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Steady fluid flows and symplectic geometry ⋆

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Abstract

We show that for an even-dimensional fluid there exists a strong relation, via the Morse theory and symplectic geometry, between the topology of the vorticity function and the existence of a stationary solution of Euler's equation. In particular, it turns out that there is no smooth steady flow on the disc provided that the vorticity function is Morse, positive and has both a local maximum and minimum in the interior of the disc.

As we show, the structure of four-dimensional steady flows is similar to that of three-dimensional flows described in Arnold's theorem. Namely, under certain hypotheses, an analytic four-dimensional steady flow is fibered into invariant tori and annuli, for such a flow gives rise to an integrable Hamiltonian system on a symplectic four-manifold. It is also proved that an odd-dimensional chaotic steady flow is always a Beltrami flow, i.e., its velocity and vorticity fields are proportional.

Key words: steady fluid flows, symplectic structure, Euler's equation, vorticity function,
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1. Introduction

Steady (or stationary) flows of a compressible or incompressible fluid often turn out to be "attractors" in phase space and, therefore, the structure of such flows gives an "approximate picture" of an arbitrary fluid motion after a long period of time.

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The stationary Euler equation describing the vorticity of a steady flow formally coincides with the equation of a stationary magnetic field in magnetohydrodynamics. Thus studying the structure of steady flows is interesting from both points of view: ideal hydrodynamics and dynamo theory.

So far, all the results on stationary flows concerned, to the best of our knowledge, the flows on two- or three-dimensional manifolds (see [A1,A2, Mo,KoL,T]). The structure of three-dimensional steady flows is described by Arnold's theorem (see [A2,A3]): an analytic stationary flow is fibered into invariant tori and annuli unless the velocity field is proportional to the vorticity field. It turns out that under a hypothesis analogous to the one of Arnold's theorem four-dimensional steady flows have a similar structure: almost the whole of the manifold is fibered into tori invariant under the flow (see Section 3). Thus four-dimensional steady flows are similar to integrable Hamiltonian systems with two degrees of freedom. The "opposite" case where the velocity and vorticity field are collinear has analogs in any odd dimension: such generalized Beltrami (or ABC) flows have a pretty complicated topology of trajectories (see Section 4).

To analyze steady flows on even-dimensional manifolds, we introduce the following notion of extra symmetry. Recall that every steady solution on an even-dimensional manifold comes along with two functions: the Bernoulli function h and the vorticity function λ (see Section 2 for more details). Assume now that the class of isovortical fields, i.e., the coadjoint orbit in the phase space of the Euler equation to which the steady solution belongs, is generic. Then every field from this class gives rise to a symplectic structure ω defined almost everywhere on the manifold. An important feature of the steady solution is that $\{h,\lambda\} = 0$, where the Poisson bracket $\{\ ,\ \}$ is taken with respect to the symplectic structure given by the steady solution.

Recall also that the vorticity function λ and the symplectic structure ω are defined for any field in the orbit. Moreover, all the pairs (λ,ω) taken for fields in the same orbit (i.e., in the same class of isovortical fields) are conjugate by volume preserving diffeomorphisms. Thus dynamical properties of the Hamiltonian flow of λ (with respect to ω) depend only on the orbit but not on a field. One may expect that for a broad class of orbits the vorticity function λ does not admit extra symmetries, i.e., the Hamiltonian vector field with the Hamiltonian λ does not have first integrals which are independent of λ. For example, it is so when the manifold is four-dimensional. We emphasize that to have no extra symmetries is a property of the entire orbit, i.e., it holds either for all the fields isovortical to a given one or for none of them. In particular, the Bernoulli function h and the vorticity function λ must be functionally dependent for a steady solution on an orbit without extra symmetries.

An immediate consequence of this analysis is that, as shown in Section 5, λ inherits many topological properties of h, provided that λ does not admit extra symmetries. In particular, we prove in Section 6 that the orbit can not contain a smooth steady solution when λ has a "wrong" topology. Since on a two-dimensional manifold no function admits extra symmetries, we obtain a broad class of orbits with no smooth steady solutions (see Section 6). This seems to be a new result in the classical area of two-dimensional hydrodynamics.

Another result which goes along the same line is that under certain hypotheses the existence of a steady solution implies that the manifold admits a complex structure making it into a domain of holomorphy in a Stein manifold. The vorticity function becomes then a plurisubharmonic function on this domain. This and some other results of Sections 5 and 6 have been announced in [GK].

We conjecture that the hydrodynamics of an orbit without extra symmetries is analogous to the two-dimensional hydrodynamics: similarly to the two-dimensional case, the dynamics of a steady flow on such an orbit is determined by the dynamics of the Hamiltonian flow of λ. One may expect that many classical results, which fail in general in higher dimensions, can be extended to an orbit without extra symmetries.

We also believe that the approach developed here, as well as our description of the topology of steady flows, may have some applications to the study of the linear and exponential instability in ideal hydrodynamics inspired by [A4].

