H) := \int_M (\nabla H, \nabla H) \omega$. One can show that functions realizing extremals of the Dirichlet functional can be regarded as stream functions of steady flows on M .

Example 2.7. For a function \bar{H} on a disk with the only critical point, maximum, inside the disk and constant on the boundary, the minimum of the Dirichlet functional is assumed on a rotationally symmetric function symplectomorphic to \bar{H} [3]. It corresponds to a steady flow on the disk whose stream function depends only on the radius and flow trajectories are concentric circles. On the other hand, for a positive function on the disk with one local maximum, one local minimum, and one saddle point there is no smooth function realizing the minimum [10]. \square

In this paper we show how very restrictive the condition of stationarity is and give a criterion for a coadjoint orbit to admit a steady solution for an appropriate metric.

2.3 Casimirs of the 2D Euler equation

The fact that the vorticity two-form ξ is “frozen into” the incompressible flow allows one to define first integrals of the hydrodynamical Euler equation valid for any Riemannian metric on M . In 2D the Euler equation on M is known to possess infinitely many so-called *enstrophy invariants* $m_\lambda(F) := \int_M \lambda(F) \omega$, where $\lambda(F)$ is an arbitrary function of the vorticity function F . In particular, the enstrophy momenta

$$m_i(F) := \int_M F^i \omega$$

are conserved quantities for any $i \in \mathbb{N}$. These Casimir invariants are fundamental in the study of nonlinear stability of 2D flows, and in particular, were the basis for Arnold’s stability criterion in ideal hydrodynamics, see [2, 5]. In the energy-Casimir method one

studies the second variation of the energy functional with an appropriately chosen combination of Casimirs.

In the case of a flow in an annulus with a vorticity function without critical points such invariants form a complete set of Casimirs [8], while for more complicated functions and domains it is not so. In Appendix 3 we give a complete description of Casimirs in the general setting of Morse vorticity functions on two-dimensional surfaces with or without boundary.

3 Coadjoint Orbits of the Symplectomorphisms Group

3.1 Simple Morse functions and measured Reeb graphs

Definition 3.1. Let M be a compact connected surface with a possibly non-empty boundary ∂M . A Morse function $F: M \rightarrow \mathbb{R}$ is called *simple* if it satisfies the following conditions:

- (i) F is constant on each connected component of the boundary ∂M ;
- (ii) F does not have critical points at the boundary;
- (iii) any F -level contains at most one critical point. (In what follows, by an F -level we mean a connected component of the set $F = \text{const.}$) □

With each simple Morse function $F: M \rightarrow \mathbb{R}$, one can associate a graph. This graph Γ_F is defined as the space of F -levels with the induced quotient topology. Each vertex of this graph corresponds either to a critical level of the function F or to a boundary component of the surface M .

The function F on M descends to a function f on the graph Γ_F . It is also convenient to assume that Γ_F is oriented: edges are oriented in the direction of increasing f .

To distinguish between vertices of Γ_F corresponding to critical levels of F and vertices corresponding to boundary components, we denote the latter by circles and call them *boundary vertices*. We shall use the notation $\partial\Gamma$ for the set of boundary vertices.

Example 3.2. Figure 2 shows level curves of a simple Morse function on a disk and the corresponding graph Γ_F . □

Definition 3.3. A *Reeb graph* (Γ, f) is an oriented connected graph Γ with a continuous function $f: \Gamma \rightarrow \mathbb{R}$ which satisfies the following properties.

- (i) All vertices of Γ are either one-valent or three-valent.

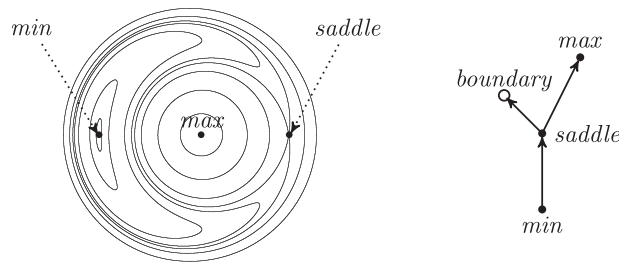


Fig. 2. A simple Morse function on a disk and the associated graph.

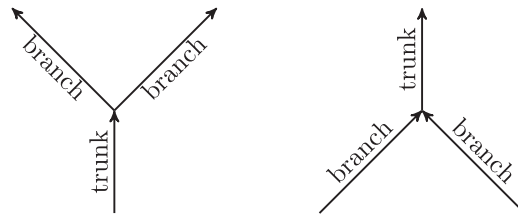


Fig. 3. Trunk and branches in a Reeb graph.

- (ii) one-valent vertices are of two types: boundary vertices and non-boundary vertices.
- (iii) For each three-valent vertex, there are either two incoming and one outgoing edge, or vice versa. \square

It is a standard result from Morse theory that the graph Γ_F associated with a simple Morse function $F: M \rightarrow \mathbb{R}$ on an orientable connected surface M is a Reeb graph in the sense of Definition 3.3. We will call this graph the Reeb graph of the function F . Reeb graphs classify simple Morse functions on M up to diffeomorphisms.

In what follows, we assume that the surface M is endowed with an area (i.e., symplectic) form ω . We are interested in the classification problem for simple Morse functions up to area-preserving (i.e., symplectic) diffeomorphisms. It turns out that this classification can be given in terms of so-called *log-smooth measures* on Reeb graphs.

Definition 3.4. Let Γ be a Reeb graph. Assume that e_0, e_1 , and e_2 are three edges of Γ which meet at a three-valent vertex v . Then e_0 is called the *trunk* of v , and e_1, e_2 are called *branches* of v if either e_0 is an outgoing edge for v , and e_1, e_2 are its incoming edges, or e_0 is an incoming edge for v , and e_1, e_2 are its outgoing edges (see Figure 3). \square

Definition 3.5. A measure μ on a Reeb graph (Γ, f) is called *log-smooth* if it has the following properties:

- (i) It has a C^∞ smooth non-vanishing density $d\mu/df$ at interior points and one valent vertices of Γ .
- (ii) At three-valent vertices, the measure μ has logarithmic singularities. More precisely, one has the following. Assume that v is a three-valent vertex of Γ . Without loss of generality assume that $f(v) = 0$ (if not, we replace f by $\tilde{f}(x) := f(x) - f(v)$). Let e_0 be the trunk of v , and let e_1, e_2 be the branches of v . Then there exist functions $\psi, \eta_0, \eta_1, \eta_2$ of one variable, smooth in the neighborhood of the origin $0 \in \mathbb{R}$ and such that for any point $x \in e_i$ sufficiently close to v , we have

$$\mu([v, x]) = \varepsilon_i \psi(f(x)) \ln |f(x)| + \eta_i(f(x)),$$

where $\varepsilon_0 = 2$, $\varepsilon_1 = \varepsilon_2 = -1$, $\psi(0) = 0$, $\psi'(0) \neq 0$, and $\eta_0 + \eta_1 + \eta_2 = 0$. □

Definition 3.6. A Reeb graph (Γ, f) endowed with a log-smooth measure μ is called a *measured Reeb graph*. □

If a surface M is endowed with an area form ω , then the Reeb graph Γ_F of any simple Morse function $F: M \rightarrow \mathbb{R}$ has a natural structure of a measured Reeb graph. The measure μ on Γ_F is defined as the pushforward of the area form on M under the natural projection $\pi: M \rightarrow \Gamma_F$.

It turns out that there is a one-to-one correspondence between simple Morse functions on M , considered up to symplectomorphisms, and measured Reeb graphs satisfying the following natural compatibility conditions:

Definition 3.7. Let M be a connected surface, possibly with boundary, endowed with a symplectic form ω . A measured Reeb graph (Γ, f, μ) is *compatible with* (M, ω) if

- (i) the genus $\dim H_1(\Gamma, \mathbb{R})$ of Γ is equal to the genus of M ;
- (ii) the number of boundary vertices of Γ is equal to the number of components of ∂M ;
- (iii) the volume of Γ with respect to the measure μ is equal to the volume of M :

$$\int_\Gamma d\mu = \int_M \omega.$$
 □

Theorem 3.8. The mapping assigning the measured Reeb graph Γ_F to a simple Morse function F provides a one-to-one correspondence between simple Morse functions on M up to a symplectomorphism and measured Reeb graphs compatible with M . □

Proof. When M is closed, this result is Theorem 3.11 of [11]. The general case can be reduced to the case of closed M by means of the symplectic cut operation. Namely, let M be a connected surface with boundary endowed with a symplectic form ω and let $F: M \rightarrow \mathbb{R}$ be a simple Morse function. Consider the topological space \tilde{M} obtained from M by contracting each connected component of ∂M into a point. Denote by $\pi: M \rightarrow \tilde{M}$ the natural projection, and by the \tilde{F} and $\tilde{\omega}$ the pushforwards to the \tilde{M} of the corresponding function F and symplectic structure ω .

Proposition 3.9. There exists a smooth structure on \tilde{M} such that the function F and symplectic structure ω descend to smooth objects \tilde{F} and $\tilde{\omega}$ on \tilde{M} . \square

Proof of Proposition 3.9. In a sense, this is a nonlinear version of the Archimedes theorem that the radial projection from a sphere to the circumscribed cylinder is area-preserving. Indeed, in a neighborhood of every boundary component ℓ of M introduce *action-angle coordinates*, that is, smooth functions

$$S: U \rightarrow \mathbb{R}_{\geq 0} = \{z \in \mathbb{R} \mid z \geq 0\}, \quad \Theta: U \rightarrow S^1 = \mathbb{R} / 2\pi\mathbb{Z},$$

such that $\omega = dS \wedge d\Theta$, $S = 0$ on ℓ , and $F = \zeta(S)$ where ζ is a smooth function in one variable such that $\zeta'(0) \neq 0$. Then in a neighborhood of the point $Z = \pi(\ell) \in \tilde{M}$, the image of this boundary component, we define a chart by taking the functions $P := \sqrt{S} \cos \Theta$, $Q := \sqrt{S} \sin \Theta$ as local coordinates. Then the function F in terms of functions P, Q near each point Z becomes $\tilde{F} = \zeta(P^2 + Q^2)$, while the symplectic structure now is $\tilde{\omega} = 2dP \wedge dQ$. \blacksquare

Now we have a functor $(M, F, \omega) \mapsto (\tilde{M}, \tilde{F}, \tilde{\omega})$ from the category of triples (M, F, ω) on surfaces with boundary to the category of triples $(\tilde{M}, \tilde{F}, \tilde{\omega})$ on closed surfaces. This functor becomes invertible upon specification of minima and maxima of the function \tilde{F} that correspond to boundary components of M . At the level of Reeb graphs, this correspondence replaces boundary vertices by marked vertices. Since measured Reeb graphs up to isomorphism completely describe simple Morse functions on closed symplectic surfaces modulo symplectomorphisms (see Theorem 3.11 of [11]), this one-to-one correspondence extends to symplectic surfaces with boundary. \blacksquare

Remark 3.10. The functor $(M, F, \omega) \mapsto (\tilde{M}, \tilde{F}, \tilde{\omega})$ may be viewed as a particular case of the symplectic cut operation introduced by Lerman [12]. \square

3.2 Antiderivatives on graphs

In order to pass from the above classification of simple Morse functions on symplectic surfaces to the classification of coadjoint orbits of the group $\text{SDiff}(M)$, we need to introduce the notion of the antiderivative of a density on a graph. Let Γ be an oriented graph, some of whose vertices are marked and called boundary vertices. Let also ρ be a *density* on Γ , that is, a finite Borel-signed measure.

