JOURNAL OF FUNCTIONAL ANALYSIS **143**, 55–97 (1997)
ARTICLE NO. FU962896

Extensions and Contractions of the Lie Algebra of q-Pseudodifferential Symbols on the Circle

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Received June 7, 1995

We construct cocycles on the Lie algebra of pseudo- and q-pseudodifferential symbols of one variable and on their close relatives: the sine-algebra and the Poisson algebra on two-torus. A "quantum" Godbillon–Vey cocycle on (pseudo)-differential operators appears in this construction as a natural generalization of the Gelfand–Fuchs 3-cocycle on periodic vector fields. A nontrivial embedding of the Virasoro algebra into (a completion of) q-pseudodifferential symbols is proposed. We describe q-analogs of the KP and KdV-hierarchies admitting an infinite number of conserved charges as well as q-deformed Gelfand–Dickey structures.

1. INTRODUCTION

The Lie algebra of symbols of pseudodifferential operators on the circle arises naturally in various contexts: it is a deformation of the Poisson algebra of functions on the two-dimensional torus, the phase space for the KP and KdV hierarchies, a generalization of the Virasoro algebra and of

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0022-1236/97 $25.00

the Lie algebra of differential operators. The latter object appeared recently in conformal field theory under the name of W_∞ algebra.

We start our consideration with the Poisson algebra $C^\infty(T^2)$ of smooth functions on a two-dimensional torus. Various deformations of this object include the sine-algebras [18], algebras of differential and pseudodifferential operators [34], q-analogs of differential operators and (finite-dimensional) $gl(n)$-approximations.

In this paper we consider so called q-deformation of the Lie algebra of pseudodifferential operators (ψDO_q) on the circle. From the quantization point of view it is the second link in the chain of deformations of the Poisson algebra to pseudodifferential operators and then to q-pseudodifferential operators. Our work is closely connected with Kassel's paper [22] where q-analogs of the algebra of purely differential operators and of the corresponding cocycles were described.

The construction can be applied to the case of pseudodifferential operators in $\mathbb{C}^n$, where one has $2n$ outer derivations coming from commutations with $\log D_i$ and $\log x_i$ (see, e.g., [3]). However, the Poisson structures related to integrable systems single out the one-dimensional case.

Some our results were generalised by Wambst [39], who calculated, in particular, the first cyclic cohomology group of the topological algebra of q-pseudodifferential symbols and found that our cocycles span linearly the whole group.

We propose a q-version of the Manin triple of differential and integral operators on the circle. The extension of this triple in the classical case by the logarithm of the derivative operator ([26]) and by the corresponding central term was considered in detail in [23]. In the non-deformed situation the corresponding (infinite dimensional) Lie–Poisson group is "unipotent" and equipped with the quadratic generalized Gelfand–Dickey Poisson structure. This structure is the main ingredient of the KP and n-KdV hierarchies ([10, 23]).

We show that the group of q-pseudodifferential operators is rather "solvable" than "unipotent". Its exponential map is still well defined and surjective as in the classical case, but it is not bijective. This allows one to define q-analogs of the hierarchies which are now commuting flows on the space of q-pseudodifferential operators. These q-hierarchies are Hamiltonian with respect to the q-deformed Gelfand–Dickey structures on the latter space, and they admit infinite number of conserved quantities.

This paper is organized as follows. In Section 2 we discuss the Poisson algebra on a torus and its approximations by finite dimensional algebras $u((m+1)/2, (m-1)/2)$. Section 3 treats various relations between the algebra of pseudodifferential operators, its q-analog, and the sine-algebra. In Section 4 we define a "q-logarithmic" outer derivation of the algebra and the corresponding cyclic 2-cocycle, as well as the Lie algebra 2-cocycle.

We construct also quantum Godbillon–Vey cocycle and higher degree generalizations. They are defined for any $q\neq 0$ and depend on q continuously. In Section 5 we study the behaviour of cocycles when the algebra is deformed. An embedding of the undeformed Virasoro algebra into an appropriately completed algebra of q-pseudodifferential operators is constructed in Section 6. The final Section 7 contains the construction of q-analogs of the KP and n-KdV hierarchies, as well as the q-deformed Poisson–Lie group of pseudodifferential symbols, inspired by the previous formalism.

2. DEFORMATIONS OF THE POISSON ALGEBRA ON TWO-TORUS

Let $\mathbb{T}^2$ be the standard torus with p, q-coordinates equipped with a symplectic form $\omega = dp \wedge dq$, and $N = C^\infty(\mathbb{T}^2)$ be the Lie algebra of periodic functions in p, q with respect to the Poisson bracket.

2.1. *Lie Cocycles on* $C^\infty(\mathbb{T}^2)$

THEOREM 2.1 (see, e.g. [28, 24]). *The group* $H^2_{\mathrm{Lie}}(C^\infty(\mathbb{T}^2), \mathbb{R})$ *is two-dimensional and generated by the* 2*-cocycles*

$$c_p(f, g) = \iint_{\mathbb{T}^2} (\{p, f\}\, g)\omega = \iint_{\mathbb{T}^2} \left(\frac{\partial f}{\partial q} \cdot g\right) \omega$$

$$c_q(f, g) = \iint_{\mathbb{T}^2} (\{q, f\}\, g)\omega = -\iint_{\mathbb{T}^2} \left(\frac{\partial f}{\partial p} \cdot g\right) \omega,$$

where $f,\ g \in N = C^\infty(\mathbb{T}^2)$.

Remark 2.1. The linear functions p and q are being considered as multivalued function on $\mathbb{T}^2$. They define Hamiltonian vector fields, which “shift the mass center” of $\mathbb{T}^2$, unlike the fields corresponding to univalued Hamiltonians.

More generally (see [28]), on an arbitrary compact symplectic manifold (M, ω) the Poisson algebra of functions $C^\infty(M)$ has the universal central extension of dimension $b_1 = \dim H_1(M, \mathbb{R})$ (i.e. $H^2_{\mathrm{Lie}}(C^\infty(M), \mathbb{R}) \simeq H_1(M, \mathbb{R})$).

THEOREM 2.2. *The group* $H^3_{\mathrm{Lie}}(C^\infty(\mathbb{T}^2), \mathbb{R})$ *is at least two-dimensional. Nontrivial* 3*-cocycles can be defined as follows*:

$$c_0(f, g, h) = \iint_{\mathbb{T}^2} (\{f, g\}\, h)\omega$$

and

$$c_1(f, g, h) = \underset{f,g,h}{\mathrm{Alt}} \iint_{\mathbb{T}^2} \left(\frac{\partial^3 f}{\partial p^3} \frac{\partial^3 g}{\partial q^3} - 3 \frac{\partial^3 f}{\partial p^2 \partial q} \frac{\partial^3 g}{\partial q^2 \partial p} + 3 \frac{\partial^3 f}{\partial p \partial q^2} \frac{\partial^3 g}{\partial q \partial p^2} - \frac{\partial^3 f}{\partial q^3} \frac{\partial^3 g}{\partial p^3} \right) h \cdot \omega,$$

where $f, g, h \in N$.

Remark 2.2. One of the most important features of the Lie algebra $C^\infty(\mathbb{T}^2)$ is the existence of an ad-invariant pairing (Killing form): $\langle f, g \rangle = \iint_{\mathbb{T}^2} (f \cdot g)\omega$. This allows one to describe the above cocycles in the following alternative way.

To prove Theorem 2.2 we introduce a map θ, which we are going to exploit intensively in Section 4:

$$\theta: C^k(N, N) \to C^{k+1}(N, \mathbb{R})$$

(where $C^k(N, V)$ stands for the additive group of V-valued k-cochains on N). Namely, $\theta(c)(f_1, ..., f_{k+1}) = \mathrm{Alt}_{f_1, ..., f_{k+1}} \langle c(f_1, ..., f_k), f_{k+1} \rangle$. Here and below Alt denote the sum over all shuffles of arguments with the corresponding signs (in the case above the sum consists of $(k+1)$ summands due to the skewsymmetry of c).

LEMMA 2.3. *Let* $c \in Z^k(N, N)$ *be a closed cochain such that*

$$\underset{f_1, ..., f_{k+2}}{\mathrm{Alt}} \langle \{f_1, f_2\}, c(f_3, ..., f_{k+2}) \rangle = 0 \tag{2.1}$$

(*the sum is over* $(k+2)(k+1)/2$ *summands*). *Then* $\theta(c) \in Z^{k+1}(N, \mathbb{R})$.

Proof. The exterior product $\boxtimes$ of the identity cochain $b = (\mathrm{id}: N \to N) \in C^1(N, N)$ with c gives a cochain in $C^{k+1}(N, N \otimes N)$

$$(b \boxtimes c)(f_1, ..., f_{k+1}) = \underset{f_1, ..., f_{k+1}}{\mathrm{Alt}} f_1 \otimes c(f_2, ..., f_{k+1}).$$

Its coboundary is

$$d(b \boxtimes c) = db \boxtimes c - b \boxtimes dc = db \boxtimes c = \{\ ,\ \} \boxtimes c$$

$$(\{\ ,\ \} \boxtimes c)(f_1, ..., f_{k+2}) = \underset{f_1, ..., f_{k+2}}{\mathrm{Alt}} \{f_1, f_2\} \otimes c(f_3, ..., f_{k+2})$$

since db is the bracket $\{\ ,\ \}$ itself. Applying the homomorphism $\langle\ ,\ \rangle$: $N \otimes N \to \mathbb{R}$ we get the cochain $\theta(c)$ with the coboundary

$$d(\theta(c))(f_1, ..., f_{k+2}) = \operatorname*{Alt}_{f_1, ..., f_{k+2}} \langle \{f_1, f_2\}, c(f_3, ..., f_{k+2}) \rangle \tag{2.2}$$

which vanishes by assumption. ▮

Example 2.1. Denote by $\xi_p \in H^1(N, N)$ (resp. ξ_q) the following outer derivation of the algebra N: $\xi_p(*) = \{p, *\}: N \to N$ (resp. $\xi_q(*) = \{q, *\}$). Then the 2-cocycle $c_p = \theta(\xi_p)$ (resp. $c_q = \theta(\xi_q)$) is $c_p(f, g) = \langle \xi_p(f), g \rangle$ (resp. $c_q(f, g) = \langle \xi_q(f), g \rangle$). The condition (2.1) follows from the skew-symmetry of ξ_p.

Example 2.2. The wedge product $d_0 = \xi_p \wedge \xi_q = \{\ ,\ \}$ is an N-valued 2-cocycle. The property (2.1) holds by the Jacobi identity. And it is easy to see, that $\theta(d_0)(f, g, h) = c_0(f, g, h)$. Note that even though d_0 is a trivial 2-cocycle (i.e., it is a 2-coboundary), its image under the homomorphism θ is a nontrivial 3-cocycle on N. The 2-cocycle d_0 corresponds to the deformation of N evolved by a dilation of the area form ω.

In Section 4 we extend this construction to the algebras of pseudodifferential operators and to their q-analogues.

A common feature of these examples is the notion of an invariant cocycle on a Lie algebra with an ad-invariant form. This notion (defined and intensively used in [29]) generalizes naturally the idea of skew-symmetric 1-cocycle. Let $\mathfrak{g}$ be a Lie algebra with an ad-invariant form $\langle\ ,\ \rangle$. One says that a cocycle $c \in C^k(\mathfrak{g}, \mathfrak{g})$ is invariant if the map $\theta'(c)$ defined through $\theta'(c)(x_0, ..., x_k) = \langle c(x_0, ..., x_{k-1}), x_k \rangle$ is skew-symmetric. One easily checks that invariant cocycles form a subcomplex in $C^*(\mathfrak{g}, \mathfrak{g})$ (denoted by $C_I^*(\mathfrak{g}, \mathfrak{g})$), and θ' (proportional to θ) defines a morphism of complexes:

$$\theta': C_I^k(\mathfrak{g}, \mathfrak{g}) \to C^{k+1}(\mathfrak{g}, \mathbb{R}).$$

So one naturally defines invariant cohomology groups (denoted by $H_I^k(\mathfrak{g}, \mathfrak{g})$) with a map, still denoted by θ',

$$\theta': H_I^k(\mathfrak{g}, \mathfrak{g}) \to H^{k+1}(\mathfrak{g}, \mathbb{R}).$$

Notice that the cohomology class of $d_0 = [\ ,\ ]$ is nontrivial in $H_I^2(\mathfrak{g}, \mathfrak{g})$, but is trivial in $H^2(\mathfrak{g}, \mathfrak{g})$. The same phenomenon occurs for semisimple classical Lie algebras: the Lie bracket defines a nontrivial element in $H_I^2(\mathfrak{g}, \mathfrak{g})$ whose image in $H^3(\mathfrak{g}, \mathbb{R})$ generates the latter group (see e.g. Section 2.2).

EXAMPLE 2.3. The 3-cocycle c_1 is obtained by the same homomorphism θ' from a nontrivial 2-cocycle $d_1 \in H^2(N, N)$; $d_1 = (\partial^3 f/\partial p^3)(\partial^3 g/\partial q^3) - 3(\partial^3 f/\partial p^2\, \partial q)(\partial^3 g/\partial q^2\, \partial p) + 3(\partial^3 f/\partial p\, \partial q^2)(\partial^3 g/\partial q\, \partial^2 p) - (\partial^3 f/\partial q^3)(\partial^3 g/\partial p^3)$. This cocycle d_1 is responsible for a formal deformation of the Poisson algebra; in fact, it is the third term of the Moyal deformation (called also deformation quantization) of the Poisson algebra N, obtained by extending the Heisenberg law $\{p, q\} = 1 \to [\hat{p}, \hat{q}] = Id$. Another notation for this cocycle is $d_1(f, g) = (\mu \nabla^3)(f \otimes g)$ (here $\nabla = ((\partial/\partial p) \otimes (\partial/\partial q) - (\partial/\partial q) \otimes (\partial/\partial p))$ and μ denotes the multiplication). One checks easily that this cocycle is invariant. Indeed, if one uses trigonometric representation for functions on $\mathbb{T}^2$, say $L_\alpha = -\exp(i(\alpha_1 q + \alpha_2 p))$, $\alpha \in \mathbb{Z}^2$, then

$$\int_{\mathbb{T}^2} d_1(L_\alpha, L_\beta)\, L_\gamma \omega = \delta_{\alpha+\beta+\gamma, 0} (\alpha \times \beta)^3,$$

where $\alpha \times \beta = \alpha_2 \beta_1 - \alpha_1 \beta_2$, and the invariance is immediate. Linear independence of the classes of c_0 and c_1 in $H^3(N, \mathbb{R})$ can also be checked through this formula. Nontriviality of these 2-cocycles is equivalent to nontriviality of the corresponding Lie algebra deformation (see, e.g. [9]).