To avoid a possible confusion, we note that the term "generic" is used here with two different meanings. Saying that an orbit or a flow is generic we mean one from a large enough set in the phase space. However, this is not quite the case for structural theorems on four-dimensional steady flows (Section 4) which are said to be proved for a "generic" pair of the Bernoulli and vorticity functions. Here "generic" (traditionally) means that this pair belongs to a large enough set among all the pairs of functions, not necessarily related to the phase space. In fact, the pair under consideration has to arise from an orbit with extra symmetries which presumably is not generic in the phase space.

In the next section we recall the mathematical model of ideal hydrodynamics and main concepts related to the fluid motion equations (see, e.g., [A1] or [AK]).

Throughout the paper M denotes a compact manifold (possibly with boundary) such that $H^1(M,\mathbb{R}) = 0$.

2. Mathematical model of ideal hydrodynamics

In this section, we briefly discuss the mathematical model describing the motion of an ideal fluid.

The starting object in inviscid incompressible hydrodynamics is a manifold M equipped with a volume form μ. The phase space is the dual space $\mathcal{G}^*$ to the Lie algebra $\mathcal{G}$ of all divergence-free vector fields on M. (If $\partial M \neq \emptyset$, the vector fields must be tangent to the boundary of M.) The dual space $\mathcal{G}^*$

carries a natural linear Poisson structure; the Euler equation of an ideal fluid is a Hamilton equation on $\mathcal{G}^*$ with respect to the Poisson structure [A3].

The Hamiltonian H comes from extra data, namely, from a Riemannian metric g on the manifold M. The construction goes as follows. The metric g gives rise to an identification of $\mathcal{G}$ and $\mathcal{G}^*$ by means of the nondegenerate quadratic form $\langle v,v\rangle = \int_M g(v,v)\mu$ on $\mathcal{G}$. By definition, H is the dual quadratic form on $\mathcal{G}^*$ (see [A3]).

The dual space $\mathcal{G}^*$ can be viewed as the vector space of exact 2-forms on M (see [MaW]). The pairing of a 2-form ω with a divergence-free vector field v is given by the formula $\langle\omega, v\rangle = \int \alpha(v)\mu$, where α is a primitive 1-form for ω, i.e., $d\alpha = \omega$. Using the assumptions that v is tangent to ∂M and $H^1(M,\mathbb{R}) = 0$, it is not hard to show that the pairing is well defined.

After this identification, the Euler equation of ideal hydrodynamics takes the following form:

$$\dot{\omega} = L_v\omega$$

(also called the Helmholtz equation), where the divergence-free vector field v (i.e. v satisfying the equation $L_v\mu = 0$) is uniquely defined by the condition $d\alpha = \omega$, where $\alpha(\cdot) = g(v,\cdot)$.

Recall that if M is odd-dimensional, then the vorticity vector field ξ of ω is given by the equation $i_\xi\mu = \omega^n$, where $\dim M = 2n$. Similarly, if M is even-dimensional, then the function $\lambda = \omega^n/\mu$, where $\dim M = 2n$, is called the vorticity function of ω (see, e.g., [AK]). If in addition ω is symplectic then instead of λ one may consider the Hamiltonian vector field ξ with the Hamiltonian λ as a possible replacement for the vorticity vector field. It is clear that the vorticity vector field or the vorticity function, taken up to a μ-preserving diffeomorphism, are invariants of the orbit of ω in $\mathcal{G}^*$.

Remark. The Euler–Helmholtz equation means that the 2-form ω is frozen into the fluid (or, in other words, it is transported by the flow of v). This implies the following "duality" of incompressible flows on odd- and even-dimensional manifolds. Namely, the vorticity vector field ξ on an odd-dimensional manifold and the covector field $d\lambda$ (i.e. the function λ) on an even-dimensional one are both frozen into the ideal fluid.

Our main goal is to describe the topology of smooth steady (or stationary) flows. By definition, such a flow is an independent of time smooth solution of the stationary Euler–Helmholtz equation: $L_v\omega = 0$. The solutions are just critical points of H on the coadjoint orbits in $\mathcal{G}^*$ [A3]. In other words, stationary flows are exactly the extremals of the energy functional among all isovortical divergence-free vector fields.

For the three-dimensional case, an almost complete description of analytic stationary flows is given by the following theorem.

Theorem 2.1 (Arnold, [A1]). *Assume that the region D is bounded by a compact analytic surface and that the field of velocities is analytic and not everywhere collinear with its curl. Then the region of the flow can be partitioned by an analytic submanifold into a finite number of cells, in each of which the flow is constructed in a standard way. Namely, the cells are of two types: those fibered into tori invariant under the flow and those fibered into surfaces invariant under the flow, diffeomorphic to the annulus $\mathbb{R} \times S^1$. On each of these tori the flow lines are either all closed or all dense, and on each annulus all flow lines are closed.*

Remark. In this theorem, it is essential that the fields of velocity v and vorticity ξ are not collinear. Since $[\xi, v] = 0$, this means that the field ξ admits an "extra symmetry" and, therefore, so does every element of the coadjoint orbit.

