Geometric hydrodynamics via Madelung transform

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We introduce a geometric framework to study Newton's equations on infinite-dimensional configuration spaces of diffeomorphisms and smooth probability densities. It turns out that several important partial differential equations of hydrodynamical origin can be described in this framework in a natural way. In particular, the Madelung transform between the Schrödinger equation and Newton's equations is a symplectomorphism of the corresponding phase spaces. Furthermore, the Madelung transform turns out to be a Kähler map when the space of densities is equipped with the Fisher–Rao information metric. We describe several dynamical applications of these results.

hydrodynamics | infinite-dimensional geometry | quantum information | Fisher–Rao | Newton's equations

Introduction

In a seminal 1966 paper, Arnold (1) showed that the Euler equations of an inviscid incompressible fluid can be reformulated as geodesic equations on an infinite-dimensional manifold of diffeomorphisms. That paper was a foundation stone for a new branch of mathematics called geometric and topological hydrodynamics (see ref. 2) and many important partial differential equations (PDEs) of mathematical physics have been shown to fit Arnold's framework. Examples include the equations of Korteweg–de Vries, Camassa–Holm, magnetohydrodynamics, and Landau–Lifschitz.

Arnold's reformulation of the Euler equations in an elegant differential-geometric language allowed an insight into both analysis and geometry of hydrodynamic equations. For example, the sectional curvature of the group of diffeomorphisms influences fluid motions via the equations of geodesic deviation, which had applications to hydrodynamic stability (see refs. 1 and 2). Furthermore, a detailed study of the analytic properties of the associated geometry of the diffeomorphism group, begun by Ebin and Marsden (3), led to sharp local well-posedness results. It is expected that further study will shed new light on challenging problems of fluid dynamics, such as regularity and persistence of 3D flows or the problem of fluid turbulence.

In this paper we propose an extension of this approach to the case of Newton's equations as a natural next step in Arnold's program. These are second-order equations that formally can be written as

$$\nabla_{\dot{q}}\dot{q} = -\nabla U(q), \qquad [1]$$

where ∇ is the covariant derivative with respect to a Riemannian metric and U is a potential function. We develop a geometric framework for Eq. **1** on the space of diffeomorphisms of a compact manifold. Using infinite-dimensional Riemannian submersion techniques we show that these equations are closely related to Newton's equations on the space of smooth probability densities. In particular, this is the case for the equations of compressible fluids. These equations have long been known to admit a Hamiltonian formulation on the dual of a semidirect product Lie algebra (see e.g., ref. 4) while their Lagrangian Arnold-type formulation was lacking since the Lagrangian was not quadratic. In our framework, however, these equations have the following simple description.

Theorem 1. *The equations for potential solutions of a compressible fluid in a compact domain are Newton's equations on the space of smooth probability densities with a potential function given by the fluid's internal energy*.

The proposed framework reveals some unexpected connections between various results in fluid dynamics, optimal transport, information geometry, and equations of mathematical physics, such as the Schrödinger equation, the Klein–Gordon equation, the Hunter–Saxton equation, and its variants (Table 1). For instance, the classical Laplace eigenproblem can be seen as the problem of determining stationary solutions of the Fisher–Rao–Newton equation on the space of densities, which in turn describes geodesics on an infinite-dimensional ellipsoid through its formulation as a Neumann problem.

An important tool in our constructions is the Madelung transform (*Definition 3* below) which turns out to have a number of surprising properties and can be viewed as a symplectomorphism, an isometry map, a Kähler map, or a generalized Hasimoto transform, depending on the context. Our study shows that the geometric features of the Madelung transform are best understood in the setting of the Fisher–Rao (information) geometry—the canonical Riemannian geometry on the space of probability densities—rather than the L^2-Wasserstein geometry.

Theorem 2. *The Madelung transform is a Kähler morphism between the cotangent bundle of the space of smooth probability densities*, *equipped with the Sasaki–Fisher–Rao metric*, *and an open subset* (*in the Fréchet topology*) *of the complex projective space of smooth wave functions equipped with the Fubini–Study metric.*[†]

In a nutshell, the Madelung transform resembles the passage from Euclidean to polar coordinates on the infinite-dimensional

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Significance

Geometry has always played a fundamental role in theoretical physics via symmetries and conservation laws. We present a geometric framework revealing a closer link between hydrodynamics and quantum mechanics than previously recognized. Newton's equations, generalized to infinite-dimensional spaces of fluid flow maps (diffeomorphisms), are used to develop a unified setting and uncover new connections between many equations of mathematical physics. These include equations of compressible fluids, motion of particles on spheres in quadratic potentials, and the Klein–Gordon and nonlinear Schrödinger equations, as well as their relation to information geometry and optimal mass transport. This work contributes toward a better understanding of geometric structures arising in hydrodynamics and quantum mechanics.

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[†]For the relevant definitions, see Eqs. **19**, **25**, and **26**.