Remark 2.3. The cohomology class of d_1 in $H^2(N, N)$ is sometimes called the Vey class after Vey's fundamental contribution to the subject [38]. It admits a global analogue for any symplectic manifold, and is often denoted by S^3_Γ. The Moyal deformation is linked with symbols of differential operators via the Weyl ordering. For arbitrary surfaces the invariance of that cocycle is conjectured, but to the best of our knowledge has not been proved yet.

Gelfand and Mathieu [12] found a nice construction of higher cocycles of the Poisson algebra N. We generalize these cocycles to the algebras of pseudodifferential or q-pseudodifferential operators and sine-algebras discussed below. This approach can be explained through the following approximation scheme.

2.2. *Approximation of* $N = C^\infty(\mathbb{T}^2)$ *by* $gl(m)$ *as* $m \to \infty$ (*following* [18])

Let us fix some odd m and consider the following two unimodular matrices in $gl(m)$:

$$E \equiv \begin{pmatrix} 1 & & & \\ & \varepsilon & 0 & \\ & 0 & \ddots & \\ & & & \varepsilon^{m-1} \end{pmatrix} \quad \text{and} \quad H \equiv \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & 0 \\ & 0 & & 1 \\ 1 & & & 0 \end{pmatrix}$$

where ε is a primitive mth root of unity (say, $\exp(4\pi i/m)$). They obey the identities $HE = \varepsilon EH$, $E^m = H^m = 1$. Then the unitary unimodular matrices

$$\mathcal{I}_{(k_1, k_2)} = \varepsilon^{k_1 \cdot k_2/2} E^{k_1} H^{k_2}$$

span the algebra of $gl(m)$. Note that $\mathcal{I}^{-1}_{(k_1, k_2)} = \mathcal{I}_{(-k_1, -k_2)}$ and $\operatorname{tr} \mathcal{I}_{(k_1, k_2)} = 0$ except for $k_1 = k_2 = 0 \pmod m$. The set of $\mathcal{I}$'s is closed under multiplication:

$$\mathcal{I}_k \mathcal{I}_l = \varepsilon^{k \times l/2} \mathcal{I}_{k+l} \qquad (k = (k_1, k_2),\ l = (l_1, l_2),\ k \times l = k_2 l_1 - k_1 l_2).$$

Therefore $\{\mathcal{I}_k\}$ satisfy the commutation relations

$$[\mathcal{I}_k, \mathcal{I}_l] = 2i \sin\left(\frac{2\pi(k \times l)}{m}\right) \mathcal{I}_{k+l} \tag{2.3}$$

The real subalgebra $\mathfrak{a}(\mathfrak{m})$ of $gl(m)$ spanned by $i\mathcal{I}_k$ is isomorphic to $u((m+1)/2, (m-1)/2)$, the Lie algebra of the group of matrices unitary with respect to the hermitian metric in $\mathbb{C}^m$ with $(m+1)/2$ positive and $(m-1)/2$ negative squares. Indeed,

$$(-1/m) \operatorname{Tr}(i\mathcal{I}_k \cdot i\mathcal{I}_l) = \delta_{k+l, 0},$$

which implies that the signature of the Killing form on $\mathfrak{a}(\mathfrak{m})/\mathbb{R}\mathcal{I}_o$ is 0. For $su(k, l)$ this signature is $(k-l)^2 - 1$.

As $m \to \infty$ this algebra goes to the algebra $[L_k, L_l] = (k \times l) L_{k+l}$ through the identification $(m/4\pi i)\mathcal{I}_k \mapsto L_k$. The latter object is exactly the Poisson algebra of Hamiltonian functions on $\mathbb{T}^2$:

$$L_{(k_1, k_2)} = -\exp(i(k_1 q + k_2 p)),$$

which is modulo constants the same as the Lie algebra of the corresponding Hamiltonian vector fields:

$$L_{(k_1, k_2)} = i \exp(i(k_1 q + k_2 p)) \left(k_1 \frac{\partial}{\partial p} - k_2 \frac{\partial}{\partial q}\right).$$

The algebra (2.3) is the non-extended case ("cyclotomic family") of an infinite dimensional "sine-algebra" [5, 6, 7]:

$$[\mathcal{I}_k, \mathcal{I}_l] = r \sin(2\pi(k \times l)/\lambda)\, \mathcal{I}_{k+l} + (a \cdot k)\delta_{k+l, 0} \tag{2.4}$$

where the constant λ is not necessarily integer, but is an arbitrary complex number, $a = (a_1, a_2)$ is a fixed plane vector, $r \in \mathbb{C}$. Here $k = (k_1, k_2)$, $l = (l_1, l_2)$ are not integers modulo m, but belong to $\mathbb{Z}^2$, and $(a \cdot k) = a_1 k_1 + a_2 k_2$.

Unlike the (simple) $sl(m)$ algebra, the (infinite dimensional) sine-algebra and the Poisson algebra $\{L_k\}$ have two nontrivial central extensions. Thus

2-cocycles on the sine-algebra with an arbitrary λ disappear when we truncate it to the finite-dimensional object. However, the situation with real- (or complex-) valued 3-cocycles is different. It is well-known that the algebra $su((m+1)/2, (m-1)/2)$ as any simple Lie algebra has the 1-dimensional group $H^3(su((m+1)/2, (m-1)/2), \mathbb{R})$.

PROPOSITION 2.4. *The Lie* 3-*cocycle on* $u((m+1)/2, (m-1)/2)$ *generating* $H^3_{Lie}(u((m+1)/2, (m-1)/2)), \mathbb{R})$ *goes to the* 3-*cocycle* c_0 (*up to a multiplicative constant*) *of Theorem* 2.2 *as* $m \to \infty$.

Proof. The cocycle on $\mathfrak{a}(\mathfrak{m})$ is given by the completely antisymmetric expression

$$\mu(A, B, C) = a\,\mathrm{Tr}([A, B]\,C), \qquad a = (4\pi i)^2\, m^{-3}.$$

Note, that due to a nondegenerate pairing on $\mathfrak{a}(\mathfrak{m}) : \langle A, B \rangle = \mathrm{Tr}(AB)$ the cocycle μ can be obtained from the invariant 2-cocycle $\eta \in Z^2(\mathfrak{a}(\mathfrak{m}), \mathfrak{a}(\mathfrak{m}))$: $\eta(A, B) = [A, B]$ via $\theta(\eta) = \mu$ (see Section 2.1). As $m \to \infty$ the commutator of matrices [,] goes to the Poisson bracket, and $a\mathrm{Tr}(\cdot)$ goes to the ad-invariant trace on N defined by $\mathrm{Tr}(f) = \int_{\mathbb{T}^2} f\omega$. Indeed, $[L_f, L_g] = L_{\{f, g\}}$, and in the limit the trace becomes the zero Fourier mode of a Hamiltonian function. ∎

The coordinate expressions of the cocycles are

$$\mu(\mathscr{I}_\alpha, \mathscr{I}_\beta, \mathscr{I}_\gamma) = \frac{m}{2\pi} \sin\left(\frac{2\pi(\alpha \times \beta)}{m}\right) \delta_{\alpha+\beta+\gamma, 0}$$

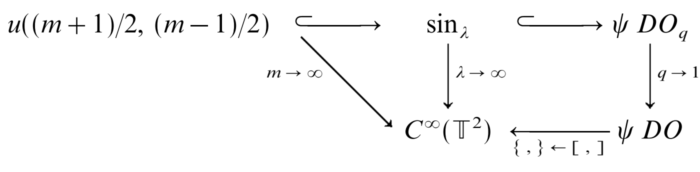
(α, β, γ are mod m), and

$$c_0(L_\alpha, L_\beta, L_\gamma) = (\alpha \times \beta)\,\delta_{\alpha+\beta+\gamma, 0}$$

3. ADJACENCY DIAGRAM OF THE ALGEBRAS $C^\infty(\mathbb{T}^2)$, $\sin_\lambda$, $\psi\, DO$, $\psi\, DO_q$

In the preceding section we showed that the 3-cocycle on $u((m+1)/2, (m-1)/2)$ survives under the limit $m \to \infty$. In the next section we will see how it deformes when one quantizes the Poisson algebra N into the algebra of pseudodifferential symbols $\psi\, DO$ or the algebra of q-analogs of pseudodifferential symbols $\psi\, DO_q$.

The diagram of the various algebras discussed in this paper is as follows



In this section we define all the objects and meaning of all arrows in this diagram. We recall also certain well- and less-known facts about the objects involved.

3.1. *Algebras* $u((m+1)/2,\ (m-1)/2)$ *and* $\sin_\lambda$

In the preceding section it was mentioned that the limit $m \to \infty$ of the structural constants of $u((m+1)/2,\ (m-1)/2)$, is to be considered in the framework of generic infinite dimensional sine-algebras $\sin_\lambda$ (with rational or irrational λ):

$$[\mathscr{I}_k, \mathscr{I}_l] = \frac{\lambda}{2\pi} \sin\left(\frac{2\pi(k \times l)}{\lambda}\right) \mathscr{I}_{k+l}.$$

3.2. *Algebra of Pseudodifferential Operators on the Line and on the Circle*

The algebra of $\psi\ DO$ is a quantization of the algebra $N = C^\infty(\mathbb{T}^2)$ where q is replaced by x and p is replaced by $\partial/\partial x$.

Definition 3.1. The ring $\psi\ DO$ of pseudodifferential symbols is the ring of formal series $A(x, \partial) = \sum_{-\infty}^{n} a_i(x)\partial^i$ with respect to ∂, where $a_i(x) \in \mathbb{C}[x, x^{-1}]$ (or $C^\infty(S^1$, $\mathbb{R}$ or $\mathbb{C}$)), and the variable ∂ corresponds to d/dx. The multiplication law in $\psi\ DO$ is given by the commutation relations

$$\partial \circ f(x) = f(x)\partial + f'(x)$$

$$\partial^{-1} \circ f(x) = f(x)\partial^{-1} - f'(x)\partial^{-2} + f''(x)\partial^{-3} - \cdots$$

These relations define the usual composition law on the subalgebra $DO \subset \psi\ DO$ of *differential* operators (i.e. on polynomials with respect to ∂) and they can be unified in one as

$$\partial^s \circ u(x) = \sum_{l \geqslant 0} \binom{s}{l} u^{(l)}(x)\partial^{s-l}, \tag{3.1}$$

where $\binom{s}{l} = (s(s-1)\cdots(s-l+1)/l!$. The product determines the natural Lie algebra structure on $\psi\ DO$: $[A, B] = A \circ B - B \circ A$. We will denote the ring $\psi\ DO$ by $\mathfrak{G}$ when we stress its Lie algebra structure.

Moreover, there are operators res: $\psi\ DO \to \mathbb{C}[x, x^{-1}]$ (or $C^\infty(S^1)$) and Tr: $\psi\ DO \to \mathbb{C}$ on the ring $\psi\ DO$: $\operatorname{res}(\sum a_i D^i) = a_{-1}(x)$, and $\operatorname{Tr} A = \int \operatorname{res} A$ (here and below we integrate over the circle S^1 if $a_{-1} \in C^\infty(S^1)$, and we set $\int x^k = \delta_{k,0}$ if $a_{-1} \in \mathbb{C}[x, x^{-1}]$). The main property of Tr is $\operatorname{Tr}[A, B] = 0$ for arbitrary A, $B \in \psi\ DO$.

Remark 3.1. The multiplication formula above is defined not only for integer values of s, but for fractional and complex values as well. One can define an associative product on $\psi\, DO$ depending on a parameter h. The Leibnitz rule for multiplication of symbols in $\psi\, DO$ gives the following general formula equivalent for $h=1$ to the rule (3.1) above:

$$A(x, \xi) \underset{h}{\circ} B(x, \xi) = \sum_{n \geqslant 0} \frac{h^n}{n!} A_{\xi}^{(n)}(x, \xi)\, B_x^{(n)}(x, \xi)$$

where $A_{\xi}^{(n)} = d^n/d\xi^n\, A(x, \xi)$, $B_x^{(n)} = d^n/dx^n\, B(x, \xi)$. Let $\Phi_t\colon \mathfrak{G} \to \mathfrak{G}$ defined through $\Phi_t(a(x)\xi^p) = a(x)\, t^{p-1}\xi^p$ be a family of bijective maps for $t \in\,]0, 1]$, singular for $t=0$. Set $[A, B]_t = \Phi_t^{-1}[\Phi_t(A), \Phi_t(B)]$. One has $[A, B]_t = \{A, B\} + O(t)$. So $\lim [A, B]_t = \{A, B\}$ as $t \to 0$. One says that the commutator of pseudodifferential symbols contracts onto the Poisson bracket.

3.3. *The Algebra of q-Pseudodifferential Symbols*

3.3.1. *Associative Algebra Structure.* The notion of the q-analog of the derivative (and any differential) operator (see, e.g., [22]) is extended here to the case of pseudodifferential operators. We follow the framework of the previous subsections stressing the difference between the classical and q-deformed case.

Let F be the algebra $\mathbb{C}[x, x^{-1}]$ and $q \in \mathbb{C}^{\times}$. It will be the main algebra of coefficients for us, the reader can easily reproduce the results for the algebra $F' = C^{\infty}(S^1)$ in the case $q \in \mathbb{C}$, $|q| = 1$ etc. We use the following notations for q-numbers:

$$(n)_q = \frac{q^n - 1}{q - 1}$$

$$\binom{m}{l}_q = \frac{(m)_q\, (m-1)_q \cdots (m-l+1)_q}{(1)_q\, (2)_q \cdots (l)_q}.$$

Define the q-analog of the derivative as

$$D_q\, f(x) = \frac{f(qx) - f(x)}{q - 1}$$

for $f \in F$. One can see that $\lim_{q \to 1} D_q = x(\partial/\partial x) = x\partial$, where $\partial = \partial/\partial x$ is the derivative operator of the preceding Section 3.2. It will be useful to define also the shift τ:

$$\tau f(x) = f(qx), \qquad \tau^{\beta} f(x) = f(q^{\beta} x).$$

Clearly, τ commutes with D_q. So, D_q is a q-derivative in the following sense:

$$D_q(fq) = D_q(f)\, g + \tau(f)\, D_q(g).$$

DEFINITION 3.2. The associative algebra $\psi\, DO_q$ of q-pseudodifferential operators is the vector space of formal series

$$\psi\, DO_q = \left\{ A(x, D_q) = \sum_{-\infty}^{n} u_i(x) D_q^i \mid u_i \in F \right\}$$

with respect to D_q. The multiplication law in $\psi\, DO_q$ is defined by the following rule: F is a subalgebra of $\psi\, DO_q$ and there are commutation relations:

$$D_q \circ u = (D_q u) + \tau(u) D_q, \qquad u \in F$$

$$D_q^{-1} \circ u = \sum_{k \geqslant 0} (-1)^k \, (\tau^{-k-1}(D_q^k u)) \, D_q^{-k-1}, \qquad u \in F, F'. \tag{3.2}$$

Each term of the product of two Laurent series in D_q is found by applying these rules finite number of times. The formula (3.2) is built so that $D_q^{-1} \circ D_q \circ u = D_q \circ D_q^{-1} \circ u = u$. For $q = 1$ these formulas recover the "classical" definition (3.1) of multiplication law in the algebra of pseudodifferential operators $\psi\, DO$.