Definition 3.11. A function $\lambda: \Gamma \setminus V \rightarrow \mathbb{R}$ defined and continuous on the graph Γ outside its set of vertices $V = V(\Gamma)$ is called an *antiderivative of the density* ρ if it has the following properties.

- (i) It has at worst jump discontinuities at vertices, which means that for any vertex $v \in V$ and any edge $e \ni v$, there exists a finite limit

$$\lim_{x \xrightarrow{e} v} \lambda(x),$$

where $x \xrightarrow{e} v$ means “as x tends to v along the edge e .”

- (ii) Assume that x, y are two interior points of some edge $e \in \Gamma$, and that e is pointing from x toward y . Then λ satisfies the Newton–Leibniz formula

$$\lambda(y) - \lambda(x) = \rho([x, y]). \quad (4)$$

- (iii) Let v be a non-boundary vertex of Γ . Then λ satisfies the Kirchhoff rule at v :

$$\sum_{e \rightarrow v} \lim_{x \rightarrow v} \lambda(x) = \sum_{e \leftarrow v} \lim_{x \rightarrow v} \lambda(x), \quad (5)$$

where the notation $e \rightarrow v$ stands for the set of edges pointing at the vertex v , and $e \leftarrow v$ stands for the set of edges pointing away from v . \square

Proposition 3.12. Let Γ be an oriented graph with marked boundary vertices.

- (i) If the graph Γ has at least one boundary vertex, then any density ρ on Γ admits an antiderivative.
- (ii) If Γ has no boundary vertices, then a density ρ on Γ admits an antiderivative if and only if $\rho(\Gamma) = 0$.
- (iii) If a density ρ on Γ admits an antiderivative, then the set of antiderivatives of ρ is an affine space whose underlying vector space is the relative homology group $H_1(\Gamma, \partial\Gamma, \mathbb{R})$, where $\partial\Gamma$ is the set of boundary vertices of Γ . \square

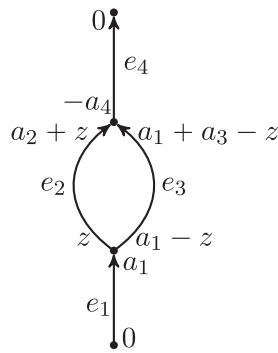


Fig. 4. The space of antiderivatives on a graph of genus one.

Example 3.13. Consider the graph Γ depicted in Figure 4. Let ρ be a density on Γ such that $\rho(e_i) = a_i$, where the numbers a_i satisfy $a_1 + a_2 + a_3 + a_4 = 0$ (so that the density ρ admits an antiderivative). The numbers near vertices in the figure stand for the limits of the antiderivative λ of ρ . The space of such antiderivatives has one parameter z , which determines how the splitting of the antiderivative between the edges e_2 and e_3 takes place (by the proposition above the space of antiderivatives is one-dimensional). \square

Proof of Proposition 3.12. Let $\lambda: \Gamma \setminus V \rightarrow \mathbb{R}$ be any function satisfying conditions (i) and (ii) of Definition 3.11. Let e be an edge of Γ going from v to w , and let $x \in e$. Denote by $\lambda^-(e)$ and $\lambda^+(e)$ the limits of $\lambda(x)$ as x tends to v and w , respectively. Consider the one-chain

$$\text{ch}(\lambda) := \sum_{e \in E} \lambda^+(e)e, \quad (6)$$

where $E = E(\Gamma)$ stands for the set of edges of Γ . The mapping ch is a one-to-one map between functions $\lambda: \Gamma \setminus V \rightarrow \mathbb{R}$ satisfying conditions (i) and (ii) of Definition 3.11 and one-chains on Γ . Then we have

$$\partial \text{ch}(\lambda) = \sum_{v \in V} \left(\sum_{e \rightarrow v} \lambda^+(e) - \sum_{e \leftarrow v} \lambda^+(e) \right) v = \chi + \sum_{v \in V} \Delta(v)v,$$

where χ is a zero-chain given by

$$\chi := - \sum_{v \in V} \sum_{e \leftarrow v} \rho(e)v,$$

and

$$\Delta(v) := \sum_{e \rightarrow v} \lambda^+(e) - \sum_{e \leftarrow v} \lambda^-(e).$$

Note that $\Delta(v) = 0$ is equivalent to the Kirchhoff rule at v . Therefore, λ is an antiderivative of ρ if and only if

$$\partial \text{ch}(\lambda) = \chi \mod \partial \Gamma. \quad (7)$$

So, a density ρ on Γ admits an antiderivative if and only if the zero-chain χ is a relative boundary modulo $\partial \Gamma$. If the set $\partial \Gamma$ is non-empty, then we have $H_0(\Gamma, \partial \Gamma, \mathbb{R}) = 0$, so the chain χ is always a relative boundary, which proves the first statement of the proposition. Further, if $\partial \Gamma$ is empty, then χ is a boundary if and only if $\varepsilon(\chi) = 0$ where ε is the augmentation map $\varepsilon(\sum c_i v_i) = \sum c_i$. We have $\varepsilon(\chi) = -\rho(\Gamma)$, which proves the second statement. To prove the third statement, note that since the mapping ch identifies appropriate functions $\lambda: \Gamma \setminus V \rightarrow \mathbb{R}$ and one-chains on Γ , it identifies antiderivatives of ρ with the set of one-chains having property (7). Now, to complete the proof, it suffices to note that the latter set is an affine space with underlying vector space $H_1(\Gamma, \partial \Gamma, \mathbb{R})$. ■

3.3 Classification of coadjoint orbits

Let M be a compact connected surface, possibly with boundary. Assume that M is endowed with a symplectic form ω . Recall that the regular dual $\mathfrak{vect}^*(M)$ of the Lie algebra $\mathfrak{vect}(M)$ of divergence-free vector fields on a surface M is identified with the space $\Omega^1(M)/d\Omega^0(M)$ of smooth one-forms modulo exact one-forms on M . The coadjoint action of a $\text{SDiff}(M)$ on $\mathfrak{vect}^*(M)$ is given by the change of coordinates in (cosets of) one-forms on M by means of a symplectic diffeomorphism: $\text{Ad}_\Phi^*[\alpha] = [\Phi^*\alpha]$. In what follows, the notation $[\alpha]$ stands for the coset of one-forms α in $\Omega^1(M)/d\Omega^0(M)$. In particular, if the form α is closed, then $[\alpha]$ is the cohomology class of α .

To describe orbits of the coadjoint action of $\text{SDiff}(M)$ on $\mathfrak{vect}^*(M)$, consider the mapping $\text{curl}: \Omega^1(M)/d\Omega^0(M) \rightarrow C^\infty(M)$ given by taking the vorticity function

$$\text{curl}[\alpha] := \frac{d\alpha}{\omega}.$$

(One can view this map as taking the vorticity of a vector field $u = \alpha^\sharp$.)

Remark 3.14. If the boundary of M is non-empty, then the mapping curl is surjective. Otherwise, its image is the space of functions with zero mean. □

An important property of the mapping curl is its equivariance with respect to the $\text{SDiff}(M)$ action: if cosets $[\alpha], [\beta] \in \mathfrak{vect}^*(M)$ belong to the same coadjoint orbit, then the functions $\text{curl}[\alpha]$ and $\text{curl}[\beta]$ are conjugated by a symplectic diffeomorphism. In particular, if $\text{curl}[\alpha]$ is a simple Morse function, then so is $\text{curl}[\beta]$.

Definition 3.15. We say that a coset of one-forms $[\alpha] \in \mathfrak{vect}^*(M)$ is *Morse-type* if $\text{curl}[\alpha]$ is a simple Morse function. A coadjoint orbit $\mathcal{O} \subset \mathfrak{vect}^*(M)$ is *Morse-type* if any coset $[\alpha] \in \mathcal{O}$ is Morse-type (equivalently, if at least one coset $[\alpha] \in \mathcal{O}$ is Morse-type). \square

Let $[\alpha] \in \mathfrak{vect}^*(M)$ be Morse-type, and let $F = \text{curl}[\alpha]$. Consider the measured Reeb graph Γ_F . Since curl is an equivariant mapping, this graph is invariant under the coadjoint action of $\text{SDiff}(M)$ on $\mathfrak{vect}^*(M)$. However, this invariant is not complete if M is not simply connected (i.e., if M is neither a sphere S^2 , nor a disk D^2). To construct a complete invariant, we endow the graph Γ_F with a *circulation function* constructed as follows. Let $\pi: M \rightarrow \Gamma_F$ be the natural projection. Take any point x lying in the interior of some edge $e \in \Gamma_F$. Then $\pi^{-1}(x)$ is a circle ℓ_x . It is naturally oriented as the boundary of the set of smaller values. The integral of α over ℓ_x does not depend on the choice of a representative $\alpha \in [\alpha]$. Thus, we obtain a function $\mathfrak{c}: \Gamma_F \setminus V(\Gamma_F) \rightarrow \mathbb{R}$ given by

$$\mathfrak{c}(x) := \int_{\ell_x} \alpha. \quad (8)$$

In the presence of a metric on M , the value $\mathfrak{c}(x)$ is the circulation over the level $\pi^{-1}(x)$ of the vector field α^\sharp dual to the one-form α .

Proposition 3.16. For any Morse-type coset $[\alpha] \in \mathfrak{vect}^*(M)$, the function \mathfrak{c} given by formula (8) is an antiderivative of the density $\rho(I) := \int_I f d\mu$ in the sense of Definition 3.11. \square

Remark 3.17. This density ρ is the pushforward of the vorticity two-form $d[\alpha]$ from the surface to the Reeb graph. \square

Proof of Proposition 3.16. The proof is straightforward and follows from the Stokes formula and additivity of the circulation integral. \blacksquare

Definition 3.18. Let (Γ, f, μ) be a measured Reeb graph. A *circulation function* \mathfrak{c} on Γ is an antiderivative of the density $\rho(I) := \int_I f d\mu$. A measured Reeb graph endowed with a circulation function is called a *circulation graph*. \square

So, with any Morse-type coset $[\alpha] \in \mathfrak{sect}^*(M)$ we associate a circulation graph. Denote this graph by $\Gamma_{[\alpha]}$.

Theorem 3.19. Let M be a compact connected symplectic surface. Then Morse-type coadjoint orbits of $\text{SDiff}(M)$ are in one-to-one correspondence with circulation graphs $(\Gamma, f, \mu, \mathfrak{c})$ compatible with M . \square

The proof of this theorem is a version of the proof of Theorem 4.6 in [11] adapted to surfaces with boundary. We present it in Appendix 1. Now we are ready to describe steady solutions on such coadjoint orbits.