Table 1. Examples of Newton's equations

Wasserstein–Otto geometry	Fisher–Rao geometry
Newton's equations on Diff(*M*)	
• Classical mechanics	• μ-Camassa–Holm
• Burgers' inviscid	• Optimal information transport
• Barotropic inviscid fluid	
Newton's equations on Dens(*M*) or T^*Dens(*M*)	
• Hamilton–Jacobi	• ∞-dim Neumann problem
• Linear Schrödinger	• Klein–Gordon
• Nonlinear Schrödinger	• Two-component Hunter–Saxton
• Vortex filament equation	

space of wave functions, such that the modulus is a probability density and the phase corresponds to a fluid vector field. We show that, after projectivization, this transform relates not only equations of hydrodynamics and quantum physics, but also the corresponding symplectic structures underlying them. Surprisingly, it is also an isometry for two well-known metrics in geometry and statistics.

For conceptual clarity and brevity of the exposition we focus here on formal aspects of the infinite-dimensional geometric constructions. Proofs of the theorems and a suitable functional-analytic setting are provided in a separate paper.[‡]

Wasserstein Geometry of the Space of Densities

In this section we recall the main notions of the Wasserstein geometry on the space of diffeomorphisms Diff(*M*) and the space of smooth probability densities Dens(*M*) and introduce Newton's equations on these spaces.

Let (M, g) be a compact Riemannian manifold with volume form μ. Define an L^2 metric on Diff(*M*) by

$$\mathsf{G}_\varphi(\dot\varphi,\dot\varphi)=\int_M \underbrace{|\dot\varphi|^2}_{\mathrm{g}_\varphi(\dot\varphi,\dot\varphi)}\,\mu. \qquad [2]$$

Given a C^1 function $U:\mathrm{Diff}(M)\to\mathbb{R}$ (a potential) Newton's equations on Diff(*M*) can be formally written as in Eq. **1**. For potential functions U depending only on the density, i.e., of the form

$$U(\varphi)=\bar U(\rho) \quad \text{with} \quad \rho=\det(D\varphi^{-1}), \qquad [3]$$

for $\bar U: C^\infty(M)\to\mathbb{R}$ and $\varphi\in\mathrm{Diff}(M)$, we obtain the following.

Theorem 3. *Newton's equation on Diff(M) for the L^2 metric in Eq.* **2** *and a potential U as in Eq.* **3** *is*

$$\nabla_{\dot\varphi}\dot\varphi=-\nabla\Big(\frac{\delta\bar U}{\delta\rho}\Big)\circ\varphi. \qquad [4]$$

In reduced variables $u=\dot\varphi\circ\varphi^{-1}$ and $\rho=\det(D\varphi^{-1})$, Eq. **4** *assumes the form*

$$\begin{cases}\dot u+\nabla_u u+\nabla\dfrac{\delta\bar U}{\delta\rho}=0\\ \dot\rho+\mathrm{div}(\rho u)=0.\end{cases} \qquad [5]$$

The system in Eq. **5** admits an invariant subset of potential solutions $u=\nabla\theta$, where $\theta\in C^\infty(M)$. We next describe the geometric origin of this observation.

Riemannian Submersion to Densities. The space of smooth probability density functions on M is

$$\mathrm{Dens}(M)=\Big\{\rho\in C^\infty(M)\mid\rho>0,\ \int_M\rho\,\mu=1\Big\}.$$

Since Dens(*M*) is an open subset of codimension one in an affine subspace of $C^\infty(M)$, the tangent bundle of Dens(*M*) is trivial: $T\mathrm{Dens}(M)=\mathrm{Dens}(M)\times C_0^\infty(M)$, where $C_0^\infty(M)$ denotes smooth mean-zero functions. Likewise, the (smooth part of the) cotangent bundle $T^*\mathrm{Dens}(M)$ is $\mathrm{Dens}(M)\times C^\infty(M)/\mathbb{R}$.

Alternatively, Dens(*M*) can be viewed as the space of left cosets of the subgroup $\mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms of M with the map $\pi(\varphi)=\det(D\varphi^{-1})$ defining a natural (left coset) projection $\pi:\mathrm{Diff}(M)\to\mathrm{Dens}(M)$. To take advantage of this setup it is useful to equip the base space with a Sobolev H^{-1}-type metric which arises in optimal mass-transport problems; cf. ref. 5.

Definition 1: The Wasserstein–Otto metric on Dens(*M*) is

$$\bar{\mathsf{G}}_\rho(\dot\rho,\dot\rho)=\int_M|\nabla\theta|^2\rho\,\mu,\qquad \theta=\mathrm{div}(\rho\nabla\dot\rho), \qquad [6]$$

where $\dot\rho\in C_0^\infty(M)$ is a tangent vector to Dens(*M*) at the point ρ. The Riemannian distance of this metric is the well-known Wasserstein distance equal to the minimal L^2 cost of transporting one density to another.