The commutation rule for D_q^n (with any integer n) and $u(x)$ join these formulae in one:

$$D_q^n \circ u = \sum_{l \geqslant 0} \binom{n}{l} (\tau^{n-l}(D_q^l u)) \, D_q^{n-l}. \tag{3.3}$$

PROPOSITION 3.1. *The q-analog of the Leibnitz rule of multiplication of two q-pseudo-differential operators $A(x, D_q)$, $B(x, D_q)$ can be written as the following operation on their symbols*

$$A(x, D_q) \circ B(x, D_q) = \sum_{k \geqslant 0} \frac{1}{(k)!} \left(\frac{d^k}{dD_q^k} A \right) * (D_q^k B) \tag{3.4}$$

where $(d^k/dD_q^k)\, A(x, D_q)$ is the q-derivative of A with respect to the second argument (for $A = f(x) D_q^\alpha$ it is $(d^k/dD_q^k) A = (d^k/dD_q^k)(f D_q^\alpha) = ((\alpha)!/(\alpha-k)!)\, f D_q^{\alpha-k}$), and $D_q^k B(x, D_q)$ is the q-derivative of B defined above, and finally, the multiplication $*$ of symbols is defined with the following commutation rule of the generators:

$$f * D_q = f D_q, \quad D_q * f = \tau(f) D_q, \qquad D_q^{-1} * f = \tau^{-1}(f) D_q^{-1}.$$

Proof. Straightforward verification of the formula (3.4) for the product $D_q^n \circ u(x)$ gives the same answer as (3.3). ∎

3.3.2. *Non-commutative Residue for $\psi\, DO_q$.*

(1) The operation $res: \psi\, DO_q \to F$ is defined by

$$\operatorname{res}\left(\sum_{i=-\infty}^{n} u_i(x)\, D_q^i\right) = u_{-1}(x)$$

generalizing definition for classical pseudodifferential symbols [40].

(2) The integral along the circle S^1 gives a linear functional $\int: F \to \mathbb{C}$, $\int x^n = \delta_{n,0}$, on the function algebra F satisfying $\int D_q f = 0$ and $\int \tau(f) = \int f$ for all $f \in F$. We have also an "integration by parts" formula $\int f\tau^{-1}(D_q(g)) = -\int D_q(f)\, g$.

3.3.3. *Inner Product on $\psi\, DO_q$.* Define in $\psi\, DO_q$ the element $T = (q-1)\, D_q + 1$ and denote its inverse

$$E \equiv T^{-1} = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(q-1)^i}\, D_q^{-i} = \frac{1}{(q-1)\, D_q + 1}.$$

LEMMA 3.2. *The automorphism $\tau: \psi\, DO_q \to \psi\, DO_q$, $fD_q^p \mapsto \tau(f)\, D_q^p$ is inner. Precisely, $\tau(A) = TAT^{-1} = E^{-1}AE$.*

Proof. Indeed, for any $f \in F$

$$\begin{aligned} TfT^{-1} &= ((q-1)\, D_q + 1)\, fT^{-1} \\ &= \{(q-1)(\tau(f)\, D_q + D_q(f)) + f\}\, T^{-1} \\ &= \{\tau(f)((q-1)\, D_q + 1) + (q-1)\, D_q(f) - \tau(f) + f\}\, T^{-1} \\ &= \tau(f)\, TT^{-1} = \tau(f) \end{aligned}$$

and T commutes with D_q. ∎

Define the Lie algebra $\mathfrak{G}_q$ as the set $\psi\, DO_q$ of all q-pseudodifferential symbols equipped with the commutator bracket $[A, B] = A \circ B - B \circ A$.

THEOREM 3.3. *The bilinear form $\langle A, B\rangle = \int \operatorname{res}(ABE)$ is an ad-invariant symmetric non-degenerate form on the Lie algebra $\mathfrak{G}_q$.*

To prove it consider the bilinear form $(A, B) \mapsto \int \operatorname{res}(A \circ B)$ on the algebra $\psi\, DO_q$. It is not symmetric but it has the following version of the symmetry property.

PROPOSITION 3.4. *For any $A, B \in \psi\, DO_q$*

$$\int \operatorname{res}(A \circ B) = \int \operatorname{res}(B \circ \tau(A)),$$

where the map $\tau: \psi DO_q \to \psi DO_q$ *is the following automorphism of the algebra* ψDO_q:

$$\tau(fD_q^b) = \tau(f)D_q^b$$

Proof. Explicitly the form $\int$ res is given by

$$\int \operatorname{res}(fD_q^a \circ gD_q^b) = \int \operatorname{res}\left(f \sum_{k \geqslant 0} \binom{a}{k} \tau^{a-k}(D_q^k g D_q^{a-k+b}\right)$$
$$= \binom{a}{a+b+1} \int f\tau^{-b-1}(D_q^{a+b+1}g)$$

for $a+b+1 \geqslant 0$, and it vanishes otherwise. On the other hand assuming $a+b+1 \geqslant 0$ we find also that

$$\int \operatorname{res}(\tau^{-1}(g) D_q^b \circ fD_q^a) = \binom{b}{a+b+1} \int \tau^{-1}(g)\, \tau^{-a-1}(D_q^{a+b+1}f)$$
$$= \frac{b(b-1)\cdots(-a)}{(a+b+1)!} \int \tau^{-b-1}(g)\, \tau^{-a-b-1}(D_q^{a+b+1}f)$$
$$= \frac{(-b)(-b+1)\cdots a}{(a+b+1)!} \int D_q^{a+b+1}(\tau^{-b-1}g) f$$

which coincides with the expression above. ▮

Now Theorem 3.3 immediately follows from Lemma 3.2.

3.3.4. *Twisted Loop Algebra.* The algebra ψDO_q can also be described as a twisted loop algebra following [21]. This construction, in short, is as follows.

For an associative algebra A and its automorphism σ one defines the twisted loop algebra $A_\sigma[x, x^{-1}]$. As a vector space this is the space of Laurent polynomials $\sum_{j \in \mathbb{Z}} x^j a_j$, $a_j \in F$. The associative product is defined by the rule:

$$(x^m a)(x^n b) = x^{m+n} \sigma^n(a) b.$$

Of course, when σ is the identity automorphism, one recovers the usual notion of the loop algebra $A[x, x^{-1}]$. (Those algebras are purely algebraic versions of the well-known cross-product in functional analysis.)

With any trace Tr on A one can associate a central extension of $A_\sigma[x, x^{-1}]$ defined by the following 2-cocycle:

$$\psi_{\sigma, Tr}(x^m a, x^n b) = -\psi_{\sigma, Tr}(x^n b, x^m a)$$
$$= Tr((1 + \sigma + \cdots + \sigma^{m-1})(\sigma^{-m}(a) b))\, \delta_{m+n, 0}$$

if m is positive, and 0 if $m = 0$.

In order to obtain q-pseudodifferential symbols take $A = \mathbb{C}[\xi, \xi^{-1}]]$ and for some $q \in \mathbb{C}^\times$ set $\sigma_q(\xi) = q\xi + 1$ so that $\sigma_q^n(\xi) = q^n \xi + (n)_q$.

Then we get an isomorphism of associative algebras

$$I\colon A_\sigma[x, x^{-1}] \to \psi DO_q$$

defined on generators as follows:

$$I(x^n) = x^n \qquad \text{and} \qquad I(\xi) = D_q.$$

In the particular case of $q = 1$, one recovers the associative algebra of pseudodifferential operators by assigning to ξ the operator $f \mapsto x \cdot \partial f / \partial x$.

3.4. *The Sine-Algebra and* ψDO_q

Here we identify q-analogs of ψDO and sine-algebra, following M. Golenishcheva–Kutuzova (see [15]), and show that the invariant trace on ψDO_q, we have constructed above, becomes the canonical invariant trace on the sine-algebra. We are grateful to M. Golenishcheva–Kutuzova for explaining to us this alternative approach to the trace description ([16]).

To define an associative product on elements of the sine-algebra, consider the "quantum torus", i.e. the C^*-algebra A_h generated by two unitary operators U_1 and U_2, satisfying the relation $U_2 U_1 = qU_1 U_2$ for $q = e^{ih}$.

An arbitrary element of A_h can be written in the form of a formal series $f = \sum_{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}} f_{(n_1, n_2)} U_1^{n_1} U_2^{n_2}$. The Lie algebra structure on A_h is defined by

$$\begin{aligned}[f, g] &= f * g - g * f \\ &= \sum_{n, m \in \mathbb{Z}^2} f_{(n_1, n_2)}\, g_{(m_1, m_2)} (q^{m_1 n_2} - q^{m_2 n_1})\, U_1^{n_1 + m_1} U_2^{n_2 + m_2}.\end{aligned}$$

One can construct a two-dimensional central extension of this Lie algebra following the prescription of Section 2.1 above: the sine-algebra A_h admits an ad-invariant trace $\eta(\sum_{(n_1, n_2)} f_{(n_1, n_2)} U_1^{n_1} U_2^{n_2}) = f_{(0,0)}$. This trace allows one to associate a 2-cocycle with scalar values to every outer derivation of the algebra.

Let $a = (a_1, a_2) \in \mathbb{C}$ and L_a be a derivation defined by $L_{(a_1, a_2)}(U_1) = a_1 U_1$, $L_{(a_1, a_2)}(U_2) = a_2 U_2$. Then the corresponding 2-cocycle is nothing but $c(f, g) = \eta(L_a f * g)$. Having chosen the basis in $\hat{A}_h = A_h \oplus \mathbb{C}c$ in the form $\mathcal{I}_{k_1 k_2} = q^{k_1 \cdot k_2/2} U_1^{k_1} U_2^{k_2}$ we come to the commutation relations of the sine-algebra (2.4) with $q = e^{4\pi i/\lambda}$, see [15].

Consider now the homomorphism $\phi\colon \sin_\lambda \hookrightarrow \mathfrak{G}_q$ defined through $\phi(U_1) = x$, $\phi(U_2) = D_q + 1/(q-1)$.

PROPOSITION 3.5. *The* ϕ*-pullback of the ad-invariant trace on* $\mathfrak{G}_q$ *induces the ad-invariant trace* η *on the sine-algebra* $\sin_\lambda$ *(with* $q = e^{4\pi i/\lambda}$*) up to a constant multiple.*

Proof. Indeed,

$$\begin{aligned}
\operatorname{Tr} \phi(U_1^k U_2^l) &= \operatorname{Tr}\left(x^k\left(D_q+\frac{1}{q-1}\right)^l\right) \\
&= \int \operatorname{res}\left(x^k\left(D_q+\frac{1}{q-1}\right)^l \frac{1}{(q-1)D_q+1)}\right) \\
&= \frac{1}{q-1}\int \operatorname{res}\left(x^k\left(D_q+\frac{1}{q-1}\right)^{l-1}\right) \\
&= \frac{\delta_{l,0}}{q-1}\int x^k = \frac{1}{q-1}\,\delta_{l,0}\delta_{k,0} \quad \blacksquare
\end{aligned}$$

Remark 3.2. The invariant inner product on both algebras is defined by the trace via

$$\langle A, B\rangle = \operatorname{Tr}(AB).$$

3.5. *Bialgebra Structure*

We recall that the Lie algebra $\mathfrak{G}$ of classical pseudodifferential symbols can be equipped with a Lie bialgebra structure. Indeed, the algebra $\mathfrak{G}$ as a linear space is a direct sum of two natural subalgebras: $\mathfrak{G}_+ = \mathfrak{G}_{DO}$ consisting of differential operators $\sum_{j\geqslant 0} a_j D^j$, and $\mathfrak{G}_- = \mathfrak{G}_{Int}$ consisting of integral symbols $\sum_{j=-1}^{-\infty} a_j D^j$.

The triple of algebras $(\mathfrak{G}, \mathfrak{G}_+, \mathfrak{G}_-)$ is a *Manin triple* (or, equivalently, $\mathfrak{G}_{Int} = \mathfrak{G}_-$ is a *Lie bialgebra*) (see, e.g., [4]). This means that (i) the algebra $\mathfrak{G}$ has an ad-invariant nondegenerate inner product ("Killing form"):

$$\langle A, B\rangle = \int \operatorname{res}(A \circ B)$$

for $A, B \in \mathfrak{G}$, (ii) as a linear space $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$, and (iii) both the subalgebras $\mathfrak{G}_+$ and $\mathfrak{G}_-$ are isotropic with respect to this Killing form. The Lie group G_{Int} corresponding to the Lie bialgebra $\mathfrak{G}_{Int} = \mathfrak{G}_-$ has a natural Lie–Poisson structure.

Similarly, the Lie algebra $S\mathfrak{G}_q = \{A = \sum_{k=-\infty}^{n} u_k(x) T^k \mid \operatorname{Tr} A = 0\}$, where $T = (q-1)D_q + 1$, is a sum of two isotropic subalgebras: of q-differential $(\mathfrak{G}_q^+ = \{\sum_{k=0}^{n} u_k(x) T^k \mid u_0 \in x\mathbb{C}[x]\})$ and of q-integral operators $(\mathfrak{G}_q^- = \{\sum_{k=-\infty}^{0} u_k(x) T^k \mid u_0 \in x^{-1}\mathbb{C}[x^{-1}]\})$ which are dual to each other with respect to the invariant inner product on $S\mathfrak{G}_q$. Indeed, the commutation relation $Tx = qxT$ implies that $(T^m x^n)_{m,n}$ and $(x^{-n}T^{-m})_{m,n}$ are two bases dual to each other with respect to the ad-invariant form on $\mathfrak{G}_q$. Therefore, $(S\mathfrak{G}_q, \mathfrak{G}_q^+, \mathfrak{G}_q^-)$ is also a Manin triple and $S\mathfrak{G}_q$ is a Lie bialgebra.