In the next section, we state an analog of this theorem for a four-dimensional manifold.

3. The structure of four-dimensional steady flows

The main result of this section shows that steady flows of a four-dimensional fluid are very similar to integrable Hamiltonian systems with two degrees of freedom.

Recall that the equation of a stationary flow has the form $L_v\omega = 0$ or, equivalently, $d(i_v\omega) = 0$ by the homotopy formula $L_v = di_v + i_v d$. This shows that the form $i_v\omega$ is closed. Since $H^1(M, \mathbb{R}) = 0$, there exists a function h, called the Bernoulli function, such that $i_v\omega = dh$. As a consequence, the velocity field v is tangent to the levels of h, i.e., $L_v h = 0$. (In the three-dimensional case, this observation alone implies the existence of tori and annuli in Arnold's theorem [A1].)

Further (except for the next section) we will mainly work with an even-dimensional manifold M^{2n}. In this case, besides h there is one more invariant function on M: $\lambda(x) = \omega^n/\mu$, called the vorticity function. The function λ is invariant since $L_v\omega = 0$ and $L_v\mu = 0$. This means that λ and h are first integrals of the flow of v on M.

Let $\pi = (h, \lambda) : M \to \mathbb{R}^2$ and Γ be the set formed by all $x \in M$ such that either $\lambda(x) = 0$ or $\pi(x)$ is a critical value of π. In other words, Γ is the union of the zero level Λ of λ and the preimage of the set of critical values of π.

Theorem 3.1. *Let M^4 be a closed orientable four-dimensional manifold. The open set $U = M \setminus \Gamma$ is invariant under the flow of v. Every connected component of U is fibered into two-dimensional tori invariant under the flow. On each of these tori the flow lines are either all closed or all dense.*

Proof. The form ω is symplectic on the complement of the set $\Lambda = \{\lambda = 0\}$. Let ξ be the Hamiltonian vector field on $M \setminus \Lambda$ with the Hamiltonian λ. Observe that the Poisson bracket of h and λ is identically zero on $M \setminus \Lambda$: $\{h,\lambda\} = 0$ since $L_v\lambda = 0$ and thus $[v,\xi] = 0$. Therefore, the flows of v and ξ together give rise to an $\mathbb{R}^2$-action on $M \setminus \Lambda$ and π is, in fact, its momentum mapping. Since π is invariant with respect to the action and U is the union of π-levels, U is also invariant with respect to the $\mathbb{R}^2$-action and, in particular, with respect to the flow of v.

Recall (see, e.g., [A2]) that the orbits of the action coincide with the connected components of π-levels. In particular, every orbit lies entirely either in U or in Γ. By definition, the projection $\pi|_U : U \to \pi(U)$ is a proper submersion. Hence each orbit in U is a smooth closed surface and so it is either a torus or a Klein bottle. Furthermore, this surface is cooriented by $dh \wedge d\lambda$. As a result, we see that the surface is orientable, i.e., a torus. Therefore, π fibers every connected component of U into tori.

On each orbit, the flow of ξ acts transitively on integral curves of v and thus the latter are either all closed or all dense in the orbit. □

Remark. In addition to Theorem 3.1, a little can be said about the behavior of v outside U. For example, let Z be the set of $x \in M$ such that $\pi(x)$ is a critical value of π, i.e., $\Gamma = \Lambda \cup Z$. Then the set $\Lambda \setminus Z$ is the union of tori and v is tangent to them. To see this, observe that on $\Lambda \setminus Z$ the field v is tangent to the π-levels, i.e., to smooth orientable surfaces. Since $i_v\omega = dh \neq 0$ and thus $v \neq 0$, each of π-levels is a torus.

Note that for a "generic" pair of h and λ the set U is open and dense in M. Thus Theorem 3.1 gives an almost complete description of the flow of v.

To keep the tradition and to cover certain examples we now turn to the real-analytic version of the theorem.

Recall that a subset of a real-analytic manifold is called semi-analytic if locally it may be defined by a finite number of real-analytic equations and inequalities.

Theorem 3.2. *Let M be as in Theorem 3.1. Assume in addition that all the data (i.e., M, μ and the metric), as well as ω, are real-analytic, and $dh \wedge d\lambda \neq 0$ somewhere on M. Then Γ is a semi-analytic subset nowhere dense in M, and $U = M \setminus \Gamma$ has a finite number of connected components. Every connected component is fibered into two-dimensional tori invariant under the flow. On each of these tori the flow lines are either all closed or all dense.*

This result is a consequence of Theorem 3.1 and well-known properties of real-analytic manifolds summarized in the following lemma.

Lemma 3.3. *Let M and N be compact connected real-analytic manifolds (possibly with boundary) and $f : M \to N$ a real-analytic map. Then*

(i) Any semi-analytic subset X of M divides M into a finite number of connected components.