Theorem 4 (cf. ref. 6). *The left coset projection π is a Riemannian submersion with respect to the L^2 metric on Diff(M) and the Wasserstein–Otto metric on Dens(M).*

An illustration is given in Fig. 1.

Theorem 5. *Newton's equations on Dens(M) for the Wasserstein–Otto metric and a potential function $\bar U$ correspond to Hamilton's equations on T^*Dens(M),*

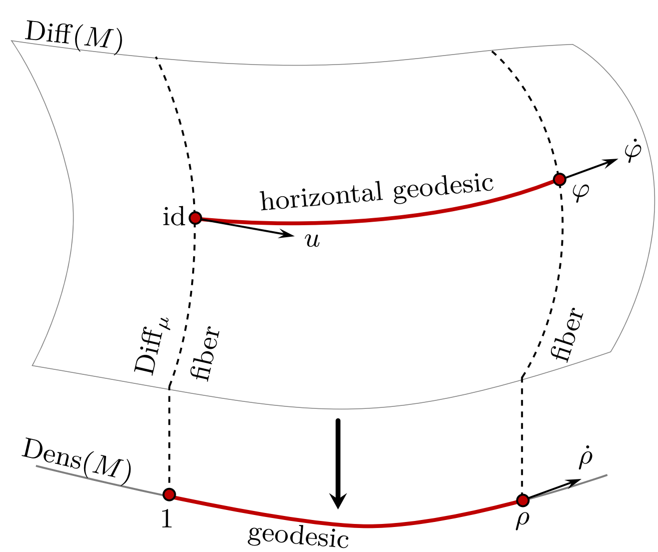

Fig. 1. Riemannian submersion from Diff(*M*) to Dens(*M*) in the L^2 and H^1 setting; compare *Theorems 4* and *9*. The projection to densities, i.e., the map π or Π, respectively, takes horizontal geodesics on Diff(*M*) to geodesics on Dens(*M*). In the L^2 metric on Diff(*M*) the former are potential solutions $u=\nabla\theta$ of Eq. **5**. For the H^1 metric, horizontal geodesics depend on the choice of F in Eq. **11**, but they all project to "great circles," which are Fisher–Rao geodesics on Dens(*M*).

[‡]Khesin B, Misiolek G, Modin K (2018) Geometry of the Madelung transform. arXiv preprint, Chalmers #3, 1-29.

$$\begin{cases} \dot{\theta} + \frac{1}{2}|\nabla\theta|^2 + \frac{\delta \bar{U}}{\delta\rho} = 0 \\ \dot{\rho} + \operatorname{div}(\rho\nabla\theta) = 0, \end{cases} \quad [7]$$

where the Hamiltonian is

$$H(\rho,\theta) = \frac{1}{2}\int_M |\nabla\theta|^2 \rho\,\mu + \bar{U}(\rho).$$

Solutions to these equations correspond to the potential solutions of Newton's equations Eq. **4** (*or Eq.* **5**) *on Diff*(M).

Example: classical mechanics. Given a C^∞ function V on M we can define an associated potential function on Dens(M):

$$\bar{U}(\rho) = \int_M V\rho\,\mu. \quad [8]$$

Proposition 6. *Newton's equations on Diff*(M) *for a potential of the form in Eq.* **8** *describe the flow of Newton's equation on* M *with potential function* V. *In particular,*

(*i*) *If* $t \mapsto \varphi(t,\cdot)$ *is a solution to Eq.* **4**, *then for each fixed* $x \in M$ *the curve* $t \mapsto \varphi(t,x)$ *satisfies Newton's equation on* M *with potential* V.
(*ii*) *The vector field* $u = \dot{\varphi}\circ\varphi$ *in Eq.* **5** *satisfies the inviscid potential Burgers equation*

$$\dot{u} + \nabla_u u + \nabla V = 0.$$

Corollary 7 (cf. ref. 7). *The momentum variable* θ *in Eq.* **7** *on* $T^*\mathrm{Dens}(M)$ *satisfies the Hamilton–Jacobi equation for the classical mechanics Hamiltonian on* T^*M,

$$H(x,p) = \frac{1}{2}\mathsf{g}_x(p^\sharp, p^\sharp) + V(x),$$

where $\sharp$ *is the isomorphism defined by the metric g on* M.

Example: barotropic fluids. Using *Theorem 3* one may present the equations of compressible fluids in M in this generalized Arnold framework with quadratic kinetic energy and without a semidirect product structure. To this end, consider a potential function

$$\bar{U}(\rho) = \int_M e(\rho)\rho\,\mu, \quad [9]$$

where e is a smooth real-valued function.

Proposition 8 (cf. refs. 8 and 9). *Newton's equations on Diff*(M) *for a potential in Eq.* **9** *describe motions of a compressible barotropic fluid with internal energy* e. *In particular, the system Eq.* **5** *takes the form of the compressible Euler equations*

$$\begin{cases} \dot{u} + \nabla_u u + \frac{1}{\rho}\nabla P(\rho) = 0 \\ \dot{\rho} + \operatorname{div}(\rho u) = 0, \end{cases} \quad [10]$$

with the pressure function $P(\rho) = e'(\rho)\rho^2$.