Remark 3.3. A likewise Manin triple there exists for the extension $\tilde{\mathfrak{G}}$ of $\mathfrak{G}$ by a central and the corresponding cocentral terms (see Section 4). The corresponding Poisson structures on the nonextended and extended Lie groups G_{Int} are respectively the Benney and quadratic Gelfand–Dickey structures [23], see Section 7.1. The latter Poisson structure is called quadratic W-algebra. Repeating the same arguments for the extension of the Lie algebra $S\mathfrak{G}_q$ by the central and cocentral terms, we get q-deformation of the Gelfand–Dickey (or W-) Poisson structure (see Section 7.2).

3.6. *Quantum and Classical W_∞-Algebras*

The Lie algebras of W_∞-type (having generators of all spins up to infinity) appeared recently in conformal field theory as the large n limit of Zamolodchikov's quadratic W_n-algebras. Deep relations observed between these objects often look very natural and transparent after translation of them into the language of symbols. A "short dictionary" for some of these objects is just like this.

The (quantum) $W_{1+\infty}$ algebra is isomorphic to the algebra $\hat{\mathfrak{G}}_{DO}$ which is the central extension of the algebra $\mathfrak{G}_{DO}$ of differential operators on the circle [33].

The quantum W_∞ algebra is the truncated $\hat{\mathfrak{G}}_{DO}$ where zero order differential operators (i.e. functions) are discarded. Historically this algebra appeared to be spanned by "compositions" of the Virasoro (i.e. first order) generators. In this construction the space of functions does not appear. This explains the funny from mathematical viewpoint notation $W_{1+\infty}$ for extension of W_∞ by functions.

Classical w_∞ algebra can be obtained from W_∞ by replacing all commutators by Poisson brackets. In the language above this is nothing but the Poisson algebra of symbols of differential operators.

The symbols of differential operators are functions on a cylinder, the cotangent bundle T^*S^1 of the circle. These symbols are polynomial along the fiber p. Generalization of DO to $\psi\ DO$ invites one to consider symbols being Laurent series in p with coefficients periodic in q (and so defined on $\{T^*S^1\backslash S^1\}$). Introducing an exponential coordinate along the fiber $p=e^{i\cdot s}$ we come to a complex two-torus with coordinates s and q. In these coordinates the outer derivation $\{\log p, -\}$ becomes $\{s, -\}$, the outer derivation of the Poisson algebra N. Moreover, the coordinates s and q are symmetric in this representation. This explains the introduction of $\log \partial$ in the next section.

Any symbol of a differential operator is a (Hamiltonian) function on T^*S^1 and it defines a Hamiltonian vector field on this manifold. In such a way the group and the algebra of area preserving (or symplectic) diffeomorphisms on T^*S^1 appears in the context of w_∞. Moreover, any diffeomorphism of S^1 can be lifted to a symplectomorphism of T^*S^1. In physical terms this means that the group′ $\mathrm{Diff}(S^1)$ is a symmetry of W_∞ algebra (see e.g. [19]).

Finally, note that in the same way one can treat algebras of (pseudo) differential operators on a higher dimensional compact manifold M^n. The algebra of Hamiltonian vector fields on T^*M^n appears as its classical limit, and the group $\mathrm{Diff}(M)$ as its symmetry (cf. [32]).

4. LIE ALGEBRA COCYCLES ON $\mathfrak{G}$ AND $\mathfrak{G}_q$

4.1. *Two-Cocycles*

Recall that the commutation rule

$$[\partial^\alpha, u(x)] = \sum_{k \geqslant 1} \binom{\alpha}{k} u^{(k)}(x) \partial^{\alpha - k}$$

makes sense not only for a positive integer α, but also for any complex values of α. Differentiating this identity in α at $\alpha = 0$ and using undoubtful

$$\left.\frac{d}{d\alpha}\right|_{\alpha=0} \partial^\alpha = \log \partial \cdot \partial^\alpha |_{\alpha=0} = \log \partial$$

we get the following commutation relation for a formal symbol $\log \partial$ and a function $u(x)$

$$[\log \partial, u(x)] = \sum_{k \geqslant 1} \frac{(-1)^{k+1}}{k} u^{(k)} \partial^{-k}$$

(or, more generally $[\log \partial, u(x) \partial^n] = \sum_{k \geqslant 1} ((-1)^{k+1}/k)\, u^{(k)} \partial^{n-k})$. Note that the result of the commutation lies in $\psi\, DO$. In such a way the symbol $\log \partial$ defines an outer derivation $\xi_{\log \partial}$ of the ring $\psi\, DO$, $\xi_{\log \partial} \in H^1(\mathfrak{G}, \mathfrak{G})$, where $\mathfrak{G}$ is the Lie algebra of pseudodifferential symbols. Pairing the result of derivation with another symbol we get a complex-valued Lie algebra 2-cocycle $c \in H^2(\mathfrak{G}, \mathbb{C})$.

THEOREM 4.1 [26]. *The following 2-cocycle*

$$c(A, B) = \int \mathrm{res}([\log \partial, A] * B) = \int \mathrm{res} \left(\sum_{k \geqslant 1} \frac{(-1)^{k+1}}{k} A_x^{(k)} \partial^{-k} \circ B \right), \qquad (4.1)$$

gives a nontrivial central extension $\hat{\mathfrak{G}}$ of the Lie algebra $\mathfrak{G}$. (Here A and B are arbitrary pseudodifferential symbols on S^1.) The restriction of this cocycle to the subalgebra of differential operators $\mathfrak{G}_{DO}$ gives the nontrivial central extension of $\mathfrak{G}_{DO}$.

The restriction of this cocycle to the subalgebra of vector fields (considered as differential operators of the first order) is the Gelfand–Fuchs cocycle of the Virasoro algebra. This observation actually implies non-triviality of the cocycle on $\mathfrak{G}$ and $\mathfrak{G}_{DO}$. The "logarithmic" cocycle on the Lie algebra $\mathfrak{G}_{DO}$ can be also identified with the central charge of the $W_{1+\infty}$ algebra in conformal field theory ([1]).

Remark 4.1. Assume for a moment that $\log \partial$ were an element of the algebra $\mathfrak{G}$. Then we could define not only the commutator $[\log \partial, A]$ but also a product $\log \partial \circ A$, and rewrite the cocycle $c(A, B) = \int \operatorname{res}([\log \partial, A] \circ B)$ as $c(A, B) = \int \operatorname{res}(\log \partial \circ [A, B])$. The last form means that the cocycle $c(A, B)$ is a 2-coboundary (and hence is trivial) because it is a linear function of the commutator: $c(A, B) = \langle \log \partial, [A, B] \rangle$. Recalling $\log \partial \notin \mathfrak{G}$, we get a heuristic proof of non-triviality of the cocycle.

Remark 4.2. Notice also that the Lie algebra $\mathfrak{G}_{DO}$ of differential operators on the circle has exactly one central extension ([8, 31, 40]), while the Lie algebra of pseudodifferential operators has exactly two independent central extensions (personal communication by B. L. Feigin). The formula for the second cocycle can be written in a similar to (4.1) form (see [23]):

$$c'(A, B) = \int \operatorname{res}([\log x, A] \circ B)$$

for coefficients $a(x) \in \mathbb{C}[x, x^{-1}]$ or

$$c'(A, B) = \int \operatorname{res}([x, A] \circ B)$$

for coefficients $a(x) \in C^{\infty}(S^1)$.

Here x is a natural coordinate on the universal covering of S^1, considered as a multivalued function on S^1. Note that here, as well as in the case of $\log \partial$, the formal symbol x is not an element of $\mathfrak{G}$ but the commutator $[x, A]$ is.

4.2. *Derivations of the Algebra of q-Pseudodifferential Operators*

Analogously to the classical case, described in Section 4.1, we define here derivations of the deformed algebra $\mathfrak{G}_q$ by the symbols $\log D_q$ and $\log x$.

4.2.1. *The Derivation* $\log D_q$. The classical case again gives us a hint how to define the action of the operator $\log D_q$ on q-pseudodifferential symbols $A(x, D_q) \in \mathfrak{G}_q$. We consider the corresponding Lie group of q-symbols and its tangent structure.

PROPOSITION 4.2. *q-pseudodifferential symbols of the form* $u = D_q^{\alpha} + \sum_{k \geq 1} u_k(x) D_q^{\alpha - k}$ *constitute an infinite dimensional Lie group* G_q *acting by* *automorphisms on the associative algebra* $\psi\, DO_q$, $A \mapsto uAu^{-1}$, $u \in G_q$. *The corresponding tangent Lie algebra* $\mathfrak{G}_q$ *is generated by integral q-operators and a symbol* $\log D_q$: $\mathfrak{G}_q = \{\sum_{j=-\infty}^{-1} a_j(x) D_q^j + \lambda\mu \log D_q\}$, *where the commutation relations for* ΨDO_q *were defined above and commutator of* $\log D_q$ *with a symbol* $A = x^n D_q^p$ *is*

$$[\log D_q, x^n D_q^p] \overset{def}{=} (\log q)\, n x^n D_q^p + \sum_{k \geqslant 1} \frac{(-1)^{k-1}}{k} (\tau^{-k}(D_q^k x^n))\, D_q^{p-k} \tag{4.2}$$

$$= (\log q)\, n x^{n\,p}_{\;\,q} + \sum_{k \geqslant 1} \frac{(-1)^{k-1}}{k} q^{-kn} (n)_q^k\, x^n D_q^{p-k}$$

Remark 4.3. Analogously to the classical case, the algebra of integral operators is closed under the commutations with $\log D_q$. The main and crucial difference with the classics is the presence of the first term $(k=0)$ in the formula for the $\log D_q$. This shows that the group G_q of q-analogs is rather "solvable" and its properties are much different from the "unipotent" G_{Int} in the classical case.

Proof of Proposition 4.2. Indeed,

$$\lim_{\varepsilon \to 0} \frac{D_q^{\varepsilon}(u D_q^p) D_q^{-\varepsilon} - u D_q^p}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \sum_{k \geqslant 0} \binom{\varepsilon}{k} (\tau^{\varepsilon - k}(D_q^k u))\, D_q^{p-k} - u D_q^p \right\}$$

$$= \lim_{\varepsilon \to 0} \frac{\tau^{\varepsilon} u - u}{\varepsilon} D_q^p + \sum_{k \geqslant 1} \frac{(-1)^{k-1}}{k} (\tau^{-k}(D_q^k x^n))\, D_q^{p-k}. \quad \blacksquare$$

The formula for the action of $\log D_q$ can be written also as the following version of the Leibnitz formula:

$$[\log D_q, A] = (\log q)\, nA + \sum_{k \geqslant 1} \frac{1}{k!} \left(\frac{d^{k-1}}{dD_q^{k-1}} D_q^{-1} \right) * (D_q^k A) \tag{4.3}$$

where $A = x^n D_q$. Note that the two logarithmic terms do not cancel out in (4.3) as it was in the classical case (see [K-K]), but generate the first right hand side term.

PROPOSITION 4.3. *The action of* $\log D_q$ *is a derivation of the associative algebra* $\psi\, DO_q$:

$$[\log D_q, A \circ B] = A \circ [\log D_q, B] + [\log D_q, A] \circ B.$$

Proof. This immediately follows from the fact that $\log D_q$ is a tangent vector to the Lie group G_q. $\blacksquare$

The inner product of Theorem 3.3 allows one to define a 2-cocycle on the corresponding Lie algebra:

$$c_D(A, B) = \int \operatorname{res}([\log D_q, A]\, BE).$$

PROPOSITION 4.4. *The 2-cochain $c_D(A, B)$ is antisymmetric and satisfies the cocycle identity.*

Proof. First note that $\int \operatorname{res}([\log D_q, A]) = 0$ for any $A \in \mathfrak{G}_q$ by virtue of (4.2). Indeed, $\int nx^n = 0$ for all $n \in \mathbb{Z}$ and $\int \tau^{-k}(D_q^k f) = 0$ for all $k > 0$. This implies skew-symmetry:

$$\begin{aligned} 0 &= \int \operatorname{res}([\log D_q, ABE]) \\ &= \int \operatorname{res}([\log D_q, A]\, BE) + \int \operatorname{res}(A[\log D_q, B]\, E) \\ &= c_D(A, B) + c_D(B, A) \end{aligned}$$

because of $[\log D_q, E] = 0$ and by Theorem 3.3. The cyclic cocycle property follows from the antisymmetry identity and Proposition 4.3. ∎

Remark 4.4. The restriction of this Lie 2-cocycle to the Lie algebra of q-differential operators (containing polynomials in D_q only) gives a non-trivial cocycle, cohomologous to the q-deformation of the Kac–Peterson cocycle [20] constructed by Kassel [22].

PROPOSITION 4.5. *In the Kac–Radul construction* (*see Section* 3.3.4) *for the trace on the associative algebra* $A = \mathbb{C}[\xi, \xi^{-1}]]$

$$\operatorname{Tr}_D\left(\sum_{p=-\infty}^{N} a_p \xi^p\right) = \sum_{p=0}^{N} a_p \frac{(-1)^p}{(q-1)^{p+1}} \left\{\log q + \sum_{k=1}^{p} \frac{(1-q)^k}{k}\right\},$$

the 2-cocycle $\psi_{\sigma, \mathrm{Tr}}$ induces on $\mathfrak{G}_q$ the same cocycle c_D described above.

Proof. Compute explicitly the cocycle c_D:

$$\begin{aligned} &c_D(x^m \sigma^m(D_q^a), x^n D_q^b) \\ &\quad = \int \operatorname{res}(x^n D_q^b \circ [\log D_q, x^m]\, \sigma^m(D_q)^a \circ E) \\ &\quad = \delta_{n+m, 0} \operatorname{res}\left\{\sigma^m(D_q)^{a+b}\left(m \log q + \sum_{k>0} \frac{(-1)^{k-1}}{k}\, q^{-km} (m)_q^k\, D_q^{-k}\right) E\right\}. \end{aligned}$$

On the other hand

$$\psi(x^m\sigma^m(\xi^a), x^n\xi^b) = \delta_{n+m,0}\operatorname{Tr}_D(1+\sigma+\cdots+\sigma^{m-1})\xi^{a+b}$$

for $m>0$. This gives an equation to determine Tr_D

$$\begin{aligned}\operatorname{Tr}_D(1+\sigma+\cdots+\sigma^{m-1})\xi^c\\ = \operatorname{res}\{\sigma^m(\xi)^c\,(m\log q+\log(1+q^{-m}(m)_q\,\xi^{-1}))\,E\}.\end{aligned} \tag{4.4}$$

Put $m=1$ and define Tr_D via the right hand side

$$\operatorname{Tr}_D\xi^c = \operatorname{res}\left\{(q\xi+1)^c\left(\log q+\sum_{k>0}\frac{(-1)^{k-1}}{k}q^{-k}\xi^{-k}\right)((q-1)\xi+1)^{-1}\right\}$$

in particular, $\operatorname{Tr}_D\xi^c=0$ if $c<0$.