(ii) The image $f(X)$ is a semi-analytic subset of N provided that $\dim N \leq 2$.

(iii) Assume that the rank of f is equal to $\dim N$ *at at least one point of M and Y is a nowhere dense semi-analytic subset of N, then the preimage $f^{-1}(Y)$ is semi-analytic and nowhere dense in M.*

Proof of Lemma 3.3. Assertions (i) and (ii) are classical results due to Lojasievicz [Lo]. To prove (iii), consider the set K of critical points of f. The set $f^{-1}(Y) \cap (M \setminus K)$ is nowhere dense because the restriction of f to $M \setminus K$ is a submersion. Since $\operatorname{rk} f = \dim N$ somewhere on M, the set K is, in turn, nowhere dense in M. Thus $f^{-1}(Y)$ is nowhere dense for it is the union of two sets each of which is nowhere dense. It is clear by definition that $f^{-1}(Y)$ is semi-analytic. □

Proof of Theorem 3.2. Note that under the hypothesis of the theorem the map π is real analytic, and we can take $f = \pi$. Let, as above, K be the critical set of π, then $\pi(K)$ is semi-analytic by (ii) and nowhere dense by the Sard lemma. The same is true for the union Y of $\pi(K)$ and the line $\lambda = 0$. Therefore by (iii), $\Gamma = \pi^{-1}(Y)$ is semi-analytic and nowhere dense in M. Applying (i) to $X = \Gamma$, we see that U is dense in M and U has a finite number of connected components.

To complete the proof, it suffices to apply Theorem 3.1. □

Let now M be an orientable compact real-analytic four-manifold with, maybe, nonempty boundary. Assume all other hypotheses of Theorem 3.2 to hold.

Theorem 3.4. *There exists a semi-analytic set Γ nowhere dense in M such that $U = M \setminus \Gamma$ is invariant with respect to the flow of v; the set U has a finite number of connected components, and on every component of U the flow is organized in a standard way. Namely, the components are of two types: those fibered into tori invariant under the flow and those fibered into annuli* $[0, 1] \times S^1$, *again invariant under the flow. On each of the tori the flow lines are either all closed or all dense; on every annulus all the flow lines are closed.*

Proof. Here we just briefly outline the proof for it follows the same line as the proof of Theorem 3.3. Let again K be the critical set of π and C the critical set of $\pi|_{\partial M}$. As above, the union Y of the sets $\pi(K)$, $\pi(C)$ and the line $\lambda = 0$ is a semi-analytic set nowhere dense in $\mathbb{R}^2$. Therefore, $\Gamma = \pi^{-1}(Y)$ is nowhere dense, semi-analytic and invariant with respect to the flow.

Although now we may not have an $\mathbb{R}^2$-action since M is a manifold with boundary, we do have a local $\mathbb{R}^2$-action on $M \setminus \partial M$. Furthermore, the maps $\pi|_U$ and $\pi|_{\partial M \cap U}$ are still proper submersions onto their images. Consider the orbit A_x of the local action through $x \in U$. The same argument as in the proof of Theorem 3.1 shows that A_x is either a torus or an annulus. In the former case, the integral curves of v are all closed or all dense on A_x. Observe that $L = A_x \cap \partial M$ is invariant under the flow of ξ, and thus A_x is an annulus if and only if it meets ∂M. By the definition of U, the field ξ is transversal to ∂M along L. This implies that L is the union of two closed integral curves of v. Since we have a locally well-defined $\mathbb{R}^2$-action, all the v-flow lines on A_x must be closed.

Let U_0 be a connected component of U. It is left to show that the orbits A_x, $x \in U_0$, are either all tori or all annuli. To see this notice that for all $x \in U$ the levels $F_x = \pi^{-1}(\pi(x))$ are transversal to ∂M. Thus the connected components A_x of F_x are diffeomorphic to each other for all $x \in U_0$.

A more formal argument goes as follows. Pick two points x and x' in U_0. Let x_t, $t \in [0, 1]$, be a smooth curves joining x and x' in U_0. Denote by P the subset of $[0, 1]$ formed by all t such that A_{x_t} is a torus. Clearly, A_{x_t} is an annulus if $t \in [0, 1] \setminus P$. Since the orbits are transversal to ∂M, the sets P and $[0, 1] \setminus P$ are both open. Thus one of these sets is empty and the other coincides with $[0, 1]$. This means that A_x and $A_{x'}$ are either both tori or both annuli. Now it is easy to show that U_0 is indeed fibered into tori or annuli. The theorem is proved. □

Putting the results of this section in a few words, one may say that the vector field v is a completely integrable Hamiltonian system on U: its independent first integrals are the Hamiltonian h and the vorticity function λ. Then the real analyticity assumption implies that the complement $\Gamma = M \setminus U$ is not too bad a set: it is semi-analytic and nowhere dense. In particular, U has a finite number of connected components.