Remark 1: If M is the 2-sphere and $e(\rho) = \rho/2$, we recover the shallow water equations describing surface motion of an ideal fluid when the wavelength is large compared with the depth (as in the case of tidal waves). In this interpretation u is the surface (horizontal) velocity and ρ is the height of the water. The above setting can be readily used to prove a local existence result for a barotropic compressible fluid in the setting of tame Fréchet spaces, e.g., as in ref. 10.

Fisher–Rao Geometry of the Space of Densities

In this section we introduce a different Riemannian structure on Diff(M). It is given by a Sobolev H^1 inner product on vector fields and induces on Dens(M) an infinite-dimensional analogue of the Fisher–Rao information metric. The setting resembles that between the L^2 metric Eq. **2** and Wasserstein–Otto metric Eq. **6** in the previous section but with some notable differences.

As before, let (M,g) be a compact Riemannian manifold with volume form μ. Define an H^1 metric on Diff(M) by

$$\mathsf{G}_\varphi(\dot{\varphi},\dot{\varphi}) = \int_M \mathsf{g}(-\Delta u, u)\mu + F(u,u), \quad \dot{\varphi} = u\circ\varphi, \quad [11]$$

where Δ is the Hodge Laplacian on vector fields and $F(u,v)$ is a positive-definite quadratic form depending only on the vertical (divergence-free) components of u and v.

Remark 2: In the applications below we focus on Newton's equations on Dens(M), corresponding to horizontal geodesics on Diff(M), for which only the first term $\int \mathsf{g}(-\Delta u, v)\mu$ in Eq. **11** is relevant.[§]

Definition 2: The Fisher–Rao metric on Dens(M) is

$$\bar{\mathsf{G}}_\rho(\dot{\rho},\dot{\rho}) = \int_M \frac{\dot{\rho}^2}{\rho}\mu, \quad [12]$$

where $\dot{\rho} \in C_0^\infty(M)$ is a tangent vector to $\mathrm{Dens}(M)$ at ρ.

Consider now the (right coset) projection $\Pi : \mathrm{Diff}(M) \to \mathrm{Dens}(M)$ from diffeomorphisms to probability densities given by $\Pi(\varphi) = \det(D\varphi)$. In analogy with *Theorem 4* we have

Theorem 9. *The right coset projection* Π *is a Riemannian submersion with respect to the* H^1 *metric Eq.* **11** *on Diff*(M) *and the Fisher–Rao metric Eq.* **12** *on Dens*(M). *Furthermore, equipped with this metric Dens*(M) *is isometric to a subset of the unit sphere in a Hilbert space with the round metric.*

Remark 3: The above isometry is given by the square-root map $\rho \mapsto \sqrt{\rho}$; see ref. 11 for details.

An illustration of *Theorem 9* is again given by Fig. 1.

Observe that the Riemannian metric on Diff(M) in *Theorem 4* is right-invariant with respect to $\mathrm{Diff}_\mu(M)$ and thus automatically descends to the right quotient $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$. On the other hand, the metric in *Theorem 9* is right-invariant [under certain natural conditions on $F(u,v)$] and descends to the left quotient $\mathrm{Diff}_\mu(M)\backslash\mathrm{Diff}(M)$. Since right-invariance is retained after taking the quotient, the Fisher–Rao metric on Dens(M) is also right-invariant under Diff(M). From this perspective the Fisher–Rao metric Eq. **12** provides more structure than the Wasserstein–Otto metric Eq. **6**.

Theorem 10. *Newton's equations on Dens*(M) *for the Fisher–Rao metric Eq.* **12** *and a potential* $\bar{U}$ *have the form*

$$\ddot{\rho} - \frac{\dot{\rho}^2}{2\rho} + \rho\frac{\delta\bar{U}}{\delta\rho} = \lambda\rho, \quad [13]$$

where λ *is a Lagrange multiplier for the constraint* $\int_M \rho\,\mu = 1$. *Its solutions correspond to horizontal solutions of Newton's equations on Diff*(M) *for the* H^1 *metric Eq.* **11** *and the potential Eq.* **3**.

One can also express Eq. **13** as Hamilton's equations on $T^*\mathrm{Dens}(M)$ with a momentum variable $\theta = \dot{\rho}/\rho$.

Example: μ-Camassa–Holm equation. The one-dimensional periodic μCH (also known as μHS) equation

$$\mu(\dot{u}) - \dot{u}_{xx} - 2u_x u_{xx} - u u_{xxx} + 2\mu(u)u_x = 0, \quad [14]$$

where $\mu(u) = \int_{S^1} u\,dx$, is a nonlinear PDE which describes a director field in the presence of an external (e.g., magnetic) force.

[§] In ref. 13 the term $F(u,v)$ is chosen as $\sum_i \int_M (u\cdot e_i)\mu \int_M (v\cdot e_i)\mu$, where $\{e_i\}$ is any L^2-orthogonal basis for the space of harmonic vector fields.