We want to prove (4.4) for $m>1$. Summing up these equations with weights $\sum_{c=0}^{b}\binom{b}{c}(q-1)^c$ we get an equivalent form

$$\begin{aligned}\operatorname{Tr}_D(1+\sigma+\cdots+\sigma^{m-1})(1+(q-1)D)^b\\ = \operatorname{res}\{(1+(q-1)\sigma^m(\xi))^b\log(q^m+(m)_q\xi^{-1})\,E\}.\end{aligned}$$

Since $\sigma(1+(q-1)\xi)=q(1+(q-1)\xi)$ this is the same as

$$(m)_{q^b}\operatorname{Tr}_D(1+(q-1)\xi)^b = q^{mb}\operatorname{res}\{(1+(q-1)\xi)^{b-1}\log(q^m+(m)_q\,\xi^{-1})\}.$$

The definition of Tr_D gives

$$\begin{aligned}\operatorname{Tr}_D(1+(q-1)\xi)^b &= \operatorname{res}\{(1+(q-1)(q\xi+1))^b\log(q+\xi^{-1})\,E\}\\ &= q^b\operatorname{res}\{(1+(q-1)\xi)^{b-1}\log(q+\xi^{-1})\}.\end{aligned}$$

So the identity to prove is

$$\begin{aligned}(m)_{q^{-b}}\operatorname{res}\{(1+(q-1)\xi)^{b-1}\log(q+\xi^{-1})\}\\ = \operatorname{res}\{(1+(q-1)\xi)^{b-1}\log(q^m+(m)_q\,\xi^{-1})\}\end{aligned}$$

for $b\in\mathbb{Z}_{>0}$. Computing the residue by differentiation with respect to q^{-m} we see that the both sides are equal to $b^{-1}(q^{-mb}-1)(1-q)^{-1}$. This gives a simple expression for Tr_D

$$\operatorname{Tr}_D(1+(q-1)\xi)^b = \frac{(b)_q}{b}\qquad\text{for}\quad b>0,$$

$$\operatorname{Tr}_D 1 = \frac{\log q}{q-1},$$

whence

$$\operatorname{Tr}_D \xi^p = (-1)^p \frac{\log q}{(q-1)^{p+1}} + \frac{1}{(q-1)^p} \sum_{k=1}^{p} (-1)^{p-k} \binom{p}{k} \frac{(k)_q}{k}$$
$$= \frac{(-1)^p}{(q-1)^{p+1}} \left(\log q + \sum_{k=1}^{p} (-1)^k \binom{p}{k} \left(\frac{q^k}{k} - \frac{1}{k} \right) \right) \tag{4.5}$$

for $p \in \mathbb{Z}_{\geqslant 0}$. Denote $f(q) = \sum_{k=1}^{p} (-1)^k \binom{p}{k} q^k/k$. Since $qf'(q) = (1-q)^p - 1$ we can write $f'(q) = -\sum_{n=0}^{p-1} (1-q)^n$. Plugging $f(q) - f(1) = \sum_{n=1}^{p} (1-q)^n/n$ to (4.5) we obtain

$$\operatorname{Tr}_D \xi^p = \frac{(-1)^p}{(q-1)^{p+1}} \left(\log q + \sum_{k=1}^{p} \frac{(1-q)^k}{k} \right). \quad \blacksquare$$

Remark 4.5. For q close to 1 and $p \geqslant 0$ this trace can be presented as

$$\operatorname{Tr}_D \xi^p = \sum_{m=0}^{\infty} \frac{(1-q)^m}{m+p+1}.$$

In particular, when $q \to 1$ the trace Tr_D is non-singular and $\lim_{q \to 1} \operatorname{Tr}_D \xi^p = 1/(p+1)$.

4.2.2. *The derivation* $\log x$. Similarly, one can define a twisted derivation l_x:

$$l_x(fD_q^a) = af(x) D_q^{a-1} = \frac{d}{dD_q} (fD_q^a)$$

for any $A \in \psi DO_q$.

PROPOSITION 4.6. *The map l_x is a twisted derivation of the associative algebra ψDO_q:*

$$l_x(AB) = l_x(A)\, \tau(B) + A \cdot l_x(B)$$

Proof is a straightforward calculation. $\blacksquare$

COROLLARY 4.7. *The map* $[\log x, -]$ *determined by*

$$[\log x, A] = -l_x(A) E^{-1} = -l_x(A)((q-1)D_q + 1)$$

is a derivation of the associative algebra ψDO_q.

Proof. Indeed,

$$\begin{aligned}[\log x, AB] &= -l_x(AB)E^{-1} \\ &= -l_x(A)\, E^{-1}BEE^{-1} - A\cdot l_x(B)E^{-1} \\ &= [\log x, A]B + A[\log x, B]. \quad \blacksquare\end{aligned}$$

PROPOSITION 4.8. *The bilinear form*

$$c_x(A, B) = \int \operatorname{res}(A[\log x, B]E) = -\int \operatorname{res}(A \circ l_x(B))$$

is a skew-symmetric cyclic 2*-cocycle.*

Proof. Indeed,

$$0 = \int \operatorname{res}(l_x(AB)) = \int \operatorname{res}(l_x(A)\,\tau(B)) + \int \operatorname{res}(Al_x(B))$$

$$= \int \operatorname{res}(Bl_x(A)) + \int \operatorname{res}(Al_x(B))$$

implies that c_x is skew-symmetric. Also,

$$c_x(A, BC) = -\int \operatorname{res}(ABl_x(C)) - \int \operatorname{res}(Al_x(B)\,\tau(C))$$

$$= c_x(AB, C) + c_x(CA, B)$$

is the cyclic cocycle property. ∎

When restricted to pure differential operators, the cocycle c_x vanishes.

Remark 4.6. In the case of the algebra $C^\infty(S^1)$ we can introduce a coordinate θ on the universal covering $\mathbb{R}^1$ of the circle S^1 so that $x = e^{i\theta}$. This is not a function from $C^\infty(S^1)$, but for any $A \in \mathfrak{G}_q$ we can form a commutator $[\theta, A] \in \mathfrak{G}_q$.

Now we find the commutator of the two derivations.

PROPOSITION 4.9. *For any* $A \in \mathfrak{G}_q$

$$[\log D_q, [\log x, A]] - [\log x, [\log D_q, A]] = [D_q^{-1}, A].$$

Proof. Assume $A = x^n D_q^a$. The left hand side is

$$[\log D_q, [\log x, x^n D_q^a]] - [\log x, [\log D_q, x^n D_q^a]]$$
$$= -a \left\{ (\log q)\, nx^n + \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} q^{-kn} (n)_q^k x^n D_q^{-k} \right\} (D_q^{a-1} + (q-1) D_q^a)$$
$$+ a(\log q)\, nx^n (D_q^{a-1} + (q-1) D_q^a)$$
$$+ \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} q^{-kn} (n)_q^k (a-k)\, x^n (D_q^{a-k-1} + (q-1) D_q^{a-k})$$
$$= x^n (D_q^{a-1} + (q-1) D_q^a) \sum_{k \geq 1} (-1)^k q^{-kn} (n)_q^k D_q^{-k}$$
$$= -(n)_q\, x^n (D_q^{a-1} + (q-1) D_q^a)(q^n D_q + (n)_q)^{-1}.$$

The right hand side is

$$[D_q^{-1}, x^n D_q^a] = x^n (q^n D_q + (n)_q)^{-1} D_q^a - x^n D_q^{a-1}$$
$$= x^n (q^n D_q + (n)_q)^{-1} D_q^{a-1} ((1-q^n) D_q - (n)_q)$$
$$= (n)_q\, x^n (q^n D_q + (n)_q)^{-1} D_q^{a-1} ((1-q) D_q - 1)$$

which coincides with the expression above. ▮

COROLLARY 4.10. *One can equip the vector space* $\bar{\mathfrak{G}}_q = \mathfrak{G}_q \oplus \mathbb{C} \log D_q \oplus \mathbb{C} \log x$ *with a Lie algebra structure such that* $\mathfrak{G}_q$ *is an ideal and*

$$[\log D_q, \log x] = D_q^{-1}.$$

There exists an exact sequence of Lie algebras $0 \to \mathfrak{G}_q \to \bar{\mathfrak{G}}_q \to \mathbb{C}^2 \to 0$.

4.3. 3-*Cocycles on* $\mathfrak{G}$ *and* $\mathfrak{G}_q$

Since the Lie algebra $\mathfrak{G}_q$ has an invariant form, it inherits also the standard 3-cocycle $c_0^q(L, M, N) = \int \operatorname{res}([L, M]\, NE)$ defined for $q \neq 1$, 0. For $q = 1$ there is a cocycle $c_0^1(L, M, N) = \int \operatorname{res}([L, M]\, N)$ on $\mathfrak{G}$.

PROPOSITION 4.11. *The cocycle* c_0^q *is non-singular as* $q \to 1$ *and, moreover,*

$$\lim_{q \to 1} c_0^q = c_0^1.$$

Proof. It suffices to check this for $L = x^l D_q^a$, $M = x^m D_q^b$, $N = D_q^c \circ x^n$. Change for $q \neq 1$ the variable D_q to $T = (q-1) D_q + 1$, then ψDO_q consists of formal power series $\sum_{k=-\infty}^{K} u_k(x) T^k$. Define $\operatorname{res}_T \colon \psi DO_q \to F$,

$\mathrm{res}_T(\sum_{k=-\infty}^{K} u_k(x)\, T^k) = u_{-1}(x)$, then $\mathrm{res}_T = (q-1)\,\mathrm{res}$. Substituting $D_q = T(1-T^{-1})/(q-1)$ we find

$$\int \mathrm{res}([L, M]\, NE)$$

$$= \frac{1}{q-1} \int \mathrm{res}_T([L, M]\, NT^{-1})$$

$$= \frac{1}{(q-1)^{a+b+c+1}}$$

$$\times \int \mathrm{res}_T\{x^n[x^l T^a(1-T^{-1})^a, x^m T^b(1-T^{-1})^b]\, T^c(1-T^{-1})^c\, T^{-1}\}$$

$$= \frac{1}{(q-1)^{a+b+c+1}}$$

$$\times \int \mathrm{res}_T\{x^{l+m+n}((q^m T)^a\, (1-q^{-m}T^{-1})^a\, T^b(1-T^{-1})^b$$

$$-(q^l T)^b\, (1-q^{-l}T^{-1})^b\, T^a(1-T^{-1})^a)\, T^{c-1}(1-T^{-1})^c\}$$

$$= \frac{\delta_{l+m+n,0}}{(q-1)^{a+b+c+1}}$$

$$\times \mathrm{res}_T \left\{ q^{ma} T^{a+b+c-1} \sum_{k \geqslant 0} (-1)^k \binom{a}{k} \right.$$

$$\times q^{-mk} T^{-k} \cdot \sum_{p \geqslant 0} (-1)^p \binom{b+c}{p} T^{-p}$$

$$\left. - q^{lb} T^{a+b+c+1} \sum_{k \geqslant 0} (-1)^k \binom{b}{k} q^{-lk} T^{-k} \cdot \sum_{p \geqslant 0} (-1)^p \binom{a+c}{p} T^{-p} \right\}$$

$$= (-1)^{a+b+c} \frac{\delta_{l+m+n,0}}{(q-1)^{a+b+c+1}}$$

$$\times \left\{ \sum_{\substack{k+p=a+b+c \\ k,p \geqslant 0}} \binom{a}{k}\binom{b+c}{p} q^{m(a-k)} - \sum_{\substack{k+p=a+b+c \\ k,p \geqslant 0}} \binom{b}{k}\binom{a+c}{p} q^{l(b-k)} \right\}$$

For $s \in \mathbb{Z}_{\geqslant 0}$, $a \in \mathbb{Z}$ introduce the function

$$f_a(a) = \sum_{\substack{k+p=s \\ k,p \geqslant 0}} \binom{a}{k}\binom{s-a}{p} z^{a-k}.$$

It is the coefficient at t^s in the decomposition

$$(z+t)^a (1+t)^{s-a} = \sum_{m\geqslant 0} t^m \sum_{\substack{k+p=s\\ k,p\geqslant 0}} \binom{a}{k}\binom{s-a}{p} z^{a-k}. \tag{4.6}$$

Differentiating the last relation with respect to z we find

$$f_a(1)=1,$$

$$\frac{d^n}{dz^n} f_a(z)\bigg|_{z=1} = n!\binom{a}{n}\binom{s-n}{s},$$

in particular, $f_a^{(n)}(1)=0$ for $1\leqslant n\leqslant s$. Therefore, $f_a(a)=1+(z-1)^{s+1} g_a(z)$ for some Laurent polynomial $g_a(z)$, such that $g_a(1)=(-1)^s\binom{a}{s+1}$.

Assuming $s=a+b+c\geqslant 0$ and plugging $f_a(q^m)-f_b(q^l)$ into $c_0^q(L, M, N)$ we see that the limit exists and

$$\lim_{q\to 1}\int \operatorname{res}([L, M]NE)$$

$$=\delta_{l+m+n,0}\left\{\binom{a}{a+b+c+1} m^{a+b+c+1} - \binom{b}{a+b+c+1} l^{a+b+c+1}\right\}.$$

Now we calculate the value of c_0^1 on $L=x^lD^a$, $M=x^mD^b$, $N=D^c\circ x^n$ when $q=1$:

$$\int \operatorname{res}([L, M]N)$$

$$=\int \operatorname{res}([x^lD^a, x^mD^b]\, D^cx^n)$$

$$=\int \operatorname{res}\{x^{l+m+n}((D+m)^a D^b-(D+l)^b D^a)\, D^c\}$$

$$=\delta_{l+m+n,0} \operatorname{res}\left\{\sum_{k\geqslant 0}\binom{a}{k} m^k D^{a-k+b+c} - \sum_{k\geqslant 0}\binom{b}{k} l^k D^{b-k+a+c}\right\}$$

$$=\delta_{l+m+n,0}\left\{\binom{a}{a+b+c+1} m^{a+b+c+1} - \binom{b}{a+b+c+1} l^{a+b+c+1}\right\}$$

which coincides with the previous limit. ▮

Remark 4.7. If a, b, $a+c$, $b+c\geqslant 0$, then $c_0^q(x^lD_q^a, x^mD_q^b, D_q^c\circ x^n)=0$. Indeed, as follows from (4.6) $f_a(z)=1$ for $0\leqslant a\leqslant s$. In particular, the restriction of c_0^q to the subalgebra DO_q vanishes identically.