Remark. For an arbitrary even dimensional manifold M^{2n} we can assert that M is a union of $(2n-2)$- (or lower) dimensional submanifolds, such that the steady vector field v is tangent to them. These submanifolds are obtained as intersections of the levels h = const. and λ = const. and have the zero Euler characteristic.

Remark. For an arbitrary odd-dimensional M^{2n+1}, instead of the function $\lambda = \omega^n/\mu$ (and the covector field $d\lambda$) we define the vorticity vector field ξ by $i_\xi \mu = \omega^n$. The fields ξ and v commute and, thus, give rise to an $\mathbb{R}^2$-action on M^{2n+1}. So in this case, a steady flow gives rise to a foliation of dimension 2, unlike the foliation of codimension 2 in the even-dimensional case.

4. Higher-dimensional Beltrami flows

In this section, we consider an odd-dimensional compact manifold M^{2m+1} equipped with a volume form μ, and an analytic divergence-free steady flow v on M.

Definition 4.1. A subset Γ^{2n} in M^{2n+1} is said to be a rational-analytic hypersurface if Γ can be given as the zero level surface of the ratio of two real-analytic functions:

$$\Gamma = \{x \in M | \phi(x)/\psi(x) = 0\}.$$

Definition 4.2. A trajectory of the field v is called chaotic if it is not contained in any rational-analytic hypersurface in M^{2n+1}.

Example. A generic trajectory of an ergodic flow is chaotic.

Theorem 4.3. *An analytic steady flow v with at least one chaotic trajectory is proportional to its vorticity ξ, i.e., $\xi = \beta \cdot v$, where $\beta \in \mathbb{R} \setminus \{0\}$.*

Remark. Recall that in the odd-dimensional case the vorticity field is defined by the relation $i_\xi \mu = \omega^n$, where the two-form $\omega = d\alpha$ is such that $\alpha(\cdot) = (v, \cdot)$. Thus, by Theorem 4.3, the field v with a chaotic trajectory is an "eigenvector" of the operator curl : $v \mapsto \xi$, even though for $n > 1$ this operator is not linear! It is natural to call such a field v a generalized Beltrami flow (or a generalized ABC-flow). The theorem shows that higher-dimensional ABC-flows, as well as three dimensional ones, have quite a complicated structure. On the contrary, a non-Beltrami steady flow is fibered by a family of hypersurfaces invariant under the flow and, therefore, real mixing is impossible for it. Mixing occurs only if at least one chaotic trajectory exists, i.e. only for generalized Beltrami flows. The proof of the theorem closely follows the argument used by V.I. Arnold for $n = 1$ [A1].

Proof. The vorticity field ξ commutes with v. The fields ξ and v are both tangent to the "Bernoulli surfaces", i.e. to the level hypersurfaces of the analytic Bernoulli function. Thus, if the Bernoulli function h is non-constant, then trajectories of v lie on level hypersurfaces of h. (Note that similarly to the three-dimensional case, nonsingular Bernoulli surfaces have the zero Euler characteristic because the tangent field v has no singular points on them.)

Let now the function h be constant. This means that the fields ξ and v are collinear. Consider a function $\rho(x) = v^2/\xi^2$ (or $\xi^2 = \rho(x) \cdot v^2$). Due to commutativity of ξ and v, the function $\rho(x)$ is invariant under the flow of v and, therefore, v is tangent to the level surfaces of ρ. Since v has a chaotic trajectory, $\rho(x) = \text{const}$. (Note that the Bernoulli function h is analytic, and the function ρ is the ratio of two analytic functions.) Hence the functions h and ρ are both constant and the fields ξ and v are locally proportional: $\xi = \beta \cdot v$, where $\beta = \pm 1/\sqrt{\rho} = \text{const}$. (In other words, v is an "eigenvector" of the curl-operator: $\operatorname{curl} v = \beta \cdot v$.) □

Example. The Hopf vector field $(x_2, -x_1, x_4, -x_3, \dots, x_{2m+2}, -x_{2m+1})$ is an example of an eigenvector field for the curl operator on $S^{2m+1} \subset \mathbb{R}^{2m+2}$ without chaotic trajectories. The theorem above claims that the existence of such a trajectory makes the vector field be an "eigenvector" of curl. It would be very interesting to find a nontrivial example of a higher-dimensional ABC-flow and compare its ergodic properties with those in the three-dimensional case [H].

5. Topology of the vorticity function

Let a two-form ω be an even-dimensional stationary solution on a coadjoint orbit $\mathcal{O} \subset \mathcal{G}^*$. In other words, $\mathcal{O}$ is the set of all flows isovortical with ω. In this section, we study the topology of the vorticity function $\lambda = \omega^n/\mu$ of the steady flow ω. We describe some special features of such λ under a mild condition that the pair (λ, ω) does not admit "too many symmetries". On the other hand, it is clear that topological invariants of λ, such as a number of its critical points, their indexes, etc., depend only on the orbit. This simple observation will enable us to find orbits with no stationary solutions at all (see Section 6).