4 Steady Fluid Flows and Coadjoint Orbits

4.1 Orbits admitting steady flows

Definition 4.1. Let (M, ω) be a compact symplectic surface, possibly with boundary. We say that a coadjoint orbit $\mathcal{O} \subset \mathfrak{sect}^*(M)$ *admits a steady flow* if there exists a metric g on M compatible with the symplectic structure ω and such that the corresponding Euler equation has a steady solution on \mathcal{O} . \square

Below we give a necessary and sufficient condition for a Morse-type coadjoint orbit to admit a steady flow. Recall that coadjoint orbits of $\text{SDiff}(M)$ are described by measured Reeb graphs endowed with a circulation function. Let Γ be a measured Reeb graph, and let \mathfrak{c} be a circulation function on Γ . Let also $v \in V(\Gamma)$ be a three-valent vertex of Γ , and let e_0, e_1, e_2 be the edges of Γ adjacent to v . Denote

$$c_i(v) := \lim_{x \xrightarrow{e_i} v} \mathfrak{c}(x). \quad (9)$$

Definition 4.2. Let Γ be a measured Reeb graph. We say that a circulation function \mathfrak{c} on Γ is *balanced* if for any three-valent vertex $v \in \Gamma$, the numbers $c_0(v), c_1(v), c_2(v)$ defined by formula (9) are non-zero and have the same sign. \square

Remark 4.3. If e_0 is the trunk of v , and e_1, e_2 are branches of v , then, by the Kirchhoff rule (5), we have $c_0(v) = c_1(v) + c_2(v)$, so $c_0(v), c_1(v), c_2(v)$ are non-zero and have the same sign if and only if $c_1(v)$ and $c_2(v)$ are non-zero and have the same sign. \square

Theorem 4.4. Let (M, ω) be a compact symplectic surface, possibly with boundary, and let $\mathcal{O} \subset \mathfrak{sect}^*(M)$ be a Morse-type coadjoint orbit. Then \mathcal{O} admits a steady flow if and only if the corresponding circulation function is balanced. \square

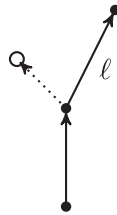


Fig. 5. A directed path ℓ joining a local minimum with a local maximum.

Corollary 4.5. If the Reeb graph of the vorticity function does not have three-valent vertices, then the orbit admits a steady flow. \square

There are only four such graphs: a segment with no, either, or two marked points. They correspond to vorticity functions with one maximum and one minimum on the sphere, disk, or cylinder. All corresponding orbits have steady flows for the metric of constant curvature. Once the vorticity function has saddle points, the existence conditions become quite non-trivial.

Example 4.6 (see Theorem 5.6 of [10]). Assume that M is diffeomorphic to a disk D^2 . Let $\mathcal{O} \subset \text{vect}^*(M)$ be a Morse-type coadjoint orbit such that the corresponding vorticity function F has a local minimum and a local maximum in $M \setminus \partial M$. Then, if $F > 0$ in $M \setminus \partial M$, the orbit \mathcal{O} does not contain steady solutions of the Euler equation.

Indeed, let $(\Gamma, f, \mu, \mathfrak{c})$ be the circulation graph corresponding to the orbit \mathcal{O} . The graph Γ has exactly one boundary vertex (since M is a disk), at least one non-boundary one-valent vertex with an incoming edge (local maximum of the vorticity), and at least one non-boundary one-valent vertex with an outgoing edge (local minimum of the vorticity). Using these three conditions, one can easily show that there exists a directed path ℓ in Γ which joins two non-boundary one-valent vertices (see Figure 5).

Let us consider the behavior of the circulation function \mathfrak{c} along the path ℓ . First, the Newton–Leibniz formula (4) and positivity of the vorticity imply that \mathfrak{c} is strictly increasing at all points of ℓ except, possibly, three-valent vertices, where \mathfrak{c} has jump discontinuities. Secondly, by the Kirchhoff rule (5), the function \mathfrak{c} vanishes at endpoints of ℓ . Together, these two conditions imply that the function \mathfrak{c} must change sign at some three-valent vertex $v \in \ell$. So, by Theorem 4.4, the orbit \mathcal{O} does not admit steady solutions of the Euler equation, q.e.d.

The assumption that M is a disk is not important. It suffices to require that the boundary ∂M of M has one connected component. \square

Remark 4.7. The above theorem provides the necessary and sufficient conditions for the existence of a steady solution on a Morse-type orbit for some metric compatible with a given area two-form ω . In fact, the necessary condition holds for any metric and, roughly speaking, the fact that the circulation function is balanced is a consequence of an analysis of the hyperbolic figure-eight levels of the vorticity function F for steady flows along with the condition $H = H(F)$ of dependence of the stream and vorticity functions (see Proposition 4.8 and Lemma 4.10).

The sufficiency (proved in the next section) is based on a technical procedure divided into local and global steps. The local part consists of constructing stream functions H and metrics g such that the triples (F, g, H) (or, equivalently, triples (α, J, H)) solve the steady Euler equation in the vicinity of all maxima and minima of F (Proposition 4.13), in a neighborhood of a figure-eight level (Propositions 4.15 and 4.16) and in regions fibered by regular levels of F (Propositions 4.12 and 4.17). The global part consists of gluing all the previous local constructions by using the structure of the corresponding circulation graphs (Proposition 4.20) and appropriate normal forms for metrics, complex structures, and stream functions in the vicinity of the critical points of F . \square

Proof of the “only if” part of Theorem 4.4. We need to prove that if a Morse-type coadjoint orbit admits a steady flow, then the corresponding circulation function is balanced. Recall that if $[\alpha] \in \mathfrak{seect}^*(M)$ is a steady solution of the Euler equation, then the vorticity function $F = \text{curl}[\alpha]$ is a first integral for the fluid velocity field u . In other words, the fluid velocity field u of a steady flow is tangent to every level of the vorticity F , and in particular, to hyperbolic figure-eight levels. The proof of necessity is based on the following proposition, which we are proving later.

Proposition 4.8. Assume that (M, ω) is a compact symplectic surface, possibly with boundary, and let $[\alpha] \in \mathfrak{seect}^*(M)$ be a fixed point of the Euler equation such that the corresponding vorticity $F = \text{curl}[\alpha]$ is a simple Morse function. Let also \mathcal{E} be a hyperbolic figure-eight level of F , and let $O \in \mathcal{E}$ be the singular point. Then:

- (i) the fluid velocity field u does not vanish in $\mathcal{E} \setminus \{O\}$;
- (ii) u induces the same orientation on both connected components of $\mathcal{E} \setminus \{O\}$ (see Figure 6). (Recall that each level of F is naturally oriented as the boundary of the set of smaller values. This allows us to compare the orientations induced by u on different connected components of $\mathcal{E} \setminus \{O\}$.) \square



Fig. 6. Neighborhood of a hyperbolic level of the vorticity function.

Now let $\alpha \in [\alpha]$ be a co-closed representative one-form for a steady solution $[\alpha] \in \mathfrak{sect}^*(M)$ of the Euler equation. Take any three-valent vertex v of the circulation graph $\Gamma_{[\alpha]}$ corresponding to the coset $[\alpha]$. Let e_0 be the trunk of v , and let e_1, e_2 be the branches of v . Let also \mathcal{E} be the figure-eight level of the vorticity function $F = \text{curl}[\alpha]$ corresponding to the vertex v (see Figure 6). Then for $i = 1, 2$ we have

$$c_i(v) = \int_{\ell_i} \alpha,$$

where $c_i(v)$ is defined by formula (9). According to Proposition 4.8, either the direction of the fluid velocity field u coincides with the natural orientation on both arcs ℓ_1, ℓ_2 of \mathcal{E} , or the field u is opposite to the orientation of both arcs. In the case where orientations coincide, one can take the natural time parameter t on integral trajectories of u as a parameter on ℓ_1, ℓ_2 . Then, for $i = 1, 2$, we have

$$c_i(v) = \int_{\ell_i} \alpha = \int_{-\infty}^{+\infty} \alpha(\dot{\ell}_i(t)) dt = \int_{-\infty}^{+\infty} \alpha(u(t)) dt = \int_{-\infty}^{+\infty} (u(t), u(t)) dt,$$

where $(,)$ is the dot product given by the metric on M . So, in this case the numbers $c_i(v)$ are both positive. Similarly, in the case of opposite orientation of u and the arcs, $c_i(v)$ are both negative. Thus, in both cases the numbers $c_i(v)$ are non-zero and have the same sign, as desired. This completes the proof of the “only if” part of the theorem modulo the proposition above. ■

Proof of Proposition 4.8. Consider a tubular neighborhood U of \mathcal{E} , as depicted in Figure 6. Since the fluid velocity vector field u is symplectic, the one-form $\beta := i_u \omega$ is closed. Further, since u is tangent to the arcs ℓ_1 and ℓ_2 , we have $\beta|_{\ell_1} = \beta|_{\ell_2} = 0$, and, therefore,

$$\int_{\ell_1} \beta = \int_{\ell_2} \beta = 0,$$

which implies that the one-form β is exact in U , that is, $i_u \omega = dH$ for a suitable smooth function $H: U \rightarrow \mathbb{R}$ (the stream function).

The first statement of the proposition is immediate: if one assumes that the field u vanishes at some point of $\mathcal{E} \setminus \{O\}$, say on the arc ℓ_1 , then dH vanishes at that point and hence on the whole arc ℓ_1 , which contradicts that O is an isolated critical point of H .

To prove the second statement we need two auxiliary lemmas. The first one is known as the Morse–Darboux lemma. This lemma is a particular case of Le lemme de Morse isochore due to Colin de Verdière and Vey [9].

Lemma 4.9. Assume that (M, ω) is a symplectic surface, and let $F: M \rightarrow \mathbb{R}$ be a smooth function. Let also $O \in M \setminus \partial M$ be a Morse critical point of F . Then, in a neighborhood of O , there exist coordinates (P, Q) such that $\omega = dP \wedge dQ$, and $F = \zeta(S)$ where $S = \frac{1}{2}(P^2 + Q^2)$ or $S = PQ$. The function ζ of one variable is smooth in the neighborhood of the origin $0 \in \mathbb{R}$, and $\zeta'(0) \neq 0$. \square

Now, let $[\alpha] \in \mathfrak{vect}^*(M)$ be a fixed point of the Euler equation, and let $\alpha \in [\alpha]$ be a co-closed representative, a co-closed one-form α in this coset. The co-closedness condition implies that the corresponding fluid velocity field $u = \alpha^\sharp$ is divergence-free, that is, symplectic or locally Hamiltonian, and the corresponding locally defined Hamiltonian is called the stream function. Let also $O \in M$ be a hyperbolic Morse critical point of the vorticity $F = \text{curl}[\alpha]$. Then in a neighborhood of O there exists a stream function H , such that $i_u \omega = dH$.