Two outer derivations $\xi = \operatorname{ad} \log D_q$ and $\eta = \operatorname{ad} \log x \in H^1(\mathfrak{G}_q, \mathfrak{G}_q)$ produce not only two complex valued two-cocycles. They also induce a 2-cocycle $f = \eta \wedge \xi = m(\eta \boxtimes \xi - \xi \boxtimes \eta)$ in $Z^2(\mathfrak{G}_q, \mathfrak{G}_q)$: it can be considered as the cup product of those two classes.

Remark 4.8. In contrast to the classical case of Poisson algebra, the cocycle f is non-trivial, at least for $q = 1$, as the following argument shows. Since $\mathfrak{G}$ is isomorphic to its own dual as a representation of itself, one has $f \in H^2(\mathfrak{G}, \mathfrak{G}^*)$ and one can check that its restriction to the Lie algebra of vector fields $\mathfrak{A}(S^1)$ is non-trivial: one has $I^*(f) \in H^2(\mathfrak{A}(S^1), \mathfrak{G})$ but as an $\mathfrak{A}(S^1)$-module $\mathfrak{G}^*$ splits into a direct sum $\mathfrak{G}^* = \bigoplus_{n \in \mathbb{Z}} \mathscr{F}_n$, where $\mathscr{F}_n$ is the space of n-densities. Then $I^*(f) \in H^2(\mathfrak{A}(S^1), \mathscr{F}_2)$ and one computes easily

$$I^*(f)\left(a \frac{\partial}{\partial x}, b \frac{\partial}{\partial x}\right) = (a''b' - a'b'')\, dx^2.$$

The non-triviality of the cohomology class of this cocycle can either be proved directly using Fuchs' computations [9] for non trivial representations of $\mathfrak{A}(S^1)$ or either remark that $\mathscr{F}_2 = \mathfrak{A}(S^1)^*$ and this class $I^*(f)$ then induces the Godbillon–Vey generator in $H^3(\mathfrak{A}(S^1), \mathbb{R})$.

The cocycle f will give rise to a nontrivial complex valued three-cocycle on the Lie algebra of pseudodifferential symbols $\mathfrak{G}_q$ via the map θ.

THEOREM 4.12. *The complex valued cochain on $\mathfrak{G}_q$ defined by*

$$c_{GV}(L, M, N) = \operatorname*{Alt}_{L, M, N} \int \operatorname{res}((\xi(M)\, N\eta(L) - M\xi(N)\, \eta(L) - LMND_q^{-1})\, E)$$

$$= \operatorname*{Alt}_{L, M, N} \int \operatorname{res}(M\xi(N)\, l_x(L) - \xi(M)\, Nl_x(L) - LMND_q^{-1} + (q-1)\, LMNE)$$

is a three-cocycle, defined for $q \neq 0, 1$ and having a limit as $q \to 1$ which is a non trivial cocycle on $\mathfrak{G}$

$$c_{GV}(L, M, N) = \operatorname*{Alt}_{L, M, N} \int \operatorname{res}(\xi(M)\, N_\eta(L) - M\xi(N)\, \eta(L) - LMND^{-1}).$$

Proof. Consider the 3-cochain

$$\theta(f)(L, M, N) = \operatorname*{Alt}_{L, M, N} \operatorname{Tr}(\eta(L)\, \xi(M) N - \xi(L)\, \eta(M) N)$$

(sum over 6 permutations). Its coboundary is given by the Lemma 2.3

$$d\theta(f)(L, M, N, P)$$

$$= \operatorname*{Alt}_{L, M, N, P} \operatorname{Tr}(\eta(L)\,\xi(M\,[N, P] - \xi(L)\,\eta(M)[N, P]) \quad \text{(12 shuffles)}$$

$$= \operatorname*{Alt}_{L, M, N, P} \operatorname{Tr}(\eta(L)\,\xi(M)\,NP - \xi(L)\,\eta(M)\,NP) \qquad \text{(24 permutations)}$$

$$\overset{(1)}{=} \tfrac{1}{2} \operatorname*{Alt}_{L, M, N, P} \operatorname{Tr}([\eta, \xi](L)\,MNP) \qquad \text{(24 permutations)}$$

$$= -\tfrac{1}{2} \operatorname*{Alt}_{L, M, N, P} \operatorname{Tr}([D_q^{-1}, L]\,MNP) \qquad \text{(24 permutations)}$$

$$= - \operatorname*{Alt}_{L, M, N, P} \operatorname{Tr}(D_q^{-1} LMNP) \qquad \text{(24 permutations)}$$

$$\overset{(2)}{=} -da(L, M, N, P)$$

where

$$a(L, M, N) = \operatorname*{Alt}_{L, M, N} \operatorname{Tr}(D_q^{-1} LMN) \qquad \text{(sum over 6 permutations).}$$

Here we used Proposition 4.9 and relation (1) is deduced by "integration by parts", that is from the property $\operatorname{Tr}(\xi(A)) = 0$, $\operatorname{Tr}(\eta(A)) = 0$ for any $A \in \mathfrak{G}_q$. Equation (2) illustrates the difference between odd and even cochains.

This implies that

$$c_{GV}(L, M, N) = \theta(f)(L, M, N) + a(L, M, N)$$

is a 3-cocycle well defined for $q \neq 0, 1$. The formula

$$c_{GV}(L, M, N) = \operatorname*{Alt}_{L, M, N} \int \operatorname{res}(M\xi(N)\, l_x(L) - \xi(M)\, Nl_x(L) \\ - LMND_q^{-1}((q-1)\, D_q + 1)^{-1})$$

shows that the cocycle has a limit as $q \to 1$ iff its last summand

$$\operatorname*{Alt}_{L, M, N} \int \operatorname{res}(LMND_q^{-1}((q-1)\, D_q + 1)^{-1})$$

has a limit. Since

$$\frac{1}{D_q((q-1)\, D_q + 1)} = \frac{1}{D_q} - \frac{q-1}{(q-1)\, D_q + 1} \tag{4.7}$$

we have

$$\begin{aligned}\operatorname*{Alt}_{L,M,N}&\int \operatorname{res}(LMND_q^{-1}E)\\ &=\operatorname*{Alt}_{L,M,N}\int \operatorname{res}(LMND_q^{-1})-(q-1)\operatorname*{Alt}_{L,M,N}\int \operatorname{res}(LMNE)\\ &=\operatorname*{Alt}_{L,M,N}\int \operatorname{res}(LMND_q^{-1})-3(q-1)\,c_0^q(L,M,N).\end{aligned}$$

By Proposition 4.11, c_0^q is not singular as q tends to 1. Therefore,

$$\begin{aligned}c_{GV}(L,M,N)=\operatorname*{Alt}_{L,M,N}&\int \operatorname{res}(M\xi(N)\,l_x(L)-\xi(M)\,Nl_x(L)-LMND_q^{-1})\\ &+3(q-1)\,c_0^q(L,M,N)\end{aligned}$$

is not singular either and

$$\lim_{q\to 1} c_{GV}(L,M,N)=\operatorname*{Alt}_{L,M,N}\int \operatorname{res}(M\xi(N)\,l_x(L)-\xi(M)\,Nl_x(L)-LMND^{-1}).$$

A direct proof of the cocycle identity for the latter cocycle on $\mathfrak{G}$ can be obtained also via the same reasoning as for $q\neq 1$.

Non-triviality of the cocycle for $q=1$ follows from explicit calculation. Computing it on the first order difference operators $L=x^lD_q$, $M=x^mD_q$, $N=x^nD_q$ we get

$$c_{GV}(x^lD_q,x^mD_q,x^nD_q)=\tfrac{1}{2}\delta_{l+m+n,0}\operatorname*{Alt}_{l,m,n} q^{-n}(m)_q\,(n)_q^2.$$

As $q\to 1$ the operator D_q tends to $D=x\partial_x$ and L, M, N tend to vector fields on S^1. Therefore, the restriction of this cocycle at $q=1$ to the algebra of vector fields $\mathfrak{A}(S^1)$ is

$$c_{GV}(fD,gD,hD)=\frac{1}{2}\int_{S^1}\begin{vmatrix} f & Df & D^2f\\ g & Dg & D^2g\\ h & Dh & D^2h\end{vmatrix}\frac{dx}{x}$$

which is the generator of the $H^3(\mathfrak{A}(S^1),\mathbb{C})$ found in [11]. ∎

The Godbillon–Vey form of this cocycle after restriction to vector fields suggests the name “quantum” Godbillon–Vey 3-cocycle for $c_{GV}(L,M,N)$.

PROPOSITION 4.13. *The cocycle c_{GV} extends to a 3-cocycle on the Lie algebra $\mathfrak{G}_q \oplus \mathbb{R} \log D_q$ by the formula*

$$c_{GV}(\log D_q, L, M) = 3 \int \operatorname{res}\{(L[\log D_q, M] D_q^{-1} + L D_q^{-1} [\log D_q, M]) E\}$$

for $L, M \in \psi DO_q$, $q \neq 0, 1$. As $q \to 1$ the cocycle is not singular and

$$\lim_{q \to 1} c_{GV}(\log D_q, L, M) = 3 \int \operatorname{res}\{L[\log D, M] D^{-1} + L D^{-1} [\log D, M]\}.$$

Proof. Since $\operatorname{Tr}(D_q^\alpha L D_q^{-\alpha}) = \operatorname{Tr} L$ for $\alpha \in \mathbb{R}$,

$$\begin{aligned} c_{GV}([\log D_q, L], M, N) &+ c_{GV}(L, [\log D_q, M], N) \\ &+ c_{GV}(L, M, [\log D_q, N]) \\ &= \frac{d}{d\alpha} c_{GV}(D_q^\alpha L D_q^{-\alpha}, D_q^\alpha M D_q^{-\alpha}, D_q^\alpha N D_q^{-\alpha})|_{\alpha=0} \\ &= \operatorname*{Alt}_{L, M, N} \operatorname{Tr}((\xi(M) N - M \xi(N))[\eta, \xi](L)) \\ &= \operatorname*{Alt}_{L, M, N} \operatorname{Tr}(\xi(L)[D_q^{-1}, M] N - [D_q^{-1}, L]\, \xi(M) N) \\ &= d a_2(L, M, N) \end{aligned}$$

the cochain a_2 being defined as

$$a_2(L, M) = 3 \operatorname{Tr}(\xi(L)\, D_q^{-1} M + D_q^{-1} \xi(L) M)$$

(we used the obvious $\xi(D_q^{-1}) = 0$). Therefore, extending c_{GV} to a cochain on $\mathfrak{G}_q \oplus \mathbb{R} \log D_q$ by $c_{GV}(\log D_q, L, M) = -a_2(L, M)$ we obtain a cocycle.

Using again (4.7) we transform the cocycle to

$$\begin{aligned} c_{GV}&(\log D_q, L, M) \\ &= 3 \int \operatorname{res}\{(L[\log D_q, M] + [\log D_q, M] L)(D_q^{-1} - (q-1) E)\} \\ &= 3 \int \operatorname{res}\{(L[\log D_q, M] + [\log D_q, M] L) D_q^{-1}\} \\ &\quad - 6(q-1)\, c_D(M, L). \end{aligned}$$

This is non-singular because c_D is not singular by Proposition 4.5, whence the formula for the limit follows. ∎

4.4 *Higher Cocycles on* $\mathfrak{G}$ *and* $\mathfrak{G}_q$

Gelfand and Mathieu constructed a family of Lie algebra cocycles in the following situation. Let A be an associative algebra and $\mathfrak{h}$ be an abelian Lie subalgebra of Der A. Let Tr: $A \to \mathbb{C}$ be $\mathfrak{h}$-invariant trace. Then there are some cocycles from $Z^k((A, [\ ,\ ]), \mathbb{C})$ associated with these data (see [12]). We consider similar case of 2-dimensional subspace $\mathfrak{h} \subset \operatorname{Der} \mathfrak{A}$ such that its image in Out $A = \operatorname{Der} A/\operatorname{Inn} A$ is an abelian Lie subalgebra. That is, we consider two derivations ξ, η such that their commutator is inner. The results are formulated in the particular case $A = \psi DO_q$, $\xi = \operatorname{ad} \log D_q$, $\eta = \operatorname{ad} \log x$.

Proposition 4.14. *For any odd $n \geqslant 1$ there is a cocycle in $Z^n(\mathfrak{G}_q, \mathbb{C})$*

$$c^{(n)}(X_1, ..., X_n) = \operatorname*{Alt}_{X_1, ..., X_n} \operatorname{Tr}(X_1 \cdots X_n).$$

For $n \geqslant 3$ it is continuous at $q = 1$:

$$\lim_{q \to 1} c^{(n)}(X_1, ..., X_n) = c^{(n)}|_{q=1} (\lim_{q \to 1} X_1, ..., \lim_{q \to 1} X_n).$$

For $n \geqslant 3$ (and for $n = 1$ if $q = 1$) its restriction to DO_q vanishes.

Proof. The first claim is standard (cf. [12]). The other claims follow from a particular case $n = 3$, considered in Proposition 4.11 and Remark 4.7. ∎

Proposition 4.15. *For any even $n \geqslant 2$ there are cocycles in $Z^n(\mathfrak{G}_q, \mathbb{C})$*

$$c_D^{(n)}(X_1, ..., X_n) = \operatorname*{Alt}_{X_1, ..., X_n} \operatorname{Tr}([\log D_q, X_1] X_2 \cdots X_n),$$

$$c_x^{(n)}(X_1, ..., X_n) = \operatorname*{Alt}_{X_1, ..., X_n} \operatorname{Tr}([\log x, X_1] X_2 \cdots X_n).$$

They are continuous at $q = 1$:

$$\lim_{q \to 1} c_D^{(n)} = c_D^{(n)}|_{q=1}, \qquad \lim_{q \to 1} c_x^{(n)} = c_x^{(n)}|_{q=1}.$$

The restriction of $c_x^{(n)}$ to DO_q vanishes.