It was derived in ref. 12 as an Euler–Arnold equation on the group $\mathrm{Diff}(S^1)$ of orientation-preserving circle diffeomorphisms equipped with a right-invariant H^1 Sobolev metric. It is known to be bi-Hamiltonian and to possess smooth, as well as cusped, soliton solutions.

Proposition 11 (cf. refs. 12 and 13). *The μCH equation Eq.* **14** *is Newton's equation on* $\mathrm{Diff}(S^1)$ *for the* H^1 *metric* Eq. **11** *and vanishing potential* $\bar{U} \equiv 0$. *The horizontal (mean-zero) solutions of the μCH equation describe geodesics of the Fisher–Rao metric Eq.* **12** *on* $\mathrm{Dens}(S^1)$.

Example: infinite-dimensional Neumann problem. The classical Neumann problem describes the motion of a particle on a sphere in the presence of a quadratic potential. It is a completely integrable system equivalent (up to time reparameterization) to the geodesic motion on an ellipsoid; cf., e.g., ref. 14.

For an infinite-dimensional analogue of this problem consider the unit sphere $S^\infty(M) = \{f \in C^\infty(M) \mid \int_M f^2 \mu = 1\}$ with the metric inherited from $L^2(M, \mathbb{R})$ and quadratic potential

$$V(f) = \frac{1}{2} \int_M |\nabla f|^2 \mu. \qquad [15]$$

Proposition 12. *Newton's equation for the Neumann problem on* $S^\infty(M)$ *with potential Eq.* **15** *is*

$$\ddot{f} - \Delta f = -\lambda f, \qquad \lambda = \int_M (\dot{f}^2 + f \Delta f) \mu, \qquad [16]$$

where λ is the Lagrange multiplier with constraint $\int_M f^2 \mu = 1$.

It turns out that this problem also admits a natural interpretation as a Newton's equation on Dens(M) with respect to the Fisher–Rao metric. To describe it consider Fisher's information functional

$$I(\rho) = \frac{1}{2} \int_M \frac{|\nabla \rho|^2}{\rho} \mu. \qquad [17]$$

Proposition 13. *The Neumann problem Eq.* **16** *on* $S^\infty(M)$ *corresponds (up to time scaling by* 4*) to Newton's equation Eq.* **13** *on Dens(M) with respect to the Fisher–Rao metric and Fisher's information functional Eq.* **17** *as a potential function. The map* $\rho \mapsto \sqrt{\rho} =: f$ *is a local diffeomorphism between the two representations.*

Remark 4: Stationary solutions to the Neumann problem on $S^\infty(M)$ correspond to the principal axes of the ellipsoid $-\int_M f \Delta f \mu = 1$ and have a natural interpretation as Laplace eigenfunctions on M. If M is the 4-torus equipped with the (pseudo-)Riemannian Minkowski metric, then the stationary solutions of the corresponding Minkowski–Neumann problem are solutions of the periodic Klein–Gordon equation

$$\ddot{f} - \Delta f = -m^2 f, \qquad m \in \mathbb{R}. \qquad [18]$$

This equation describes spinless scalar particles (such as the Higgs boson) and plays a role in quantum field theory. *Proposition 13* shows that it can be viewed as describing stationary potential solutions of a hydrodynamical system on $\mathrm{Diff}(\mathbb{T}^4)$.

Geometric Properties of the Madelung Transform

In 1927 Madelung (15) gave a hydrodynamical formulation of the Schrödinger equation. Using the framework developed in this paper one can exhibit a number of surprising geometric properties of an important transformation that he introduced.

Definition 3: Let ρ and θ be real-valued functions on M with $\rho > 0$. The Madelung transform is the mapping $\Phi : (\rho, \theta) \mapsto \psi$ defined by

$$\Phi(\rho, \theta) = \sqrt{\rho e^{i\theta}}. \qquad [19]$$

Observe that Φ is a complex extension of the square-root map described in *Theorem 9* (compare *Remark 3*).

Madelung Transform as a Symplectomorphism. Denote by $PC^\infty(M, \mathbb{C})$ the complex projective space of smooth complex-valued functions on M. Its elements are cosets $[\psi]$ of the complex L^2 sphere of smooth functions, where $\psi' \in [\psi]$ if and only if $\psi' = e^{i\alpha} \psi$ for some $\alpha \in \mathbb{R}$. The subspace $PC^\infty(M, \mathbb{C} \setminus \{0\})$ is a submanifold of $PC^\infty(M, \mathbb{C})$.

Theorem 14. *The Madelung transform Eq.* **19** *induces a map*

$$\Phi : T^*\mathrm{Dens}(M) \to PC^\infty(M, \mathbb{C} \backslash \{0\}) \qquad [20]$$

which is a symplectomorphism (in the Fréchet topology) with respect to the canonical symplectic structure of $T^*\mathrm{Dens}(M)$ *and the complex projective structure of* $PC^\infty(M, \mathbb{C})$.