Lemma 4.10. Assume that (M, ω) is a symplectic surface, and let $[\alpha] \in \mathfrak{vect}^*(M)$ be a fixed point of the Euler equation. Let also $O \in M \setminus \partial M$ be a hyperbolic Morse critical point of the vorticity $F = \text{curl}[\alpha]$. Then O is also a hyperbolic Morse critical point for the stream function H . Moreover, in coordinates P, Q from Lemma 4.9, the Taylor expansion of H at O reads $H = a + bPQ + \dots$, where a, b are constants, $b \neq 0$, and dots denote higher-order terms. \square

Proof. Since $[\alpha]$ is a fixed point for the Euler equation, we have $\{F, H\} = 0$, where $\{, \}$ is the Poisson bracket defined by the symplectic structure ω . Applying Lemma 4.9 to the vorticity F at the point O , we find local coordinates P, Q such that $\omega = dP \wedge dQ$, and $F = \zeta(PQ)$. Since $\zeta'(0) \neq 0$, we have $PQ = \zeta^{-1}(F)$, and

$$\{PQ, H\} = (\zeta^{-1})'(F) \cdot \{F, H\} = 0.$$

The latter equation implies that the Taylor expansion of H at O reads $H = a + bPQ + cP^2Q^2 + \dots$, where $a, b, c \in \mathbb{R}$ are constants, and dots denote higher-order terms. Assume

that $b = 0$. Then $H = a + cP^2Q^2 + \dots$, and a straightforward computation shows that

$$F = -\Delta H = 8cg_{12}(O)PQ - 2cg_{11}(O)P^2 - 2cg_{22}(O)Q^2 + \dots, \quad (10)$$

where Δ is the Laplace–Beltrami operator, and g_{ij} are components of the metric g in P, Q coordinates. On the other hand, we have

$$F = \zeta(PQ) = \zeta(0) + \zeta'(0)PQ + \dots$$

Comparing the latter formula with (10), we conclude that $c = 0$, so $\zeta'(0) = 0$, which is a contradiction. Therefore, our assumption is false, and we have $H = a + bPQ + \dots$, where $b \neq 0$, q.e.d. ■

Now, let us prove the second statement of Proposition 4.8. In a neighborhood of the point O , the fluid velocity field u has the form

$$u = \omega^{-1}dH = b \left(P \frac{\partial}{\partial p} - Q \frac{\partial}{\partial q} \right) + \dots,$$

Similarly, we have

$$X_F = \omega^{-1}dF = \zeta'(0) \left(P \frac{\partial}{\partial p} - Q \frac{\partial}{\partial q} \right) + \dots,$$

so, modulo higher-order terms, we have $u \approx (b/\zeta'(0))X_F$. Now, note that the vector field X_F induces the same orientation on ℓ_1, ℓ_2 (it is opposite to the natural orientation). Therefore, the same is true for the field u . This completes the proof of the proposition. ■

4.2 Construction of steady flows

In this section we prove that if a circulation function corresponding to a Morse-type coadjoint orbit $\mathcal{O} \subset \mathfrak{vect}^*(M)$ is balanced, then the orbit \mathcal{O} admits a steady flow, that is, there exists a metric g on M compatible with the symplectic structure ω and such that the corresponding Euler equation has a steady solution on \mathcal{O} . We divide the proof into several statements (see Remark 4.7 for the scheme of the proof). The first lemma is well-known being a particular case of the Arnold–Liouville theorem:

Lemma 4.11. Assume that (C, ω) is a compact symplectic cylinder, and let $F: C \rightarrow \mathbb{R}$ be a smooth function constant on the boundary of C . Assume also that $dF \neq 0$ on C . Then there exist functions

$$S: C \rightarrow \mathbb{R}, \quad \Theta: C \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z},$$

called the *action-angle variables*, such that $\omega = dS \wedge d\Theta$, and $F = \zeta(S)$ where ζ is a smooth function in one variable such that $\zeta'(S) \neq 0$. \square

Recall that to construct a steady flow with a prescribed vorticity F on a symplectic surface (M, ω) it suffices to define an F -steady triple (α, J, H) on M , see Definition 2.4. Here α is a one-form, J is an almost complex structure compatible with ω , and $H = H(F)$ satisfying $J^*\alpha = -dH$ and $d\alpha = F\omega$.

Proposition 4.12. Assume that (C, ω) is a symplectic cylinder, and let $F: C \rightarrow \mathbb{R}$ be a smooth function constant on the boundary of C . Assume also that $dF \neq 0$ on C . Then there exists an F -steady triple (α, J, H) on C . Moreover, for any F -level $\ell \subset C$ and any constant c , the form α can be chosen in such a way that $\int_\ell \alpha = c$. \square

Proof. We work in action-angle coordinates S, Θ from Lemma 4.11. Without loss of generality, we may assume that $S = 0$ on ℓ . By setting

$$\alpha := \eta(S)d\Theta, \quad H := -\int \eta(S)dS, \quad \text{and} \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (11)$$

we provide $J^*\alpha = -dH$ for any smooth function $\eta(S)$. Now by specifying this function to be

$$\eta(S) := \frac{c}{2\pi} + \int_0^S \zeta(\xi)d\xi$$

we also obtain $d\alpha = F\omega$ for $F = \zeta(S)$, as well as $\int_\ell \alpha = c$, as desired. \blacksquare

Proposition 4.13. Let (M, ω) be a symplectic surface, and let $F: M \rightarrow \mathbb{R}$ be a smooth function on M . Assume that $O \in M \setminus \partial M$ is a non-degenerate minimum or maximum singular point of F . Then there exists an F -steady triple (α, J, H) in the neighborhood of O . \square

Proof. This proposition can be thought of as a modification of the one above for $c = 0$, but requires an independent proof of smoothness at O . By Lemma 4.9, there exist local coordinates P, Q in the neighborhood of O such that $\omega = dP \wedge dQ$ and $F = \zeta(S)$ where $S = \frac{1}{2}(P^2 + Q^2)$. We use the polar coordinates $(\sqrt{2S}, \Theta)$ related to (P, Q) as action-angle variables, $\omega = dP \wedge dQ = dS \wedge d\Theta$. Then we set

$$\eta(S) := \int_0^S \zeta(\xi)d\xi,$$

and define a steady triple (α, J, H) using formulas (11). \blacksquare

Before we turn to constructing F -steady triples near hyperbolic levels, let us point out the following key property of the sign of dH/dF .

Lemma 4.14. For a steady flow α with a stream function $H = H(F)$ expressed in terms of vorticity F , the sign of the derivative of the stream function with respect to vorticity is minus the sign of the circulation function. More precisely, for any nonsingular F -level $\ell \subset M$

$$\text{sign} \frac{dH}{dF} = -\text{sign} \int_{\ell} \alpha. \quad \square$$

Equivalently, in terms of the Reeb graph (Γ, f) of the function F on M and the circulation function $c(x)$ on the graph, defined by integrals of the one-form α over levels $\ell_x = \pi^{-1}(x)$ for interior points $x \in \Gamma$, one has

$$\text{sign} \frac{dH}{dF}(f(x)) = -\text{sign} c(x). \quad (12)$$

The proof in terms of F -steady triples directly follows from the relation $J^*\alpha = -dH$.

Now we study the vicinity of hyperbolic levels of vorticity functions.

Proposition 4.15. Let (M, ω) be a symplectic surface, possibly with boundary, and let $F: M \rightarrow \mathbb{R}$ be a smooth function on M . Assume that $O \in M \setminus \partial M$ is a non-degenerate hyperbolic singular point of F . Then there exists an F -steady triple (α, J, H) in the neighborhood of O . Moreover, for any $\varepsilon = \pm 1$, the triple (α, J, H) can be chosen in such a way that $\text{sign}(dH/dF) = \varepsilon$. \square

Proof. By Lemma 4.9, there exist local coordinates P, Q in the neighborhood of O such that $\omega = dP \wedge dQ$, and $F = \zeta(S)$ where $S = PQ$, and $\zeta'(0) \neq 0$. Without loss of generality, we may assume that $\zeta'(0) > 0$ (if not, we replace the chart P, Q with $Q, -P$). Let

$$\alpha := (\eta(S)Q + cP)dP - (\eta(S)P + cQ)dQ \quad \text{for} \quad \eta(S) := -\frac{1}{2S} \int_0^S \zeta(\xi) d\xi,$$

where c is any constant such that $\text{sign} c = \varepsilon$ and $|c| > |\eta(S)|$ in a sufficiently small neighborhood of the point O . Then we have

$$d\alpha = -2(S\eta'(S) + \eta(S)) dP \wedge dQ = F\omega,$$

as desired. Further, for $\varepsilon = 1$ we set

$$J := \frac{1}{\sqrt{c^2 - \eta(S)^2}} \begin{pmatrix} \eta(S) & -c \\ c & -\eta(S) \end{pmatrix}, \quad H(S) := \int \sqrt{c^2 - \eta(S)^2} \, dS.$$

Then J is a complex structure compatible with ω , and $J^*\alpha = -dH$. Furthermore, we obtain

$$\text{sign} \frac{dH}{dF} = \text{sign} \left(\frac{dH}{dS} : \frac{dF}{dS} \right) = \text{sign} \frac{\sqrt{c^2 - \eta(S)^2}}{\zeta'(S)} = 1,$$

as required. Similarly, for $\varepsilon = -1$ we simply need to change the signs of J and H . ■

Proposition 4.16. Let (M, ω) be a compact symplectic surface, possibly with boundary, and let $F: M \rightarrow \mathbb{R}$ be a simple Morse function on M . Let also \mathcal{E} be a figure-eight level of F . Then there exists a steady F -triple (α, J, H) in the neighborhood of \mathcal{E} . Moreover, for any non-zero numbers $c_1, c_2 \in \mathbb{R}$ having the same sign, the triple (α, J, H) can be chosen in such a way that

$$\int_{\ell_i} \alpha = c_i,$$

where ℓ_i are the loops constituting the level \mathcal{E} (see Figure 6). □

The proof of this proposition is a rather technical explicit construction, and we give it in Appendix 2.

Proposition 4.17. Let (C, ω) be a symplectic cylinder with boundary $\partial C = \ell_2 - \ell_1$, and let $F: C \rightarrow \mathbb{R}$ be a smooth function without critical points on C and constant on the boundary components: $F|_{\ell_1} = c_1 < c_2 = F|_{\ell_2}$. Let also α_1, α_2 be one-forms defined in neighborhoods of the cycles ℓ_1, ℓ_2 respectively such that

$$d\alpha_1 = d\alpha_2 = F\omega. \tag{13}$$

Assume also that in those neighborhoods the expression $\omega^{-1}(\alpha_i, dF)$ does not vanish, that is, one-forms α_i and dF are linearly independent at every point of the neighborhoods. Furthermore, consider the function

$$\eta(z) := \int_{\ell_1} \alpha_1 + \int_{C_z} F\omega,$$

where C_z is the set of points where $F \leq z$. We assume that

$$\eta(C_2) = \int_{\ell_2} \alpha_2, \quad (14)$$

and that $\eta(z)$ does not change sign in the interval $[c_1, c_2]$.