Proof. The fist claim follows from results of Gelfand and Mathieu [12]. Continuity of c_D follows from the particular case $n = 2$ (see Proposition 4.5 and Remark 4.5). The claims about c_x follow from the presentation

$$c_x^{(n)}(X_1, ..., X_n) = -\operatorname*{Alt}_{X_1, ..., X_n} \int \operatorname{res}(X_2 \cdots X_n l_x(X_1))$$

valid for any q. ∎

THEOREM 4.16. *For any odd $n \geqslant 3$ there is a cocycle in $Z^n(\mathfrak{G}_q, \mathbb{C})$*

$$c^{(n)}_{x,D}(X_1, ..., X_n) = \operatorname*{Alt}_{X_1, ..., X_n} \operatorname{Tr}\left(\sum_{i=1}^{(n-1)/2} \xi(X_1) X_2 \cdots \eta(X_{2i}) \cdots X_n \right.$$
$$\left. - \xi(X_1) X_2 \cdots X_{n-1} \eta(X_n) - D_q^{-1} X_1 X_2 \cdots X_n \right).$$

It is continuous at $q = 1$:

$$\lim_{q \to 1} c^{(n)}_{x,D}(X_1, ..., X_n) = c^{(n)}_{x,D}|_{q=1} (\lim_{q \to 1} X_1, ..., \lim_{q \to 1} X_n).$$

Proof. Consider the n-cochain

$$b_n(X_1, ..., X_n)$$
$$= \operatorname*{Alt}_{X_1, ..., X_n} \operatorname{Tr}\left(\sum_{i=1}^{(n-1)/2} \xi(X_1)\, X_2 \cdots \eta(X_{2i}) \cdots X_n - \xi(X_1)\, X_2 \cdots X_{n-1} \eta(X_n) \right)$$

which can be written as

$$b_n = \operatorname{Tr}\left(\sum_{i-1}^{(n-1)/2} \xi \cup m^{2(i-1)} \cup \eta \cup m^{n-2i} - \xi \cup m^{n-2} \cup \eta \right),$$

where $m^k(Y_1, ..., Y_k) = \operatorname{Alt}_{Y_1, ..., Y_k} Y_1 \cdots Y_k$ and $a \cup b \overset{\text{def}}{=} \mu(a \boxtimes b)$, μ being the multiplication. Since $m^k = m^1 \cup m^1 \cup \cdots \cup m^1$ and $dm^1 = m^2$, we have $dm^k = m^{k+1}$ for odd k, and $dm^k = 0$ for even k. Therefore,

$$db_n = \operatorname{Tr}\left(\sum_{i=1}^{(n-1)/2} \xi \cup m^{2(i-1)} \cup \eta \cup m^{n+1-2i} + \xi \cup m^{n-1} \cup \eta \right)$$
$$= \operatorname{Tr} \sum_{i=1}^{(n+1)/2} \xi \cup m^{2(i-1)} \cup \eta \cup m^{n+1-2i}$$
$$= \tfrac{1}{2} \operatorname{Tr}([\xi, \eta] \cup m^n)$$

due to the following identity

$$\tfrac{1}{2} \operatorname{Alt} \operatorname{Tr}((\xi\eta)(X_1) X_2 \cdots X_{n+1} - (\eta\xi)(X_1) X_2 \cdots X_{n+1})$$
$$= \tfrac{1}{2} \operatorname{Alt} \operatorname{Tr}\left(- \sum_{i=2}^{n+1} \eta(X_1) X_2 \cdots \xi(X_i) \cdots X_{n+1} \right.$$
$$\left. + \sum_{j=2}^{n+1} \xi(X_1) X_2 \cdots \eta(X_j) \cdots X_{n+1} \right)$$

$$=\tfrac{1}{2}\,\mathrm{Alt}\,\mathrm{Tr}\left(-\sum_{i=2}^{n+1}(-1)^{i-1}\,\xi(X_1)X_2\cdots\eta(X_{n-i+3})\cdots X_{n+1}\right.$$
$$\left.+\sum_{j=2}^{n+1}\xi(X_1)X_2\cdots\eta(X_j)\cdots X_{n+1}\right)$$
$$=\tfrac{1}{2}\,\mathrm{Alt}\,\mathrm{Tr}\sum_{j=2}^{n+1}[(-1)^{n+3-j}+1]\,\xi(X_1)X_2\cdots\eta(X_j)\cdots X_{n+1}$$
$$=\mathrm{Alt}\,\mathrm{Tr}\sum_{i=1}^{(n+1)/2}\xi(X_1)X_2\cdots\eta(X_{2i})\cdots X_{n+1}.$$

Hence,

$$db_n=\tfrac{1}{2}\,\mathrm{Tr}(\mathrm{ad}\,D_q^{-1}\cup m^n)$$

and we get

$$db_n(X_1, ..., X_{n+1})=\tfrac{1}{2}\mathop{\mathrm{Alt}}_{X_1, ..., X_{n+1}}\mathrm{Tr}(D_q^{-1}X_1X_2\cdots X_{n+1}-X_1D_q^{-1}X_2\cdots X_{n+1})$$
$$=\mathop{\mathrm{Alt}}_{X_1, ..., X_{n+1}}\mathrm{Tr}(D_q^{-1}X_1X_2\cdots X_{n+1})$$
$$=da_n(X_1, ..., X_{n+1}),$$

where

$$a_n(X_1, ..., X_n)=\mathop{\mathrm{Alt}}_{X_1, ..., X_n}\mathrm{Tr}(D_q^{-1}X_1X_2\cdots X_n).$$

This implies that $c_{x,D}^{(n)}=b_n-a_n$ is a cocycle.

Using the presentation

$$b_n(X_1, ..., X_n)$$
$$=\mathop{\mathrm{Alt}}_{X_1, ..., X_n}\int\mathrm{res}\left(-\sum_{i=1}^{(n-1)/2}X_{2i+1}\cdots X_n\xi(X_1)\,X_2\cdots l_x(X_{2i})\right.$$
$$\left.+\xi(X_1)X_2\cdots X_{n-1}l_x(X_n)\right)$$

we see that b_n is continuous when q tends to 1. The continuity of a_n folows from identity (4.7) and the continuity of $c^{(n)}$ as in the proof of Theorem 4.12. ∎

Results of Feigin [8] give some support to the conjecture of non-triviality of the cocycles above.

5. DEFORMATIONS OF THE COCYCLES

One can ask how the cohomology is modified under formal deformations. So let $\mathcal{G}_t$ be a formal deformation of a Lie algebra $\mathcal{G} = \mathcal{G}_0$ and M_t a formal deformation of a $\mathcal{G}$ module $M = M_0$. In the finite dimensional case, one has the "Fuchs principle" which states that the cohomology cannot increase in dimension through deformation: $\dim H^*(\mathcal{G}_t; M_t) \leqslant \dim H^*(\mathcal{G}_0; M_0)$. The Hochschild cohomology version of this result is carefully described in the work of M. Gerstenhaber and S. D. Schack [13], where, in particular, the authors acknowledge the priority of Nijenhuis and Richardson (unpublished) for the Lie algebra cohomology case.

In many cases, the cohomology decreases effectively, and in particular, the cohomology class which leads the deformation gets killed in the cohomology of the deformation. Intuitively speaking, formal or actual deformations make cohomology simpler and simpler. But this is not always the case. We shall prove that the Vey class survives in the cohomology of the Lie algebra of pseudo-differential operators.

Let $(\mathcal{G}_t, M_t)$ be a formal deformation of a Lie algebra and module $(\mathcal{G}_0, M_0)$ as above. Let (C_t^*, d_t) and (C_0^*, d_0) be the corresponding Chevalley–Eilenberg complexes; one has a one parameter family of differentials on the same underlying graded space. Let c_t be a family of cocycles: $d_t(c_t) = 0$. If the limit $c_0 = \lim_{t \to 0} c_t$ is a d_0-cocycle (i.e., $d_0(c_0) = 0$) one says that the cohomology class $[c_t]$ contracts onto $[c_0]$.

PROPOSITION 5.1. *There exists a nontrivial cohomology class* $[\tilde{V}] \in H^2(\mathfrak{G}, \mathfrak{G})$ *which contracts onto the Vey class* $[V] \in H^2(N, N)$.

We shall describe the phenomenon of contraction of cohomology classes by using graded Lie algebra techniques as developed by P. B. A. Lecomte ([27]). Let E be the underlying vector space of the Lie algebras we consider, and $A^*(E)$ be the Richardson–Nijenhuis graded Lie algebra on E, whose bracket will be denoted $[\![\ ,\ ]\!]$. The Lie algebra structure $\mathcal{G}_0$ is then defined by some $c_0 \in A^1(E)$ such that $[\![c_0, c_0]\!] = 0$.

Let $c_t = c_0 + \sum_{i \geqslant 1}^{+\infty} t^i c_i$ be the formal deformation giving the structure on $\mathcal{G}_t$; and then $[\![c_t, c_t]\!] = 0$. This deformation being supposed to be nontrivial, the cocycle c_1 induces a nontrivial cohomology class in $H^2(\mathcal{G}_0, \mathcal{G}_0)$. Then one can construct a cocycle on $\mathcal{G}_t$ whose cohomology class contracts onto the class of c_1; simply take the formal series given by derivative of c_t: $\dot{c}_t = c_1 + \sum_{i \geqslant 2}^{+\infty} i t^{i-1} c_i$. One obviously has $[\![\dot{c}_t, c_t]\!] = 0$ by derivation of $[\![c_t, c_t]\!] = 0$ and $\dot{c}_t$ contracts onto c_1.

Let us check whether the cocycle $\dot{c}_t$ is cohomologically trivial or not for $t > 0$. If so there exists a formal series $a_t \in A^0(E)[[t]]$ such that $\dot{c}_t = [\![c_t, a_t]\!]$. But a_t must be singular at 0 since c_0 is not cohomologically trivial.

So let us suppose $a_t = 1/t(a_0 + \sum_{t \geqslant 1} t^i a_i)$. One deduces from $\dot{c}_t = [\![c_t, a_t]\!]$ that

(i) $[\![c_0, a_0]\!] = 0$

(ii) $[\![c_0, a_1]\!] + [\![c_1, a_0]\!] = c_1$

so (i) a_0 is a cocycle and (ii) the action of a_0 on cohomology leaves the class of c_1 invariant (the term $[\![c_0, a_1]\!]$ being nothing but the coboundary of a_1).

Straightforward example: let $E = k^3$ ($k = \mathbb{R}$ or $\mathbb{C}$) with basis X, Y, Z and c_0 given by $c_0(X, Y) = Z$ and other terms vanishing; so $\mathcal{G}_0$ is the 3 dimensional Heisenberg algebra. Let c_1 be given by $c_1(X, Z) = X$, $c_1(Y, Z) = -Y$, $c_1(X, Y) = 0$; so $\mathcal{G}_t$ is isomorphic to $sl(2, k)$. If a_0 is the derivation given by $a_0(X) = X$, $a_0(Y) = -Y$, $a_0(Z) = 0$, then $[\![c_1, a_0]\!] = c_1$ and so $\dot{c}_t = [\![c_t, a_t]\!]$ which proves that the class given by c_1 is killed by the deformation.

Let us consider now $(N, \{\ ,\ \})$ and the local version of the Richardson–Nijenhuis algebra $A^*_{loc}(N)$ (see [30]). So let $c_1 \in A^1_{loc}(N)$ be the Vey cocycle, defining the infinitesimal part of the Moyal bracket deformation. One should check the 1-cohomology of N. It is well known since the first work of Lichnerowicz about deformations of Lie algebras of vector fields that the group $H^1(N, N)$ is isomorphic to the first De Rham cohomology group of the manifold. This isomorphism works as follows: let α be a closed one form on the manifold, it induces a derivation $\tilde{\alpha}$ of N through the formula $\tilde{\alpha}(f) = \alpha(H_f)$. An equivalent way of describing this space is through its isomorphism with the space of symplectic vector fields modulo the space of hamiltonian vector fields (i.e., all multivalued Hamiltonian functions modulo singlevalued Hamiltonian ones). In that case we associate to each symplectic vector field the natural derivation $f \to L_X f$. But then it is straightforward to check that symplectic vector fields respect the Vey cocycles using the geometric description (see [36, 30]). So following the above notations, the condition $[\![c_0, a_0]\!] = 0$ necessarily implies $[\![c_1, a_0]\!] = 0$. Thus the above conditions can never be satisfied, and the Vey class will survive under the deformation.

It would be interesting to generalize those arguments in order to decide whether $[\tilde{V}]$ has a non trivial image in $H^3(\mathfrak{G}, \mathbb{R})$ or not. The situation can again be described through a diagram:

$$\begin{array}{ccc} H^2_I(\mathfrak{G}; \mathfrak{G}) & \xrightarrow{\theta} & H^3(\mathfrak{G}; \mathbb{R}) \\ \downarrow & & \downarrow \\ H^2_I(N; N) & \xrightarrow{\theta} & H^3(N; \mathbb{R}) \end{array}$$

where the vertical dashed arrows indicate the contractions. It is easy to check that $\theta([V])$ is non trivial in $H^3(N; \mathbb{R})$ (see Section 2.1). So the above diagram gives some confidence in the non vanishing of the corresponding class in $H^3(\mathfrak{G}; \mathbb{R})$, but the proof is unknown to the authors.

6. EMBEDDING OF THE VIRASORO ALGEBRA INTO THE COMPLETION OF $\mathfrak{G}_q$

A natural question to ask is whether the standard embedding of vector fields $\mathfrak{A}(S^1)$ into pseudodifferential symbols $\mathfrak{G}(S^1)$ can be in some sense quantized, i.e., whether there exists a Lie algebra embedding $I_q: \mathfrak{A}(S^1) \to \mathfrak{G}_q$ which gives the standard one as $q \to 1$.

If one tries to use the description of ψDO_q as a twisted loop algebra ([21]), one identifies the generators of $\mathfrak{A}(S^1)$ with the terms of the form $x^n D$, $n \in \mathbb{Z}$ belonging to $A_\sigma[x, x^{-1}]$ (see the above notations, $D := x(\partial/\partial x)$) but then one has:

$$x^n Dx^m D = x^{n+m} \sigma^m(D) D$$

and so

$$\begin{aligned}[x^n D, x^m D] &= x^{n+m}(\sigma^m(D) D - \sigma^n(D) D) \\ &= x^{n+m}[([m]_q - [n]_q) D + (q^m - q^n) D^2].\end{aligned}$$

This bracket no longer belongs to the image of $\mathfrak{A}(S^1)$; the space spanned by $(x^n D)_{n \in \mathbb{Z}}$ is "not closed" in physicist's language.