The Madelung transform is already known to be a symplectic submersion from $T^*\mathrm{Dens}(M)$ to the unit sphere of nonvanishing wave functions (see ref. 16). The stronger (symplectomorphism) property in *Theorem 14* is obtained with projectivization.

Example: linear and nonlinear Schrödinger equations. Consider a family of Schrödinger (or Gross–Pitaevsky) equations (with Planck's constant $\hbar = 1$ and mass $m = 1/2$) of the form

$$i\dot{\psi} = -\Delta \psi + V \psi + f(|\psi|^2) \psi, \qquad [21]$$

where ψ is a wave function, $V : M \to \mathbb{R}$ is a potential, and $f : \mathbb{R} \to \mathbb{R}$. Setting $f \equiv 0$ we obtain the linear Schrödinger equation with potential V. Setting $V \equiv 0$ we obtain a family of nonlinear Schrödinger (NLS) equations; typical choices are $f(a) = \kappa a$ and $f(a) = \frac{1}{2}(a-1)^2$.

Eq. **21** is a Hamiltonian equation with respect to the symplectic structure induced by the complex structure of $L^2(M, \mathbb{C})$. Indeed, minus four times the imaginary part of the Hermitian inner product defines a symplectic structure. The Hamiltonian associated with Eq. **21** is

$$H(\psi) = 2 \| \nabla \psi \|_{L^2}^2 + 2 \int_M \left(V |\psi|^2 + F(|\psi|^2) \right) \mu,$$

where $F' = f$.

Observe that the L^2 norm of a wave function satisfying Eq. **21** is conserved in time. Furthermore, the equation is equivariant with respect to a constant change of phase $\psi \mapsto e^{i\alpha} \psi$ and therefore descends to the projective space $PC^\infty(M, \mathbb{C})$. This was first suggested by Kibble (17).

Proposition 15 (cf. refs. 15 and 16). *The Madelung transform Φ maps the family of Schrödinger equations Eq.* **21** *to a family of Newton's equations Eq.* **7** *on Dens(M) equipped with the Wasserstein–Otto metric Eq.* **6** *and with potentials*

$$\bar{U}(\rho) = I(\rho) + 2 \int_M (V \rho + F(\rho)) \mu, \qquad [22]$$

where I is Fisher's information functional Eq. **17**. *Furthermore, the extension of Eq.* **5** *from potential to arbitrary vector fields, i.e., to a system on* $\mathfrak{X}(M) \times \mathrm{Dens}(M)$, *reads*

$$\begin{cases} \dot{u}+\nabla_u u+2\nabla\Big(V+f(\rho)-\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big)=0 \\ \dot{\rho}+\operatorname{div}(\rho u)=0. \end{cases} \qquad [23]$$

Corollary 16. *The Hamiltonian system on $T^*\mathrm{Dens}(M)$ for potential solutions (compare Theorem 5) of Eq.* **23** *is mapped symplectomorphically to the Schrödinger equation Eq.* **21**.

Conversely, classical PDEs of hydrodynamics can be expressed as NLS-type equations. E.g., potential solutions of the compressible Euler equations of a barotropic fluid Eq. **10** can be recovered from an NLS equation with Hamiltonian

$$H(\psi)=2\|\nabla\psi\|_{L^2}^2-I(|\psi|^2)+\int_M e(|\psi|^2)|\psi|^2\mu. \qquad [24]$$

The choice $e=0$ yields a Schrödinger formulation for potential solutions of Burgers' equation (or the Hamilton–Jacobi equation, compare *Corollary 7*) whose solutions describe geodesics of the Wasserstein–Otto metric Eq. **6** on Dens(*M*). This way the geometric framework links optimal transport for cost functions with potentials, the compressible Euler equations and the NLS-type equations described above.

Example: vortex filament equation. The celebrated vortex filament (or binormal) equation,

$$\dot{\gamma}=\gamma_x\times\gamma_{xx},$$

describes the motion of a closed curve γ in $\mathbb{R}^3$ where x is the arc-length parameter (Fig. 2). It is a localized induction approximation (LIA) of the 3D Euler equation with initial vorticity supported on γ.

The filament equation is known to be Hamiltonian with respect to the Marsden–Weinstein symplectic structure on the space of curves in $\mathbb{R}^3$ with the Hamiltonian given by the length functional, e.g., ref. 2. On the other hand, it is the equation of the 1D barotropic fluid Eq. **10** with $\rho=k^2$ and $u=2\tau$, where k and τ denote curvature and torsion of the curve γ, respectively.

Definition 4: The Hasimoto transform assigns a wave function $\psi:\mathbb{R}\to\mathbb{C}$ to a curve γ with curvature k and torsion τ, according to the formula

$$(k(x),\tau(x))\mapsto\psi(x)=k(x)\exp\big(\mathrm{i}\int^x\tau(\tilde{x})d\tilde{x}\big).$$

This map takes the vortex filament equation to the NLS equation $\mathrm{i}\psi_t+\psi_{xx}+\frac{1}{2}|\psi|^2\psi=0$. In particular, the filament equation is a completely integrable system whose first integrals are obtained from those of the 1D NLS equation.