Then there exists a one-form α on C which “interpolates” between α_i , that is, it is linearly independent with dF , coincides with α_i in a sufficiently small neighborhood of ℓ_i , and satisfies $d\alpha = F\omega$. \square

Proof. We work in action-angle coordinates S, Θ from Lemma 4.11. In the neighborhood of ℓ_i we have

$$\alpha_i = A_i(S, \Theta)dS + B_i(S, \Theta)d\Theta.$$

By taking the direct image of the second component under the projection to the S -axis and recalling that $F = \zeta(S)$ we see that

$$\int_0^{2\pi} B_i(S, \Theta)d\Theta = \eta(\zeta(S))$$

(for any S where the left-hand side is well-defined), as follows from (13), (14), and the Stokes formula. The condition of linear independence implies that the functions B_1, B_2 do not vanish. Furthermore, they have the same sign as the function η , since

$$\int_0^{2\pi} B_i(S(\ell_i), \Theta)d\Theta = \int_{\ell_i} \alpha_i = \eta(C_i).$$

Consider the convex set \mathcal{B} of smooth functions $B(S, \Theta)$ on the cylinder C which have the same sign as the function η and the same pushforward $\eta(\zeta(S))$ for the one-forms $B(S, \Theta)d\Theta$. The set \mathcal{B} is non-empty as it contains a function of S only: $B_0(S, \Theta) := \frac{1}{2\pi} \eta(\zeta(S))$. Therefore, using a partition of unity, one can construct a function $\tilde{B}(S, \Theta) \in \mathcal{B}$ which coincides with B_i in a neighborhood of ℓ_i for $i = 1, 2$. Then one can restore the desired one-form

$$\alpha := A(S, \Theta)dS + \tilde{B}(S, \Theta)d\Theta,$$

satisfying $d\alpha = F\omega$ by Poincaré’s lemma with parameters: the one-form $(\frac{\partial}{\partial S} \tilde{B}(S, \Theta) - \zeta(S))d\Theta$ is closed on levels sets of F and has zero periods, and hence exact, that is, for any fixed S represented as $d_\Theta A(S, \Theta)$ for an appropriate function $A(S, \Theta)$ on the cylinder. \blacksquare

Definition 4.18. Let (Γ, f, μ) be a measured Reeb graph, and let \mathfrak{c} be a balanced circulation function on Γ . Denote by $\tilde{V} \subset \Gamma$ the set of *exceptional points* of the graph, defined as the union of all vertices and all points x of the graph where either $f(x) = 0$ or $\mathfrak{c}(x) = 0$ (or both):

$$\tilde{V} := V(\Gamma) \cup \{x \in \Gamma \mid f(x) = 0\} \cup \{x \in \Gamma \mid \mathfrak{c}(x) = 0\}. \quad \square$$

Definition 4.19. By a *one-form* on a Reeb graph (Γ, f) we mean an expression which can locally be written as $g(f)df$, where $g(f)$ is a smooth function of f . A one-form β is *exact* if for any cycle ℓ in Γ we have $\int_{\ell} \beta = 0$. \square

Proposition 4.20. Let (Γ, f, μ) be a measured Reeb graph, \mathfrak{c} a balanced circulation function on Γ , and \tilde{V} the set of the corresponding exceptional points of the graph. Assume that in the neighborhood U_i of each exceptional point $x_i \in \tilde{V}$ there is a one-form β_i such that for each $x \in U_i \setminus \{x_i\}$, we have

$$\text{sign } \frac{\beta_i}{df} = -\text{sign } \mathfrak{c}. \quad (15)$$

Then there exists a one-form β on Γ which does not vanish in $\Gamma \setminus \tilde{V}$, coincides with β_i in a neighborhood of $x_i \in \tilde{V}$, and such that the form $f\beta$ is exact. \square

The proof of Proposition 4.20 is based on the following lemma:

Lemma 4.21. Assume that Γ is an oriented graph, and let $\varepsilon: E(\Gamma) \rightarrow \{\pm 1\}$ be a function such that for any subset $E' \subset E(\Gamma)$, we have

$$\partial \sum_{e \in E'} \varepsilon(e)e \neq 0. \quad (16)$$

Then there exists a one-coboundary $\xi: E(\Gamma) \rightarrow \mathbb{R}$ such that $\text{sign } \xi = \varepsilon$. \square

Proof. Reverse the orientation of all edges e such that $\varepsilon(e) = -1$. Then from condition (16) it follows that the obtained graph has no directed cycles. Therefore, there exists a zero-cochain η such that $\eta(w) > \eta(v)$ if there is an edge going from v to w , and it is easy to see that the one-coboundary $\xi := \delta\eta$, where δ is the coboundary operator, has the desired property. \blacksquare

Proof of Proposition 4.20. Consider the graph $\tilde{\Gamma}$ obtained from Γ by regarding all points $x \in \Gamma$ such that $f(x) = 0$ or $\mathfrak{c}(x) = 0$ as two-valent vertices. Then the product $f\mathfrak{c}$

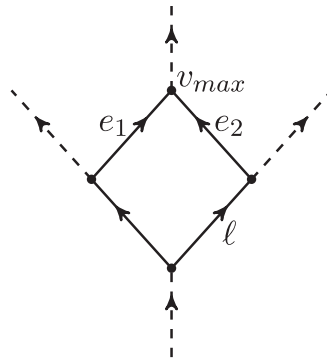


Fig. 7. A maximum of f on a cycle ℓ in a measured Reeb graph.

has the same sign at interior points of any edge $e \in \tilde{\Gamma}$. Define a function $\varepsilon: E(\tilde{\Gamma}) \rightarrow \{\pm 1\}$ by setting

$$\varepsilon(e) := -\text{sign } f|_e. \quad (17)$$

Thanks to condition (15), the forms β_i defined near vertices of the graph $\tilde{\Gamma}$ extend to a global form β such that

$$\text{sign } \frac{\beta}{df}(x) = -\text{sign } \mathfrak{c}(x)$$

for any interior point $x \in \tilde{\Gamma}$. For any edge $e \in E(\tilde{\Gamma})$, we have

$$\text{sign } \int_e f\beta = \varepsilon(e),$$

where $\varepsilon(e)$ is given by formula (17). Moreover, it is easy to see that for any one-cochain $\xi: E(\tilde{\Gamma}) \rightarrow \mathbb{R}$ such that $\text{sign } \xi(e) = \varepsilon(e)$, one can choose the form β in such a way that

$$\int_e f\beta = \xi(e). \quad (18)$$

We claim that the function $\varepsilon(e)$ satisfies condition (16). Indeed, assume that

$$\ell = \sum_{e \in E'} \varepsilon(e)e$$

is a cycle. Consider the point $v_{\max} \in \ell$ where f is maximal. Then v_{\max} has to be a three-valent vertex with two incoming edges $e_1, e_2 \in E(\tilde{\Gamma})$ (see Figure 7). Since the circulation

function \mathfrak{c} is balanced, the product $f\mathfrak{c}$ has the same sign at e_1 and e_2 . Therefore, we have $\varepsilon(e_1) = \varepsilon(e_2)$, which contradicts ℓ being a cycle.

Thus, since $\varepsilon(e)$ satisfies (16), by Lemma 4.21 the one-cochain ξ entering (18) can be chosen to be a coboundary. The latter implies that β can be chosen in such a way that $f\beta$ is exact, q.e.d. \blacksquare

Proof of the “if” part of Theorem 4.4. Let (M, ω) be a compact-connected symplectic surface, possibly with boundary. Let also $\mathcal{O} \subset \mathfrak{sect}^*(M)$ be a Morse-type coadjoint orbit, and let $(\Gamma, f, \mu, \mathfrak{c})$ be the corresponding circulation graph. Assume that the circulation function \mathfrak{c} is balanced. We need to show that there exists a metric g on M which is compatible with ω and such that the corresponding Euler equation has a fixed point on \mathcal{O} .

Take any $[\tilde{\alpha}] \in \mathcal{O}$, and let $F = \text{curl}[\tilde{\alpha}]$. Then the measured Reeb graph Γ_F associated with F coincides with (Γ, f, μ) , and we have a projection $\pi: M \rightarrow \Gamma$. As in Proposition 4.20, let \tilde{V} be the set of exceptional points,

$$\tilde{V} := V(\Gamma) \cup \{x \in \Gamma \mid f(x) = 0\} \cup \{x \in \Gamma \mid \mathfrak{c}(x) = 0\},$$

and let U_i be a neighborhood of $x_i \in \tilde{V}$. Now, we use Propositions 4.12, 4.13, 4.16 to construct an F -steady triple (α_i, J_i, H_i) in $\pi^{-1}(U_i)$ such that:

- (i) If $x_i \in \tilde{V}$ is a boundary vertex or an interior point of Γ , then

$$\int_{\ell_i} \alpha_i = \mathfrak{c}(x_i),$$

where $\ell_i = \pi^{-1}(x_i)$.

- (ii) If $x_i \in \tilde{V}$ is a three-valent vertex of Γ , and e_1, e_2 are branches of x_i , then for $j = 1, 2$ we have

$$\int_{\ell_{ij}} \alpha_i = c_j(x_i),$$

where $\ell_{ij} = \pi^{-1}(x_i) \cap \partial\pi^{-1}(e_j)$, and $c_j(x_i)$ is the limit of the circulation function $\mathfrak{c}(x)$ as x tends to x_i along the edge e_j .

It follows from our construction that for any $x \in U_i \setminus x_i$ we have

$$\int_{\ell_x} \alpha_i = \mathfrak{c}(x),$$

where $\ell_x = \pi^{-1}(x)$. Furthermore, from the relation $J^*\alpha_i = -dH_i(F)$, it follows that if α_i is linearly dependent with dF at some point Z lying on a non-singular level $\pi^{-1}(x)$, then $\alpha \equiv 0$ on $\pi^{-1}(x)$, and thus $c(x) = 0$. Therefore, α_i and dF are independent at all points of $M \setminus \pi^{-1}(\tilde{V})$ (the latter also follows from a more general formula (12) above).

Moreover, using Proposition 4.17, we extend the forms α_i to a form α defined on the whole M which is linearly independent with dF at all points of $M \setminus \pi^{-1}(\tilde{V})$ and satisfies $d\alpha = F\omega$. For any interior point $x \in \Gamma$, we have

$$\int_{\ell_x} \alpha = c(x),$$

where $\ell_x = \pi^{-1}(x)$. By Lemma 4.14 one obtains $\text{sign } \partial H_i / \partial F(f(x)) = -\text{sign } c(x)$ for any $x \in U_i \setminus \{x_i\}$. Therefore, descending the forms dH_i from $\pi^{-1}(U_i)$ to U_i , one obtains a family of one-forms β_i satisfying the condition of Proposition 4.20. So, there exists a one-form β on Γ which does not vanish in $\Gamma \setminus \tilde{V}$, coincides with β_i in a neighborhood of $x_i \in \tilde{V}$, and such that the form $f\beta$ is exact. Lifting the form β back to M , we obtain a form γ which does not vanish in $M \setminus \pi^{-1}(\tilde{V})$, coincides with dH_i in the neighborhood of the fiber $\pi^{-1}(x_i)$, and such that the form $F\gamma$ is exact. Now, we extend almost complex structures J_i to a global almost complex structure J by setting $J^*\alpha := -\gamma$, $J^*\gamma := \alpha$. Further, define a metric g on M by setting $g(u_1, u_2) := \omega(u_1, Ju_2)$. Then, from the formula $d(J^*\alpha) = -d\gamma = 0$ it follows that α is co-closed. For the fluid velocity field $u = \alpha^\sharp$, we have

$$L_u[\alpha] = [i_u d\alpha] = [Fi_u \omega] = -[FJ^*\alpha] = [F\gamma] = 0,$$

so $[\alpha]$ is a steady solution of the Euler equation lying on the orbit \mathcal{O} , as desired (we have $[\alpha] \in \mathcal{O}$ since $d\alpha = F\omega$, and the circulation function corresponding to $[\alpha]$ coincides, by construction, with the prescribed function c). ■

Remark 4.22. The metrics obtained for steady solutions via the above construction are usually hyperbolic near saddle points. Also it is worth mentioning that while the above steady solutions have stream functions near all critical points of F , they might not glue into a univalued stream function on the whole surface M .