Definition 5.1. A function f on a compact symplectic manifold (W, ω) does not admit extra symmetries if an arbitrary function g such that $\{f, g\} = 0$ is constant on connected components of levels of f (i.e., the differential dg is proportional to df everywhere on W).

Remark. On a two-dimensional symplectic manifold no functions admit extra symmetries. We conjecture a generic function on a compact symplectic manifold of any dimension does not admit extra symmetries. Note however that to the best of our knowledge in dimensions greater than four, even the existence of such a function has not been proved yet. (The question turns out to be closely related to some subtle problems of Hamiltonian dynamics. Furthermore, our conjecture can be regarded as a Hamiltonian version of the problem of generic non-integrability due to Arnold [A5].)

On the other hand, the problem can be easily dealt with when $\dim M = 4$. First, note that then the notions of complete integrability and integrability coincide. Hence it follows from results of Markus and Meyer [MMe] that a generic function on M admits no extra symmetries provided that $\partial M = \emptyset$ and

$\dim M = 4$. Using a method similar to [MMe] one may show that the same is true for functions constant on connected components of ∂M.

To define a coadjoint orbit with no extra symmetries, note that the form $\omega \in \mathcal{G}^*$ is symplectic precisely on the complement to the zero level of $\lambda = \omega^n/\mu$.

Definition 5.2. A coadjoint orbit $\mathcal{O} \subset \mathcal{G}^*$ does not admit extra symmetries if for any (or, equivalently, for some) $\omega \in \mathcal{O}$ the vorticity function λ does not admit extra symmetries on $\lambda^{-1}([a,b])$ for any two of its regular values $0 < a < b$ (or $a < b < 0$).

Our definitions are consistent: a function f on a compact symplectic manifold does not admit extra symmetries if and only if its restriction to the preimage of any segment with regular endpoints does not. If $\dim M = 4$, then orbits without extra symmetries do exist (see the remark above).

Definition 5.3. An orbit $\mathcal{O} \subset \mathcal{G}^*$ has the Morse type if for any (or, equivalently, for some) $\omega \in \mathcal{O}$ the function λ is a Morse function constant on every connected component of ∂M. The orbit is called positive (nonnegative) if $\lambda(x)$ is so for all $x \in M$.

Theorem 5.4. *Let* $\dim M \geq 4$ *and* $\mathcal{O}$ *be a Morse type orbit without extra symmetries. Assume that* $\mathcal{O}$ *contains a steady solution. Then for every* $\omega \in \mathcal{O}$ *all the critical points of the vorticity function* λ *have indexes either less than* $n+1$ *or greater than* $n-1$ *on every connected component of* $M \backslash \{\lambda = 0\}$.

Example. If $\mathcal{O}$ is as above and $\lambda > 0$ on $M \backslash \partial M$, then λ cannot have both a local maximum (index $2n$) and a local minimum (index 0) on $M \backslash \partial M$.

Proof. For the sake of simplicity we assume that $\mathcal{O}$ is a positive orbit, i.e., $\lambda > 0$ on M. Only a minor modification is required to prove the general case. Let $\omega \in \mathcal{O}$ be a stationary solution ($L_v \omega = 0$) and h a function such that $dh = i_v \omega$.

Since $\lambda = \omega^n/\mu$ does not admit extra symmetries and $\{h, \lambda\} = 0$, the function h must be constant on connected components of levels of λ.

Lemma 5.5. *The functions* λ *and* h *have the same critical points. In particular, the critical points of* h *are isolated.*

Proof. Since λ does not admit extra symmetires, $d\lambda(x) = 0$ implies that $dh(x) = 0$. The rest of the critical set of h may only be the union of some connected components of λ-levels. Let α be the differential 1-form $g(v, \cdot)$, where g is the Riemannian metric on M. Then $d\alpha = \omega$ by the definition of v, and $\alpha(v) = g(v,v) \geq 0$.

Consider the vector field w on M defined by the formula $i_w\omega = \alpha$. We have

$$L_w\omega = \omega \tag{1}$$

and

$$L_w h = \alpha(v) \geq 0, \tag{2}$$

where

$$L_w h = 0 \Leftrightarrow \alpha(v) = 0 \Leftrightarrow \alpha = 0. \tag{3}$$

If the critical set of h contains a connected component C of a λ-level, then $L_{w_x}h = 0$ for all $x \in C$ and, as a consequence of (3), $\alpha|_C = 0$. Hence, $\omega|_C = d\alpha|_C = 0$. This is impossible, because C is a hypersurface in the symplectic manifold (M, ω) and $2n = \dim M_0 \geq 4$. The lemma is proved. □

Observe now that as follows from (1), all the zeroes of the vector field w are nondegenerate. Therefore, the field w has smooth stable and unstable manifolds in a neighborhood of its every zero. The dimension of the stable manifold is greater than n because, by (1), the restriction of ω on the unstable manifold of w must be zero.