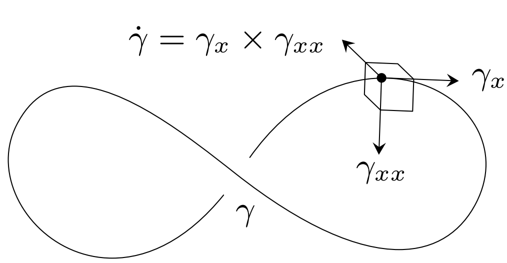

Fig. 2. Vortex filament flow: Each point of the curve γ moves in the direction of the binormal. If $k(x)$ and $\tau(x)$ are the curvature and torsion at $\gamma(x)$, then the wave function $\psi(x)=k(x)\exp(\mathrm{i}\int^x\tau(\tilde{x})d\tilde{x})$ satisfies the NLS equation, while the pair of functions $u=2\tau$ and $\rho=k^2$ satisfies Eq. **10** of the 1D barotropic fluid. The latter is a manifestation of the 1D Madelung transform.

Proposition 17. *The Hasimoto transform is the Madelung transform in 1D.*

This can be seen by comparing *Definitions 3* and *4*, which make the appearance of the Hasimoto transform seem much less surprising.

The filament equation has a higher-dimensional analog for membranes (i.e., compact oriented surfaces Σ of codimension 2 in $\mathbb{R}^n$) as a skew-mean-curvature flow

$$\dot{q}=\mathbf{J}(\mathbf{MC}(q)),$$

where $q\in\Sigma$ is any point of the membrane, $\mathbf{MC}(q)$ is the mean curvature vector to Σ at the point q, and $\mathbf{J}$ is the operator of rotation by $\pi/2$ in the positive direction in every normal space to Σ. This equation is again Hamiltonian with respect to the Marsden–Weinstein structure on membranes of codimension 2 and with a Hamiltonian function given by the $(n-2)$-dimensional volume of the membrane, e.g., ref. 18.

It would be interesting to find an analogue of the Hasimoto transform which maps the skew-mean-curvature flow to an NLS-type equation in any dimension n. The higher-dimensional Madelung transform and its symplectic property are an indication that such an analogue should exist.

Madelung Transform as a Kähler Morphism. In this section we consider again the Madelung transform as a map between $T^*\mathrm{Dens}(M)$ and $PC^\infty(M,\mathbb{C})$ but now equipped with suitable Riemannian structures.

Let $TT^*\mathrm{Dens}(M)$ be the tangent bundle of $T^*\mathrm{Dens}(M)$. Its elements can be described as 4-tuples $(\rho,\theta,\dot{\rho},\dot{\theta})$, where $\rho\in\mathrm{Dens}(M)$, $[\theta]\in C^\infty(M)/\mathbb{R}$, $\dot{\rho}\in C_0^\infty(M)$, and $\dot{\theta}\in C^\infty(M)$ subject to the constraint $\int_M\dot{\theta}\rho\,\mu=0$.

Definition 5: The Sasaki–Fisher–Rao metric on $T^*\mathrm{Dens}(M)$ is the natural lift of the Fisher–Rao metric on $\mathrm{Dens}(M)$:

$$\bar{\mathsf{G}}^*_{(\rho,[\theta])}\Big((\dot{\rho},\dot{\theta}),(\dot{\rho},\dot{\theta})\Big)=\int_M\Big(\frac{\dot{\rho}^2}{\rho}+\dot{\theta}^2\rho\Big)\mu. \qquad [25]$$

The (scaled, infinite-dimensional) Fubini–Study metric[¶] is the following canonical metric on $PC^\infty(M,\mathbb{C})$,

$$\mathsf{G}^*_\psi(\dot{\psi},\dot{\psi})=\frac{4\langle\langle\dot{\psi},\dot{\psi}\rangle\rangle_{L^2}}{\|\psi\|_{L^2}^2}-\frac{4\langle\langle\psi,\dot{\psi}\rangle\rangle_{L^2}\langle\langle\dot{\psi},\psi\rangle\rangle_{L^2}}{\|\psi\|_{L^2}^4}, \qquad [26]$$

which is the projectivization of the L^2 metric on $C^\infty(M,\mathbb{C})$.

Theorem 18. *The Madelung transform Eq.* **20** *is an isometry between $T^*\mathrm{Dens}(M)$ with the Sasaki–Fisher–Rao metric and $PC^\infty(M,\mathbb{C}\setminus\{0\})$ with the Fubini–Study metric.*

Since the Fubini–Study metric is a Kähler metric on $PC^\infty(M,\mathbb{C})$ it follows that $T^*\mathrm{Dens}(M)$ carries a natural Kähler structure compatible with its canonical symplectic structure. The associated complex structure is $J(\dot{\rho},\dot{\theta})=(-\rho\dot{\theta},\dot{\rho}/\rho)$.