Finally note that Theorem 4.4 can be extended to the cases of arbitrary (non-simple) Morse functions and to certain cases of Morse functions with critical points on the boundary. □



Fig. 8. A maximal directed path ℓ starting at a three-valent vertex v .

4.3 Steady flows on closed surfaces

Definition 4.23. Let Γ be a measured Reeb graph with no boundary vertices, and let c be a circulation function on Γ . We say that c is *totally negative* if for any three-valent vertex v of Γ , the limits $c_0(v)$, $c_1(v)$, $c_2(v)$ of c at v are negative. \square

Remark 4.24. Total negativity in particular implies that $c(x) < 0$ at any interior point $x \in \Gamma \setminus V(\Gamma)$. Indeed, for any edge $e \in \Gamma$ the total negativity condition and the Kirchhoff condition (5) at one-valent vertices imply that $c(x)$ has non-positive limits at endpoints of e . At the same time, from property (4) of circulation functions and the monotonicity of f along e it follows that the function c restricted to e is concave down. Therefore, we indeed have $c < 0$ at all points of e . \square

It turns out that for closed surfaces Theorem 4.4 can be reformulated in the following way:

Theorem 4.25. Let (M, ω) be a closed connected symplectic surface, and let $\mathcal{O} \subset \text{spect}^*(M)$ be a Morse-type coadjoint orbit. Then \mathcal{O} admits a steady flow if and only if the corresponding circulation function is totally negative. \square

Proof. It suffices to show that if Γ is a measured Reeb graph with no boundary vertices, then a circulation function c on Γ is balanced if and only if it is totally negative. Clearly, if c is totally negative, then it is balanced. Let us prove the converse statement. Let c be a balanced circulation function on Γ . Assume that there exists a three-valent vertex $v \in \Gamma$ such that $c_i(v) > 0$ for $i = 0, 1, 2$. Let us first assume that $f(v) \geq 0$. Consider any maximal directed path ℓ in Γ starting at the vertex v (see Figure 8). Then $f \geq 0$ at points of the path ℓ , and the circulation function c has non-negative limits at endpoints of ℓ , since it

"integrates" f along ℓ . However, at the top endpoint the circulation function tends to 0. So, the same argument as in Example 4.6 shows that c must change sign at some three-valent vertex $w \in \ell$, which contradicts the fact that c is balanced. Similarly, if $f(v) \leq 0$, then one considers a maximal directed path ending at v and applies the same argument to obtain a contradiction. So, we indeed have $c_i(v) < 0$ for all three-valent vertices $v \in \Gamma$, q.e.d. \blacksquare

Example 4.26. Consider the circulation graph (Γ, f, μ, c) for a Morse-type coadjoint orbit $\mathcal{O} \subset \mathfrak{vect}^*(T^2)$ on a torus depicted in Figure 4, cf. Example 3.13. Let

$$a_i := \int_{e_i} f d\mu.$$

The space of circulation functions on Γ is one-dimensional and the numbers a_i satisfy the relation $a_1 + a_2 + a_3 + a_4 = 0$ by Proposition 3.12.

The set of totally negative circulation functions on Γ is described by linear inequalities

$$z < 0, \quad a_1 - z < 0, \quad a_2 + z < 0, \quad a_1 + a_3 - z < 0,$$

that is, the circulation function c is totally negative if and only if $z \in I_1 \cap I_2$, where I_1, I_2 are open intervals $I_1 = (a_1, 0)$, $I_2 = (a_1 + a_3, -a_2)$. Using that f is monotonous along edges of Γ and the zero average condition $a_1 + a_2 + a_3 + a_4 = 0$, one can easily prove that $a_1 < 0$, and $a_1 + a_3 < -a_2$, so that the intervals I_1, I_2 are non-empty. However, these intervals do not need to intersect, so depending on a_i 's, the set of totally negative circulation functions on Γ may be empty. Furthermore, even if intervals I_1, I_2 do intersect, the set of totally negative circulation functions on Γ is only a bounded interval, while the set of all circulation functions is a line \mathbb{R}^1 . The latter implies that, roughly speaking, most of Morse-type coadjoint orbits of $\text{SDiff}(T^2)$ corresponding to the graph in Figure 4 do not admit steady solutions of the Euler equation. \square

The following theorem generalizes the conclusion of Example 4.26:

Theorem 4.27. Assume that (Γ, f, μ) is a measured Reeb graph with no boundary vertices. Then the set of totally negative circulation functions on Γ is a (possibly empty) open convex polytope in the affine space of all circulation functions. In particular, the set of totally negative circulation functions on Γ is bounded. \square

Proof. It is obvious from the definition that the set of totally negative circulation functions on Γ is an open convex polyhedron. So, it suffices to prove that the latter set is bounded. Since a circulation function c is uniquely determined by numbers $c^+(e)$ (recall that the notation $c^+(e)$ stands for the limit of the circulation function $c(x)$ as x tends to the endpoint of the edge e along e), it suffices to show that for any edge $e \in \Gamma$ there exist numbers $m, M \in \mathbb{R}$ such that for any totally negative circulation function c on Γ we have $m \leq c^+(e) \leq M$.

Consider the following poset structure on the set of edges of Γ . Say that $e_1 \leq e_2$ if there exists a directed path ℓ in Γ whose first edge is e_1 , and whose last edge is e_2 . Let us also define the height $h(e)$ of an edge $e \in \Gamma$ as the maximal length (i.e., the number of edges) of a directed path whose last edge is e . Using properties of circulation functions, the total negativity condition, and induction by height, it can be easily shown that for any edge $e \in \Gamma$ and any totally negative circulation function c on Γ one has

$$c^+(e) \geq \sum_{e' \leq e} \int_{e'} f d\mu.$$

At the same time, by total negativity condition one has $c^+(e) \leq 0$, so the set of totally negative circulation functions on Γ is indeed bounded, q.e.d. \blacksquare

Corollary 4.28. Assume that M is a closed connected symplectic surface of genus ≥ 1 . Let also (Γ, f, μ) be a measured Reeb graph compatible with M and satisfying the condition $\int_{\Gamma} f d\mu = 0$. Then the set of coadjoint orbits of $\text{SDiff}(M)$ corresponding to the graph Γ and admitting steady solutions of the Euler equation is an open convex polytope in the affine space of all coadjoint orbits of $\text{SDiff}(M)$ corresponding to Γ . \square

Remark 4.29. Loosely speaking, Corollary 4.28 states that if M is of genus ≥ 1 , then “most” Morse-type coadjoint orbits of $\text{SDiff}(M)$ do not admit steady solutions of the Euler equation. The conclusion of Corollary 4.28 remains true for the sphere as well; however, in this case the affine space of all coadjoint orbits of $\text{SDiff}(M)$ corresponding to Γ is a point (i.e., there is a one-to-one correspondence between coadjoint orbits and measured Reeb graphs), so the statement of Corollary 4.28 becomes trivial. \square

Example 4.30. Figures 1, 9, and 10 show several vorticity functions on a pretzel with different measured Reeb graphs Γ_F , as well as the corresponding polytopes of totally negative circulation functions. The numbers at edges are integrals $\int_e f d\mu$. In Figures 1 and 10, the set of circulation functions on Γ_F is parametrized by limits $c_1(v), c_2(v)$ at

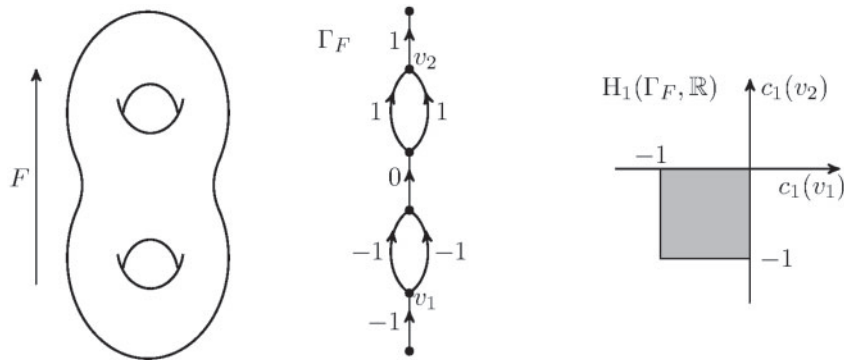


Fig. 9. An example of a vorticity F on a pretzel, its measured Reeb graph, and the corresponding polytope of totally negative circulation functions.

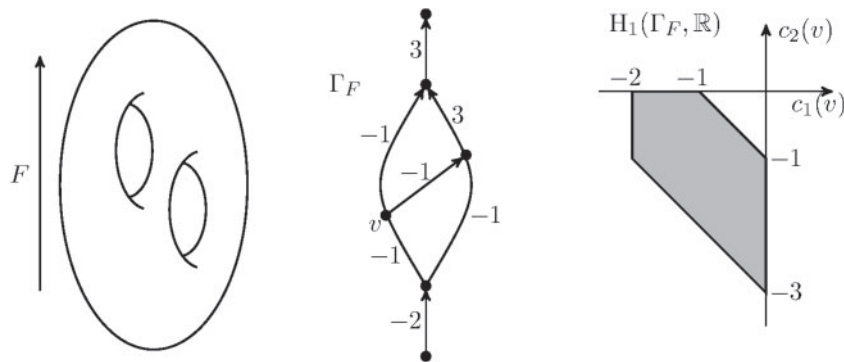


Fig. 10. A vorticity F on a pretzel with a different measured Reeb graph and the corresponding polytope of totally negative circulation functions.

branches of the vertex v . In Figure 9, the set of circulation functions is parametrized by limits $c_1(v_1), c_1(v_2)$ at left branches of the vertices v_1, v_2 . \square

Appendix 1: Proof of the Classification of Generic Coadjoint Orbits

In this appendix we prove Theorem 3.19 on the classification of generic coadjoint orbits in the group $\text{SDiff}(M)$.

Theorem A.1. (=3.19) Let M be a compact-connected symplectic surface. Then Morse-type coadjoint orbits of $\text{SDiff}(M)$ are in one-to-one correspondence with circulation graphs (Γ, f, μ, c) compatible with M . In other words, the following statements hold:

- (i) For a symplectic surface M Morse-type cosets $[\alpha], [\beta] \in \mathfrak{vect}^*(M)$ lie in the same orbit of the $\mathrm{SDiff}(M)$ coadjoint action if and only if circulation graphs $\Gamma_{[\alpha]}$ and $\Gamma_{[\beta]}$ corresponding to these cosets are isomorphic.
- (ii) For each circulation graph Γ which is compatible with M , there exists a Morse-type $[\alpha] \in \mathfrak{vect}^*(M)$ such that $\Gamma_{[\alpha]} = (\Gamma, f, \mu, c)$. (See Definition 3.7 for compatibility of a graph and a surface.) \square

We start with the following preliminary lemma.