Now we are ready to finish the proof of the theorem. The field w is gradient-like for the function h (due to (2)). Therefore, w is either gradient- or antigradient-like for λ because the λ- and h-levels coincide in a neighborhood of every critical point and λ is a Morse function. □

Remark. One may prove that all the critical points of h are nondegenerate except, maybe, for its maxima and minima.

Theorem 5.6. *Let M be diffeomorphic to the disc D^2. If a Morse type orbit $\mathcal{O} \subset \mathcal{G}^*$ contains a stationary solution, then for any $\omega \in \mathcal{O}$ the vorticity function λ can not simultaneously have a local maximum and a local minimum in M, provided that $\lambda > 0$ on $M \setminus \partial M$.*

Remark. Since $\dim M = 2$, the orbit $\mathcal{O}$ automatically does not admit extra symmetries.

The proof below is a formalization of the following argument which is evident from a physical viewpoint. Minima and maxima of the vorticity function correspond to rotations of the fluid in the opposite directions. On the other hand, the positivity of λ prescribes *a priori* a counterclockwise drift.

Proof. First, recall that h must not have maxima. Indeed, in a neighborhood of a maximum the gradient-like (for h) field w would shrink the area that contradicts Eq. (1): $L_w\omega = \omega$. Let C be the critical set of h. Observe that since h is constant on ∂M, the set C either contains the boundary ∂M or does not meet it. We claim that $M\backslash C$ is connected. To prove this, assume the contrary. Then there exists an open set $U \subset M\backslash C$ such that $\partial U \subset C$. The set U is invariant under the flow of w, because dh (and thus w) vanishes on C. On the other hand, as above, the existence of such a set U contradicts Eq. (1).

Observe now that, since w is gradient-like for h and every local minimum of λ is a local minimum of h, the field w is gradient-like for λ in a neighborhood of a local minimum of λ, whereas near a local maximum of λ, the field w must be antigradient-like for λ. Switching from being gradient-like to antigradient-like (and vice versa) may occur only on C. But C does not divide M. Hence w is either gradient-like or antigradient-like on the entire M. The theorem follows. □

Besides the restriction on indexes of critical points of λ, there are more subtle properties of the pair (ω, λ) which follow from the existence of a steady solution. Let $\mathcal{O}$ be a coadjoint orbit in $\mathcal{G}$ which does not admit extra symmetries and contains at least one exact steady solution. For the sake of simplicity, we also assume that $\mathcal{O}$ is positive, i.e., $\lambda > 0$ for every $\omega \in \mathcal{O}$. In other words, every $\omega \in \mathcal{O}$ is symplectic.

Recall that a hypersurface Γ in a symplectic manifold (W, σ) has contact type if there exists a 1-form θ on Γ such that $d\theta = \sigma|_\Gamma$ and $\theta \wedge (d\theta)^{n-1} \neq 0$ anywhere on Γ.

Remark. An example of a compact closed hypersurface in $\mathbb{R}^{2n}$ (and thus in any symplectic manifold) that does not have contact type has been found by A. Weinstein [W].

Proposition 5.7. *For every $\omega \in \mathcal{O}$, connected components of regular levels of λ have contact type.*

Proof. It is sufficient to prove the proposition for a steady solution $\omega \in \mathcal{O}$. Recall that for such an ω, connected components of λ-levels coincide with connected components of h-levels. Thus we need to show that every regular connected component Γ of an h-level has contact type. By definition, we have $d\alpha|_\Gamma = \omega|_\Gamma$, where $\alpha = g(v, \cdot)$. To show that $\alpha \wedge (d\alpha)^{n-1} \neq 0$ on Γ, observe that $\alpha \wedge (d\alpha)^{n-1} = i_w \omega^n / n$, where w is given by the equation $i_w \omega = \alpha$. By (3), w is transversal to regular levels of h and, therefore, $i_w \omega^n|_\Gamma$ is nowhere zero. The proposition follows. □

Remark. The proposition implies that under certain assumptions every connected component Γ of a regular level of λ contains at least one closed trajectory of the vector field ξ. For example, as follows from a result of C. Viterbo [V], this is correct when M is a domain in $\mathbb{R}^{2n}$ and $\mathcal{O}$ is the orbit of the standard symplectic structure provided, of course, that other the hypotheses of Proposition 5.7 hold.

6. The nonexistence of steady flows and other applications of main theorems

Applying Theorems 5.4 and 5.6, one can easily find a coadjoint orbit which does not contain a steady solution.

The case of a two-dimensional M is particularly simple. Consider a disk $M = D^2 \subset \mathbb{R}^2(x, y)$ with $\mu = dx \wedge dy$ and $\omega = \lambda \cdot \mu$, where λ is a positive Morse function on D such that $\lambda|_{\partial D} = \text{const}$. Assume also that λ has both a local maximum and a local minimum in $\operatorname{int} D$ (see, e.g., fig. 1).