Remark 5: Molitor (19) found an almost complex structure on $T^*\mathrm{Dens}(M)$ related to the Wasserstein–Otto metric and the Madelung transform. He also observed that it does not integrate to a complex structure. In contrast, our result shows that the corresponding complex structure does become integrable (and simple) when the Wasserstein–Otto metric is replaced with the Fisher–Rao metric.

¶ Also called the Bures metric in quantum physics.

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Example: two-component Hunter–Saxton equation. The two-component Hunter–Saxton (2HS) equation is a system of two equations

$$\begin{cases} \dot{u}_{xx} = -2u_x u_{xx} - u u_{xxx} + \sigma \sigma_x, \\ \dot{\sigma} = -(\sigma u)_x \end{cases} \qquad [27]$$

where u and σ are time-dependent periodic functions on the line. This system can be viewed as a high-frequency limit of the two-component Camassa–Holm equation; cf. ref. 20.

It turns out that this system is closely related to the Kähler geometry of the Madelung transformation and the Sasaki–Fisher–Rao metric Eq. **25**. Consider the semidirect product $\mathcal{G} = \mathrm{Diff}_0(S^1) \ltimes C^\infty(S^1, S^1)$, where $\mathrm{Diff}_0(S^1)$ is the group of circle diffeomorphisms fixing a prescribed point and $C^\infty(S^1, S^1)$ stands for S^1-valued maps of a circle. Define a right-invariant Riemannian metric on the group $\mathcal{G}$ given at the identity by

$$\mathtt{G}_{(\mathrm{id},0)}((u,\sigma),(u,\sigma)) = \frac{1}{4}\int_{S^1} (u_x^2 + \sigma^2)\, dx.$$

If $t \to (\varphi(t), \alpha(t))$ is a geodesic in $\mathcal{G}$, then $u = \dot{\varphi} \circ \varphi^{-1}$ and $\sigma = \dot{\alpha} \circ \varphi^{-1}$ satisfy equations in Eq. **27**. Lenells (21) showed that the map

$$(\varphi, \alpha) \mapsto \sqrt{\varphi_x\, \mathrm{e}^{\mathrm{i}\alpha}} \qquad [28]$$

from $\mathcal{G}$ to a subset of $\{\psi \in C^\infty(S^1, \mathbb{C}) \mid \|\psi\|_{L^2} = 1\}$ is an isometry. Moreover, solutions to Eq. **27** satisfying $\int_{S^1} \sigma \mathrm{d}x = 0$ correspond to geodesics in $PC^\infty(S^1, \mathbb{C})$ equipped with the Fubini–Study metric. Our results show that this isometry is a particular case of *Theorem 18*.

Proposition 19. *The two-component Hunter–Saxton Eq.* **27** *with initial data satisfying* $\int_{S^1} \sigma\, \mathrm{d}x = 0$ *is equivalent to the geodesic equation of the Sasaki–Fisher–Rao metric Eq.* **25** *on* $T^*\mathrm{Dens}(S^1)$.

The proof is based on the observation that the mapping Eq. **28** can be expressed as $(\varphi, \alpha) \mapsto \Phi(\pi(\varphi), \alpha)$, where Φ is the Madelung transform and π is the projection $\varphi \mapsto \det(D\varphi)$ specialized to the case $M = S^1$.

Remark 6: Observe that if $\sigma = 0$ at $t = 0$, then $\sigma(t) = 0$ for all t and the 2HS Eq. **27** reduces to the standard Hunter–Saxton equation. Geometrically, this is a consequence of the fact that horizontal geodesics on $T^*\mathrm{Dens}(M)$ with the Sasaki–Fisher–Rao metric descend to geodesics on $\mathrm{Dens}(M)$ with the Fisher–Rao metric.

Madelung Transform as a Momentum Map. We briefly comment on the Madelung transform from the perspective of symplectic geometry and reduction theory.

The Riemannian submersion results in *Theorems 4* and *9* can be regarded as a Hamiltonian reduction of the natural symplectic structure on $T^*\mathrm{Diff}(M)$ with respect to the action of the group $\mathrm{Diff}_\mu(M)$. From this viewpoint, Fusca (22) showed that the inverse Madelung map has an interpretation as a momentum map for an action of the semidirect product group $S = \mathrm{Diff}(M) \ltimes C^\infty(M)$ on the (Hermitian) space of smooth wave functions. The semidirect product structure itself appears naturally from the symplectomorphism property in *Theorem 14*. Furthermore, the momentum map viewpoint suggests a natural multicomponent wave-function generalization of the Madelung transform, thus allowing a geometric hydrodynamical interpretation of (nonlinear) Schrödinger equations with spin degrees of freedom or, vice versa, the full compressible Euler equations (with entropy) as a multicomponent Schrödinger equation.

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