Lemma A.2. Let M be a connected oriented surface possibly with boundary, and let F be a simple Morse function on M . Assume that $[\gamma] \in H^1(M, \mathbb{R})$ is such that the integral of γ over any F -level vanishes. Then there exists a C^∞ function $H: M \rightarrow \mathbb{R}$ constant on the boundary ∂M such that the one-form HdF is closed, and its cohomology class is equal to $[\gamma]$. Moreover, H can be chosen in such a way that the ratio H/F is a smooth function. \square

The proof follows that of Lemma 4.8 in [11].

Proof of Theorem A.1. For the first statement, it is immediate by construction, that if cosets $[\alpha]$ and $[\beta]$ lie in the same $\mathrm{SDiff}(M)$ -orbit then their circulation graphs $\Gamma_{[\alpha]}$ and $\Gamma_{[\beta]}$ are isomorphic. To prove the converse statement let $\phi: \Gamma_{[\alpha]} \rightarrow \Gamma_{[\beta]}$ be an isomorphism of circulation graphs. By Theorem 3.8, ϕ can be lifted to a symplectomorphism $\Phi: M \rightarrow M$ that maps the function $F = \mathrm{curl}[\alpha]$ to the function $G = \mathrm{curl}[\beta]$. Therefore, the one-form γ defined by $\gamma := \Phi^*\beta - \alpha$ is closed.

Assume that $\Psi: M \rightarrow M$ is a symplectomorphism which maps the function F to itself and is isotopic to the identity. Then the composition $\tilde{\Phi} = \Phi \circ \Psi^{-1}$ maps F to G , and

$$[\tilde{\Phi}^*\beta - \alpha] = [\Phi^*\beta - \Psi^*\alpha] = [\gamma] - [\Psi^*\alpha - \alpha].$$

We claim that Ψ can be chosen in such a way that $\tilde{\Phi}^*\beta - \alpha$ is exact, that is, one has the equality of the cohomology classes $[\Psi^*\alpha - \alpha] = [\gamma]$. We construct Ψ using a version of the Moser path method. Let us show that there exists a time-independent symplectic vector field X which is tangent to the boundary ∂M , preserves F , and satisfies

$$[\Psi_t^*\alpha - \alpha] = t[\gamma], \tag{A.1}$$

where Ψ_t is the phase flow of X . Differentiating (A.1) with respect to t , we get in the left-hand side

$$[\Psi_t^* L_X \alpha] = [L_X \alpha] = [i_X d\alpha] = [F \cdot i_X \omega],$$

since $L_X \alpha$ is closed and Ψ_t^* does not change its cohomology class. Thus

$$[F \cdot i_X \omega] = [\gamma]. \quad (\text{A.2})$$

Since Φ preserves the circulation function, the integrals of γ over all F -levels vanish. Therefore, by Lemma A.2, there exists a smooth function H such that $[\gamma] = [HdF]$. Now we set $X := (H/F) \omega^{-1} dF$. It is easy to see that the vector field X is symplectic, preserves the levels of F , and satisfies the equation (A.2). Therefore, its phase flow satisfies the equation (A.1), and then for the symplectomorphism $\tilde{\Phi} = \Phi \circ \Psi_1^{-1}$, where Ψ is the time-one map Ψ_1 , we have $\tilde{\Phi}^*[\beta] = \alpha$, as desired.

Now, let us prove the second statement. By Theorem 3.8, there exists a simple Morse function $F: M \rightarrow \mathbb{R}$ such that the measured Reeb graph of F is (Γ, f, μ) . Since the graph Γ admits a circulation function, from Proposition 3.12 and Remark 3.14 it follows that the function F lies in the image of the mapping curl , that is, there exists a one-form $\alpha \in \Omega^1(M)$ such that $\text{curl}[\alpha] = F$. Further, if γ is a closed one-form, then $\text{curl}[\alpha + \gamma] = F$ as well. For any one-form $\tilde{\alpha}$ such that $\text{curl}[\tilde{\alpha}] = F$, let $c_{\tilde{\alpha}}$ denote the corresponding circulation function on Γ . Consider the mapping $\rho: H^1(M, \mathbb{R}) \rightarrow H_1(\Gamma, \partial\Gamma, \mathbb{R})$ given by

$$\rho[\gamma] := \text{ch}(c_{\alpha+\gamma}) - \text{ch}(c_{\alpha}),$$

where $\text{ch}(c)$ is given by (6). Let us show that the homomorphism ρ is surjective. Indeed, ρ can be written as

$$\rho[\gamma] = \sum_e \left(\int_{\ell(e)} \gamma \right) e,$$

where $\ell(e) = \pi^{-1}(x_e)$ and $x_e \in e$ is any interior point of the edge e . Therefore, the kernel of the homomorphism ρ consists of those cohomology classes which vanish on cycles homologous to regular F -levels, and hence $\dim \text{Ker } \rho = \varkappa$, where \varkappa is the genus of the surface M . On the other hand, comparing the dimensions of the spaces involved, we have

$$\dim H^1(M, \mathbb{R}) = \begin{cases} 2\varkappa, & \text{if } M \text{ is closed,} \\ 2\varkappa + k - 1, & \text{if } M \text{ is not closed,} \end{cases}$$

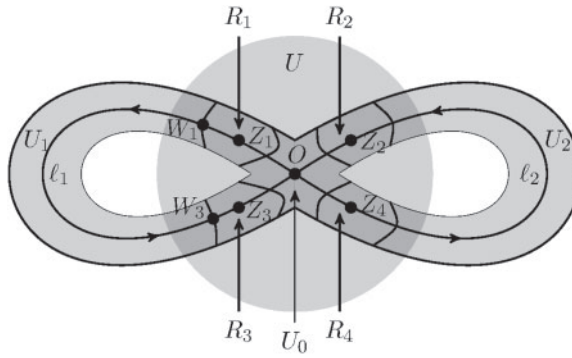


Fig. A1. Constructing a steady flow near a hyperbolic fiber.

where k is the number of boundary components of M or, equivalently, the number of boundary vertices of Γ . Further, from the exact sequence of the pair $(\Gamma, \partial\Gamma)$ we find that

$$\dim H_1(\Gamma, \partial\Gamma, \mathbb{R}) = \begin{cases} \kappa, & \text{if } k = 0, \\ \kappa + k - 1, & \text{if } k > 0. \end{cases}$$

Since $\dim H^1(M, \mathbb{R}) - \dim H_1(\Gamma, \partial\Gamma, \mathbb{R}) = \kappa$, by the dimensional argument the homomorphism ρ is indeed surjective. This implies that one can find a closed one-form γ such that $\rho[\gamma] = \text{ch}(c) - \text{ch}(c_a)$, where c is a given circulation function on Γ , and therefore $c_{\alpha+\gamma} = c$, as desired. ■

Appendix 2: Existence of Steady Triples in the Vicinity of Hyperbolic Levels

In this appendix we prove technical Proposition 4.16 that near any figure-eight level of the vorticity function there exists a steady triple.

Proposition A.3. (=4.16) Let (M, ω) be a compact symplectic surface, possibly with boundary, and let $F: M \rightarrow \mathbb{R}$ be a simple Morse function on M . Let also \mathcal{E} be a figure-eight level of F . Then there exists a steady F -triple (α, J, H) in the neighborhood of \mathcal{E} . Moreover, for any non-zero numbers $c_1, c_2 \in \mathbb{R}$ having the same sign, the triple (α, J, H) can be chosen in such a way that $\int_{\ell_i} \alpha = c_i$, where ℓ_i are the loops constituting the level \mathcal{E} (see Figure A1). □

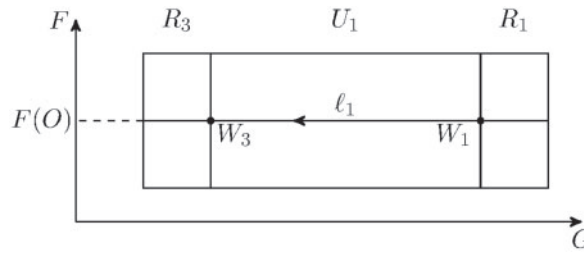


Fig. A2. Region $V = U_1 \cup R_1 \cup R_3$.

Proof. First, we apply Lemma 4.15 to construct an F -steady triple (α, J, H) such that $\text{sign}(dH/dF) = \varepsilon := -\text{sign } c_1 = -\text{sign } c_2$ in the neighborhood U of the singular point $O \in \mathcal{E}$. Since H is a function of F , it extends to the neighborhood of \mathcal{E} in a unique way.

Consider points $Z_1, Z_2, Z_3, Z_4 \in U \cap \mathcal{E}$ depicted in Figure A1. In U we have

$$\omega^{-1}(\alpha, dH) = \omega^{-1}(J^*\alpha, \alpha) = g^{-1}(\alpha, \alpha),$$

where g is the Riemannian metric given by $g(u_1, u_2) = \omega(u_1, Ju_2)$. Therefore, we have $\omega^{-1}(\alpha, dH) > 0$ at all points except the point O . In particular, the forms α and dH are linearly independent at the points Z_i . The latter implies that there exist functions G_i in neighborhoods of points Z_i which form a positive coordinate system with F (i.e., $(dG_i \wedge dF)/\omega > 0$) and such that $\alpha = A_i(F, G_i) dG_i$ for certain smooth functions A_i .

Let R_i be a small rectangle in (F, G_i) coordinates near Z_i . Taking, if necessary, a smaller neighborhood of \mathcal{E} , we may assume that the complement to R_i , $i = 1, \dots, 4$ in this neighborhood consists of three regions U_0, U_1, U_2 where $O \in U_0$ (see Figure A1).