Corollary 6.1 (of Theorem 5.6). *The coadjoint orbit through ω does not contain a steady solution.* □

A higher-dimensional version of this corollary follows from Theorem 5.4. Let $\mathcal{O}$ be a Morse type orbit which is positive (i.e., $\lambda > 0$) and without extra symmetries.

Corollary 6.2. *Assume that for some $\omega \in \mathcal{O}$ the vorticity function λ has a critical point of index $k_1 < n$ and a critical point of index $k_2 > n$, where $2n = \dim M$, then $\mathcal{O}$ contains no steady solutions.* □

Corollary 6.3. *Assume that $H^{k_1}(M, \mathbb{R}) \neq 0$ and $H^{k_2}(M, \mathbb{R}) \neq 0$ for some $k_1 < n$ and $k_2 > n$, then $\mathcal{O}$ contains no steady solutions.*

Proof. Apply the Morse inequalities. □

Remark. Here, as everywhere in this paper, the steady solution is assumed to be smooth. Note that a "generalized steady solution" with a discontinuous vorticity function may still exist and be of certain interest for applications.

It turns out that Theorems 5.4 and 5.6 are almost sharp as long as we are

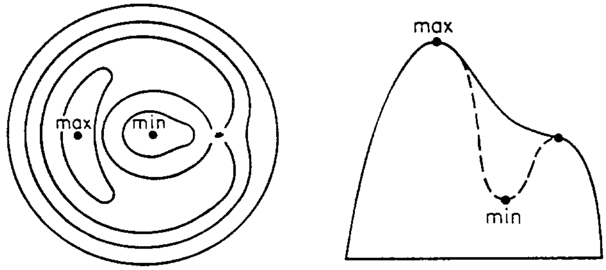

Fig. 1. Level surfaces and a profile of the vorticity function having no smooth steady flow.

not concerned about the metric. Namely, there is no general restriction on the topology of the vorticity function except that given by the theorems. Let us first consider an important, yet almost trivial, construction of steady solutions and then state the sharpness result.

Example. Let W be a Stein manifold such that $H^1(W,\mathbb{R}) = 0$ and λ a positive smooth strictly plurisubharmonic function. Consider the manifold $M = \{\lambda \leq c\}$, where c is a regular value of λ. The 2-form $\omega = -2\,\mathrm{Im}\,\partial\bar{\partial}\lambda$ is a symplectic form on W and, therefore, on M. Let us equip W with the volume form $\mu = \omega^n/\lambda$ and the metric $g = \omega(\cdot, J\cdot)$, where J is the complex structure on W. We claim that ω is a steady solution.

To prove this observe that, in fact, $\omega = d(Jd\lambda)$. Thus if we set $\alpha = Jd\lambda$ and define v by the equality $\alpha = g(v, \cdot)$, then $d\alpha = \omega$ and v is the Hamiltonian vector field with the Hamiltonian λ. By the definition of μ, we have $L_v\mu = 0$, i.e., v is divergence-free. Therefore, v is the vector field corresponding to ω by means of the identification of $\mathcal{G}$ and $\mathcal{G}^*$ given by the energy quadratic form. Since v is Hamiltonian, $L_v\omega = 0$ and ω is a steady solution.

Note that here we have $v = \xi$ and $\lambda = h$, yet the orbit through ω may admit extra symmetries.

This construction of a steady solution is particularly interesting when $\dim_{\mathbb{C}} W = 1$. Consider, for example, a positive smooth subharmonic function λ on $\mathbb{C}$, constant on the unit circle. Then our argument shows that on the unit disc D there exists a metric g and an area form μ such that λ is the vorticity function of a steady solution. In particular, the vorticity function may have saddle critical points, at least for some metrics and volume forms. □

Theorem 6.4. *Let M be a compact manifold with boundary,* $\dim M = 2n \geq 6$ *and λ a smooth positive function on M such that f is constant on connected components of ∂M and all the critical points of λ have indexes no greater than n. Assume in addition that M admits an almost complex structure. Then there exists a metric and a volume form on M such that λ is the vorticity function of a steady solution.*

Proof. As shown by Ya. Eliashberg [E], the manifold M admits a complex structure which makes λ into a plurisubharmonic function. To finish the proof it remains to apply the argument from the preceding example. □

As the following result indicates, there is apparently a deep connection between steady solutions and complex structures. Let ω be an exact steady solution on M^{2n} and λ its vorticity function. Assume also that $2n \geq 6$, the orbit $\mathcal{O}$ through ω admits no extra symmetries and that $\lambda > 0$.

Theorem 6.5. *There exists a complex structure J on M which makes λ into a* *plurisubharmonic function. Equipped with J, the manifold M is biholomorphically equivalent to a domain of holomorphy in a Stein manifold.*

Proof. Since $\lambda > 0$, the manifold M carries a symplectic structure and thus an almost-complex structure. The theorem follows from Theorems 5.4 and 6.4. □

Remark. The complex structure J can be chosen to be ω-tame (cf. [EGr]), i.e., such that ω is a $(1,1)$-form.

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