We will look for an embedding of the following kind:

$$I_q: \mathfrak{A}(S^1) \to \hat{\mathfrak{G}}_q \qquad \text{given by} \quad I_q\left(x^{n+1} \frac{\partial}{\partial x}\right) = x^n f_q(D_q)$$

into some completion of $\mathfrak{G}_q$ with analytic $f_q(D)$ approaching D when $q \to 1$.

Replace $A = \mathbb{C}[D, D^{-1}]]$ by $\hat{A} = \mathbb{C}[\log D, D][[D^{-1}]]$ with the automorphism

$$\sigma(D) = qD + 1$$

$$\sigma(D^{-1}) = (qD+1)^{-1} = \sum_{k=1}^{\infty} (-1)^{k-1} q^{-k} D^{-k}$$

$$\sigma(\log D) = \log(qD+1) = \log D + \log q + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} q^{-k} D^{-k}$$

and then let $\hat{\mathfrak{G}}_q = \{P \log D + Q \mid P,\ Q \in A_\sigma[x, x^{-1}]\}$ to be the Lie subalgebra of $\hat{A}_\sigma[x, x^{-1}]$. Set

$$f_q(D) = \frac{\log(1 + D(q-1))}{\log q}$$

$$\overset{\text{def}}{=} \frac{1}{\log q} \left\{ \log D + \log(q-1) - \sum_{k=1}^{\infty} \frac{1}{k(1-q)^k} D^{-k} \right\}$$

and $I_q\colon \mathfrak{A}(S^1) \to \hat{\mathfrak{G}}_q$ given by $I_q(x^{n+1}(\partial/\partial x)) = x^n f_q(D)$. So

$$\left[I_q\left(x^{n+1} \frac{\partial}{\partial x}\right), I_q\left(x^{m+1} \frac{\partial}{\partial x}\right) \right]$$
$$= x^{n+m}[\sigma^m(f_q(D))\, f_q(D) - \sigma^n(f_q(D))\, f_q(D)].$$

Since f_q is analytic in the neighbourhood of $D = \infty$, one has

$$\sigma^m f_q(D) = f_q(\sigma^m(D)) = f_q(q^m D + [m]_q)$$
$$= \frac{1}{\log q} \log(q^m((q-1)D + 1)).$$

So

$$f_q(\sigma^m(D)) - f_q(\sigma^n(D)) = \frac{1}{\log q} (\log q^m - \log q^n) = (m-n).$$

Finally

$$\left[I_q\left(x^{n+1} \frac{\partial}{\partial x}\right), I_q\left(x^{m+1} \frac{\partial}{\partial x}\right) \right] = (m-n) I_q\left(x^{n+m+1} \frac{\partial}{\partial x}\right)$$
$$= I_q\left(\left[x^{n+1} \frac{\partial}{\partial x}, x^{m+1} \frac{\partial}{\partial x} \right]\right).$$

Now we can check the behaviour of Kac–Radul's cocycle through this homomorphism. Using any trace on $\hat{A}$, for instance $\mathrm{Tr} = \mathrm{Tr}_D$ extended to the whole algebra by $\mathrm{Tr}(D^p(\log D)^k) = 0$ if $k > 0$, one gets:

$$\psi\left(I_q\left(x^{n+1} \frac{\partial}{\partial x}\right), I_q\left(x^{-n+1} \frac{\partial}{\partial x}\right) \right)$$
$$= \mathrm{Tr}((1 + \sigma + \cdots + \sigma^{n-1})\, \sigma^{-n}(f_q(D))\, f_q(D))$$
$$= \mathrm{Tr}\left(\sum_{i=0}^{n-1} (i - n + f_q(D))(i + f_q(D)) \right)$$

$$= \mathrm{Tr}(1) \sum_{i=0}^{n-1} (i-n)i + \mathrm{Tr}(f_q(D)) \sum_{i=0}^{n-1} (2i-n) + n\,\mathrm{Tr}(f_q(D)^2)$$
$$= \frac{n(n+1)(n-1)}{6}\,\mathrm{Tr}(1) + n\,\mathrm{Tr}(f_q(D)^2 - f_q(D)).$$

So one recovers, up to a multiple, the Virasoro cycle, the last term being a coboundary. Let us denote by $\hat{\hat{\mathfrak{G}}}_q$ the central extension of $\hat{\mathfrak{G}}_q$, defined through the same cocycle.

PROPOSITION 6.1. *There exists a Lie algebra homomorphism I_q: $\mathrm{Vir} \to \hat{\hat{\mathfrak{G}}}_q$ such that I_q tends to the canonical embedding as $q \to 1$.*

Remark 6.1. For the above isomorphism between $\mathfrak{G}_q$ and the sine-algebra $x \mapsto U_1$ and $D_q + 1/(q-1) \mapsto U_2$, the map I_q gives: $I_q(x^n(\partial/\partial x)) = \log[(q-1)\,U_2]/\log q$. So in that context, completion means adding $\log U_2$ as a formal supplementary variable.

7. q-GELFAND–DICKEY STRUCTURES, q-KP AND q-KdV HIERARCHIES

In this section we define q-analogs for the KP and n-KdV hierarchies and show their "complete integrability". The q-hierarchies in our approach are systems describing commutative flows on the space of q-pseudo-differential operators equipped with the q-deformation of the Gelfand–Dickey Poisson Structure. In both the classical and q-deformed cases the extended algebra of integral operators (responsible for the hierarchies) and the centrally extended Lie algebra of differential operators are dual to each other (they are the components of the Manin triple, see [23]).

7.1. *Classical Hierarchies and Poisson Structures*

Let Q be a pseudodifferential operator of the form $Q = \partial + u_1(x)\partial^{-1} + u_2(x)\partial^{-2} + \cdots$.

THEOREM 7.1 (see, e.g. [10]). *For any $m = 1, 2, \ldots$ the system*

$$\frac{\partial Q}{\partial t_m} = [Q, (Q^m)_+] \tag{7.1}$$

(a) *defines an evolution on the space of $\{Q\}$;*

(b) *is Hamiltonian with respect to so called first and second Gelfand–Dickey brackets;*

(c) *defines commuting flows for different m.*

(c′) *The corresponding Hamiltonians $H_m(Q) = \int \mathrm{res}(Q^m)$ are in involution and define infinite number of conserved quantities for each flow.*

Proof. We prove here the statements (a) and (c), which later on are generalized to the case of q-analogs.

(a) We need to show that the vector $[Q, (Q^m)_+]$ belongs to the tangent space of $\{Q\}$, i.e. that its differential part vanishes. Indeed, $[Q, (Q^m)_+] = -[Q, (Q^m)_-]$, and $\deg[Q, (Q^m)_-] = \deg Q + \deg(Q^m)_- = 1 = -1$, i.e this is an integer operator.

(c) Straightforward simple calculation:

$$\frac{\partial H_n(Q)}{\partial t_k} = \int \operatorname{res} \frac{\partial Q^n}{\partial t_k} = \int \operatorname{res} \sum_{j=0}^{n} Q^j \frac{\partial Q}{\partial t_k} Q^{n-j-1}$$

$$= \int \operatorname{res} \sum_{j=0}^{n} (Q^j [Q, (Q^k)_+] Q^{n-j-1}) = \int \operatorname{res}[Q^n, (Q^k)_+] = 0$$

The last identity is due to the ad-invariance of trace: the residue of any commutator is a full derivative. ▮

Remark 7.1. The classical KP equation is the compatibility equation for the flows $\{\partial\psi/\partial t_m = (Q^m)_+ \psi\}$ for $m = 2$ and 3 (see [10]):

$$\frac{\partial (Q^3)_+}{\partial t_2} - \frac{\partial (Q^2)_+}{\partial t_3} = [(Q^3)_+, (Q^2)_+]. \tag{7.2}$$

This is a system of two equations on the coefficients $\{u_1, u_2\}$, from which one function can be excluded.

Remark 7.2. In the same way the phase space of the n-KdV hierarchy is the set of differential operators $\{L = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0\}$. For any operator L there exists the only pseudodifferential operator Q such that $Q^n = L$, see [10] (notation: $Q = L^{1/n}$). Then the mth flow of the n-KdV hierarchy is the system on the coefficients of L:

$$\frac{\partial L}{\partial t_m} = [L, (L^{m/n})_+]. \tag{7.3}$$

THEOREM 7.2 (see [2, 10]). *This system*

(a) *is well-defined*

(b) *is Hamiltonian*

(c) *has infinite number of conserved quantities* $H_k(L) = \int \operatorname{res}(L^{k/n})$.

Proof. To check (a) note that by definition $[L, (L^{m/n})_+] = -[L, (L^{m/n})_-]$, and $\deg[L, (L^{m/n})_-] = \deg L + \deg(L^{m/n})_- - 1 = n - 1 - 1 = n - 2$, i.e. this is an element of the tangent space of $\{L\}$.

The statement (c) follows from theorem above and the fact that n-KdV flows are the KP flows restricted to the set of those operators $\{Q\}$, whose nth power is a purely differential operator:

$$\frac{\partial L}{\partial t_m}=\frac{\partial Q^n}{\partial t_m}=[Q^n,(Q^m)_+]=[L,(L^{m/n})_+]. \quad \blacksquare$$

The classical KdV equation is the first nontrivial equation in the hierarchy on the space $\{L=\partial^2+u(x)\}$ (here $n=2$, $m=3$).

Remark 7.3. It is natural to describe the KP- and KdV-hierarchies as Hamiltonian systems on the Poisson–Lie group $\tilde{G}_{\text{Int}}=\{\partial^\alpha(1+A)\mid \alpha\in\mathbb{C}, A=\sum_{k=-\infty}^{-1}u_k(x)\,\partial^k\}$ corresponding to the Lie bialgebra $\tilde{\mathfrak{G}}_{\text{Int}}=\tilde{\mathfrak{G}}_-=\{A+\alpha\cdot\log\partial\}$. The Poisson structure on $\tilde{G}_{\text{Int}}$ is the generalized second Gelfand–Dickey structure defined as follows. To a pseudodifferential symbol $X=\partial^-\cdot B$, $B\in\mathfrak{G}_{DO}$ regarded as a linear functional on $Q=\partial^\alpha(1+A)\in\tilde{G}_{\text{Int}}$, one associates the Hamiltonian vector $(QX)_+Q-Q(XQ)_+$ on $\tilde{G}_{\text{Int}}$. For $\alpha=n$ this quadratic Poisson algebra of functionals is called W_n-algebra.

7.2. *q-Hierarchies and q-Gelfand–Dickey Structures*

To define the q-deformations of the *GD* bracket (or of the W_n-Poisson algebras) and the deformed KP and KdV hierarchies, we make use of the Lie bialgebra structure on $S\mathfrak{G}_q$ (see Section 3.5). Similar to the non-deformed case above, there is a bialgebra structure on the extended Lie algebra $\tilde{\mathfrak{G}}_q^-=\{A+\alpha\cdot\log T\mid\alpha\in\mathbb{C}, A=\sum_{k=-\infty}^{0}u_k(x)\,T^k, u_0\in x^{-1}\mathbb{C}[x^{-1}]\}$. Indeed, the "double" extension of $S\mathfrak{G}_q$ by $\log D_q$ and by the corresponding central term form a Manin triple $(S\tilde{\mathfrak{G}}_q, \tilde{\mathfrak{G}}_q^-, \tilde{\mathfrak{G}}_q^+)$ with two isotropic subalgebras relative to the ad-invariant inner product. The Lie group $\tilde{G}_q^-=\{T^\alpha(1+A)\}$ corresponding to $\tilde{\mathfrak{G}}_q^-$ has a natural Poisson structure given by the same Gelfand–Dickey formula (here $T=(q-1)\,D_q+1$, and $\log T$ is defined just as $\log D_q$ in Proposition 4.2).

The consideration above almost literally can be applied to the associative algebra of q-pseudodifferential symbols. The phase space for a q-KP hierarchy is the set $\{Q_q=D_q+u_0(x)+u_1(x)\,D_q^{-1}+u_2(x)\,D_q^{-2}+\cdots\}$, and the corresponding system has the same form

$$\frac{\partial Q_q}{\partial t_m}=[Q_q,(Q_q^m)_+] \tag{7.4}$$

where + means taking differential part of q-pseudodifferential operators. This is a system of differential equations on the coefficients of D_q^j (q-difference operators).

THEOREM 7.3. *The system* (7.4) *for any* m *defines an evolution on the space* $\{Q_q\}$. *For any* m *there is an infinite number of conservation laws* $H_m^{(q)}(Q_q)=\int \operatorname{res}(Q_q^m E)$ *and the flows commute for different* m.

Proof. Likewise $[Q_q, (Q_q^m)_+] = -[Q_q, (Q_q^m)_-]$, and thus $\deg \partial Q_q/\partial t_m = \deg Q_q + \deg(Q_q^m)_- = 1-1=0$. Now we do not have the cancellation of the leading term in the commutator of q-symbols. This is why we consider here the space of operators $\{Q_q\}$ containing arbitrary subleading terms $u_0(x)$. The same calculation as in the classical case verifies the invariance of $H_m^{(q)}(Q_q)$. We use the *ad*-invariance of the trace Tr for q-operators (Theorem 3.3). ∎

Remark 7.4. The analogous q-KP equation is the differential system

$$\frac{\partial(Q_q^3)_+}{\partial t_2} - \frac{\partial(Q_q^2)_+}{\partial t_3} = [(Q_q^3)_+, (Q_q^2)_+]. \tag{7.5}$$

The existence of a solution is guaranteed by the Lax formulation above.

To consider a q-analog of the n-KdV hierarchy we need to define the nth root Q_q of a differential operator $L_q = D_q^n + u_{n-1} D_q^{n-1} + \cdots + u_0$. However, instead of uniqueness of the exponential map of the "unipotent" group $\bar{G}_{\mathrm{Int}}$, we have now a "solvable" group $\tilde{G}_{\mathrm{Int}}^-$ of q-operators with the surjective but not one-to-one "exponent". This means that we have a freedom in the choice of the nth-root Q_q.

Then likewise the mth flow of nth KdV hierarchy

$$\frac{\partial L_q}{\partial t_m} = [L_q, (L_q^{m/n})_+]$$

for different m the flows commute, and the conserved quantities are $H_k(L_q) = \int \operatorname{res}(L^{k/n}E) = \operatorname{Tr}(L^{k/n})$. Proof repeats the classical case and uses the ad-invariance of the trace on $\mathfrak{G}_q$ (Theorem 3.3).

ACKNOWLEDGMENTS

We thank A. N. Kirillov and M. I. Golenishcheva-Kutuzova for fruitful discussions. We also thank V. Kac and A. Radul for communicating to us a preliminary version of their work [21].

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