Consider the region $V = U_1 \cup R_1 \cup R_3$. Extend the functions G_1 and G_3 (maybe shifting one of them by a constant) from R_1 and R_3 to a function G defined in the region V in such a way that (G, F) is a positive coordinate system in U_1 . (This means that the orientation defined by $dG \wedge dF$ coincides with the symplectic orientation or the function G decreases along ℓ_1 .) In coordinates (G, F) , the region V is a rectangle depicted in Figure A2. Our aim is to extend the one-form α defined in R_1, R_3 to V . First, note that in regions R_1, R_3 we have

$$\omega^{-1}(\alpha, dF) = A_i(F, G) \omega^{-1}(dG, dF) = -A_i(F, G) \{F, G\},$$

where $\{F, G\}$ is the Poisson bracket associated with the symplectic structure ω . This, in particular, implies that $\text{sign } A_i = \text{sign}(dH/dF) = \varepsilon$. We first extend the functions $A_1(F(O), G)$ and $A_3(F(O), G)$ (where $F(O)$ is the value of F at the level \mathcal{E}) defined on arcs

$\ell_1 \cap R_1$ and $\ell_1 \cap R_3$ to a function $A^0(G)$ defined on $\ell_1 \cap V$ and such that $\text{sign } A^0 = \text{sign } A_i = \varepsilon$. Now define a function

$$A(F, G) := A^0(G) + \int_{F(O)}^F \frac{F}{\{F, G\}} dF.$$

We claim that this function coincides with A_i in the regions R_i . Indeed, by taking the exterior differential in the equation $\alpha = A_i(F, G_i) dG_i$, we get $F\omega = \frac{\partial}{\partial F} A_1(F, G) dF \wedge dG$, that is, $F = \frac{\partial}{\partial F} A_1(F, G) \cdot \{F, G\}$. Then the corresponding derivatives coincide:

$$\frac{\partial}{\partial F} A_1(F, G) = \frac{F}{\{F, G\}} = \frac{\partial}{\partial F} A(F, G), \quad (\text{A.3})$$

and together with $A(F(O), G) = A^0(G) = A_1(F(O), G)$, equation (A.3) implies that $A = A_1$ in the region R_1 . Similarly, $A = A_3$ in R_3 . Now, we extend the one-form α to V by setting $\alpha := A dG$. Then $d\alpha = F\omega$, as desired. Repeating the same procedure for the region $\tilde{V} = U_2 \cup R_2 \cup R_4$, we obtain a one-form α defined in the whole neighborhood of \mathcal{E} and satisfying $d\alpha = F\omega$. Furthermore, in V we have

$$\omega^{-1}(\alpha, dH) = \frac{dH}{dF} A(F, G) \omega^{-1}(dG, dF) > 0$$

in a sufficiently small neighborhood of $\mathcal{E} \cap V$ and, similarly, for $\mathcal{E} \cap \tilde{V}$. This allows one to extend the almost complex structure J to a neighborhood of \mathcal{E} by setting $J^*\alpha := -dH$, $J^*dH := \alpha$. Thus, the first statement is proved.

To prove the second statement denote by W_i the intersection point of ℓ_1 and the common boundary of the regions R_i and U_1 (see Figures A1 and A2). Then

$$\int_{W_1}^{W_3} \alpha = \int_{G(W_1)}^{G(W_3)} A^0(G) dG. \quad (\text{A.4})$$

By using ambiguity in the choice of a smooth function $A^0(G)$ extending the functions $A_i(F(O), G)$ and satisfying $\text{sign } A^0 = \varepsilon$, one can choose it so that the integral (A.4) is equal to any c (note that $G(W_1) > G(W_3)$) provided that $\text{sign } c = -\varepsilon$. Furthermore, by moving the points Z_1, Z_3 closer to O and shrinking the regions R_1, R_3 , one can make the integral (A.4) arbitrary close to $\int_{\ell_1} \alpha$. Since $\text{sign } c_1 = -\varepsilon$, this implies that one can choose α such that $\int_{\ell_1} \alpha = c_1$. The proof for ℓ_2 is analogous. ■

Appendix 3: Casimir Invariants of the 2D Euler Equation

Above we classified coadjoint orbits of the group $\text{SDiff}(M)$ in terms of graphs with some additional structures, see Theorem 3.19. However, for applications, it is important to describe numerical invariants of the coadjoint action, that is, Casimir functions. We begin with the description of such invariants for functions on symplectic surfaces.

Let (M, ω) be a compact-connected symplectic surface, possibly with boundary, and let F be a simple Morse function on M . With each edge e of the measured Reeb graph $\Gamma_F = (\Gamma, f, \mu)$, one can associate an infinite sequence of moments

$$m_{i,e}(F) = \int_e f^i d\mu = \int_{M_e} F^i \omega,$$

where $i = 0, 1, 2, \dots$, and $M_e = \pi^{-1}(e)$ for the natural projection $\pi: M \rightarrow \Gamma$. Obviously, the moments $m_{i,e}(F)$ are invariant under the action of $\text{SDiff}(M)$ on simple Morse functions. Moreover, they form a complete set of invariants in the following sense:

Theorem A.4. Let (M, ω) be a compact connected symplectic surface, possibly with boundary, and let F and G be simple Morse functions on M . Assume that $\phi: \Gamma_F \rightarrow \Gamma_G$ is an isomorphism of abstract-directed graphs which preserves moments on all edges. Then Γ_F and Γ_G are isomorphic as measured Reeb graphs, and there exists a symplectomorphism $\Phi: M \rightarrow M$ such that $\Phi_*F = G$. \square

Proof. Consider an edge $e = [v, w] \in \Gamma_F$. Pushing forward the measure μ on e by means of the homeomorphism $f: e \rightarrow [f(v), f(w)] \subset \mathbb{R}$, we obtain a measure μ_f on the interval $I_f = [f(v), f(w)]$, whose moments coincide with the moments of μ at e . Repeating the same construction for the measure on Γ_G , we obtain another measure μ_g , which is defined on the interval $I_g = [g(\phi(v)), g(\phi(w))]$ and has the same moments as μ_f .

Now, consider any closed interval $I \subset \mathbb{R}$ which contains both I_f and I_g . Then the measures μ_f, μ_g may be viewed as measures on the interval I supported at I_f and I_g , respectively. The moments of the measures μ_f, μ_g on I coincide, so, by the uniqueness theorem for the Hausdorff moment problem, we have $\mu_f = \mu_g$, which implies the proposition. \blacksquare

The above theorem allows one to describe Casimirs of the 2D Euler equation on M , that is, invariants of the coadjoint action of the symplectomorphism group $\text{SDiff}(M)$. Let F be a Morse vorticity function of an ideal flow with velocity v on a surface M , possibly with boundary, and let Γ be its Reeb graph. The corresponding moments $m_{i,e}(F)$

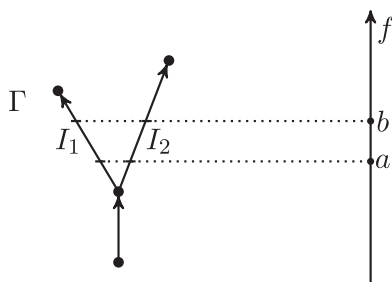


Fig. A3. Modifying the measure on the edges.

for this vorticity are natural to call *generalized enstrophies*. Then group coadjoint orbits in the vicinity of an orbit with the vorticity function F are singled out as follows.

Corollary A.5. A complete set of Casimirs of a 2D Euler equation in a neighborhood of a Morse-type coadjoint orbit is given by the moments $m_{i,e}$ for each edge $e \in \Gamma, i = 0, 1, 2, \dots$, and all circulations of the velocity v over cycles in the singular levels of F on M . \square

The (finite) set of required circulations can be sharpened by considering fewer quantities needed to describe the circulation function, as in Section 3.3.

Remark A.6. As invariants of the coadjoint action of $\text{SDiff}(M)$, one usually considers total moments

$$m_i(F) = \int_M F^i \omega = \int_\Gamma f^i d\mu,$$

where $F = \text{curl}[\alpha]$ is the vorticity function, and (Γ, f, μ) is the measured Reeb graph of F . However, the latter moments do not form a complete set of invariants even in the case of a sphere or a disk.

Consider, for example, the measured Reeb graph (Γ, f, μ) depicted in Figure A3. Let μ' be any smooth measure on \mathbb{R} supported in $[a, b]$. Define a new measure $\tilde{\mu}$ on Γ by “moving some density from one branch to another”, i.e. by setting

$$\tilde{\mu} := \begin{cases} \mu + f^*(\mu') & \text{in } I_1, \\ \mu - f^*(\mu') & \text{in } I_2, \end{cases}$$

and $\tilde{\mu} := \mu$ elsewhere. Then $(\Gamma, f, \tilde{\mu})$ is again a measured Reeb graph. Moreover, for all total moments we have

$$\int_{\Gamma} f^k d\mu = \int_{\Gamma} f^k d\tilde{\mu}.$$

However, the graphs (Γ, f, μ) and $(\Gamma, f, \tilde{\mu})$ are not isomorphic and thus correspond to two different coadjoint orbits of $\text{SDiff}(S^2)$. \square

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References

- [1] Adelson-Welskii, G. M., and A. S. Kronrode, "Sur les lignes de niveau des fonctions continues possédant des dérivées partielles." *Comptes rendus (Doklady) de l'Académie des sciences de l'URSS* 49 (1945): 235–7.
- [2] Arnold, V. I. "Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits." *Annales de l'institut Fourier* 16, no. 1 (1966): 319–61.
- [3] Arnold, V. I. "The asymptotic Hopf invariant and its applications." *Selecta Mathematica Sovietica* 5, no. 4 (1986): 327–45.
- [4] Arnold, V. I. *Arnold's Problems*. Berlin, Heidelberg, New York, Springer-Verlag, 2004.
- [5] Arnold, V. I., and B. A. Khesin. *Topological Methods in Hydrodynamics*. New York, Springer, 1998.
- [6] Bloch, A. M., H. Flaschka, and T. S. Ratiu. "A Schur-Horn-Kostant convexity theorem for the diffeomorphism group of the annulus." *Inventiones Mathematicae* 113, no. 1 (1993): 511–29.
- [7] Bloch, A. M., M. O. El Hadrami, H. Flaschka, and T. S. Ratiu. "Maximal tori of some symplectomorphism groups and applications to convexity." In *Deformation Theory and Symplectic Geometry* edited by D. Sternheimer, J. Rawnsley, and S. Gutt, 201–22. Mathematical Physics Studies, vol. 20, Dordrecht, Netherlands, Kluwer Academic Publishers, 1997.
- [8] Choffrut, A. and V. Šverák. "Local structure of the set of steady-state solutions to the 2D incompressible Euler equations." *Geometric and Functional Analysis* 22, no. 1 (2012): 136–201.
- [9] Colin de Verdière, Y. and J. Vey. "Le lemme de Morse isochore." *Topology* 18, no. 4 (1979): 283–93.
- [10] Ginzburg, V. L. and B. Khesin. "Steady fluid flows and symplectic geometry." *Journal of Geometry and Physics* 14, no. 2 (1994): 195–210.
- [11] Izosimov, A., B. Khesin, and M. Mousavi. "Coadjoint orbits of symplectic diffeomorphisms of surfaces and ideal hydrodynamics." (2015) to appear in *Annales de l'Institut Fourier*, 2016.
- [12] Lerman, E. "Symplectic cuts." *Mathematical Research Letters* 2, no. 3 (1995): 247–58.

- [13] Moffatt, H. K. "Structure and stability of solutions of the Euler equations: a Lagrangian approach." *Philosophical Transactions of the Royal Society of London A* 333, no. 1631 (1990): 321–42.
- [14] Reeb, G. "Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique." *Comptes Rendus de l'Académie des Sciences. Paris* 222 (1946): 847–9.
- [15] Serre, D. "Invariants et dégénérescence symplectique de l'équation d'Euler des fluides parfaits incompressibles." *Comptes Rendus de l'Académie des Sciences. Paris, Series A* 298, no. 14 (1984): 349–52.
- [16] Shnirelman, A. I. "Lattice theory and flows of ideal incompressible fluid." *Russian Journal of Mathematical Physics* 1 (1993): 105–14.
- [17] Yoshida, Z., P. J. Morrison, and F. Dobarro. "Singular Casimir elements of the Euler equation and equilibrium points." *Journal of Mathematical Fluid Mechanics* 16, no. 1 (2014): 41–57.