PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2005) **461**, 3505–3524
doi:10.1098/rspa.2005.1498
Published online 9 September 2005

Polar linkings, intersections and Weil pairing

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Polar homology and linkings arise as natural holomorphic analogues in algebraic geometry of the homology groups and links in topology. For complex projective manifolds, the polar k-chains are subvarieties of complex dimension k with meromorphic forms on them, while the boundary operator is defined by taking the polar divisor and the Poincaré residue on it. We also define the corresponding analogues for the intersection and linking numbers of complex submanifolds, and show that they have properties similar to those of the corresponding topological objects. Finally, we establish the relation between the holomorphic linking and the Weil pairing of functions on a complex curve and its higher-dimensional counterparts.

Keywords: linking number; Poincaré residue; polar homology; Parshin symbols; meromorphic forms

1. Introduction

Polar homology groups for complex projective manifolds can be regarded as a complex version of singular homology groups in topology. The idea of such a geometric analogue of topological homology comes from thinking of the Dolbeault (or $\bar{\partial}$) complex of $(0, k)$-forms on a complex manifold as an obvious analogue of the de Rham complex of k-forms on a smooth manifold. This poses an immediate question: 'what is the analogue of the chain complex relevant to the context of complex manifolds?'

In this paper, we consider the complex of 'polar chains' (proposed in Khesin & Rosly 2003), which consists of algebraic subvarieties and meromorphic forms on them, whose poles are of the first-order only. The boundary operator takes the polar divisor and the Poincaré residue of the corresponding form. An advantage of considering polar chains is that it allows one to use the analogy between the boundary operator in topology and the residue in algebraic geometry in a direct way. In particular, one can define analogues for the intersection and linking numbers of complex submanifolds mimicking properties of the corresponding topological notions.

Some features of the above analogy between boundaries and residues can also be found in the recent literature. In particular, this correspondence leads to an

Received 4 January 2005
Accepted 14 April 2005

explicit construction of a group cocycle for the double-loop groups (Frenkel & Khesin 1996). It is useful in the study of the Poisson structure on the moduli spaces of holomorphic bundles on complex surfaces (see the description of Lagrangian submanifolds in Thomas (1997) and Donaldson & Thomas (1998) and that of symplectic leaves in Khesin & Rosly (2001)). Moreover, the approach of Donaldson & Thomas (1998) of transferring differential geometric constructions into the context of complex analytic (or algebraic) geometry could lead one to a 'complexification of geometry' in a sense similar to the 'complexification of topology' pursued here.

Below, we briefly mention one of the main motivations, which come from mathematical physics: a study of the holomorphic Chern–Simons gauge theory on a Calabi–Yau threefold suggested in Witten (1995). The latter context leads us immediately to a search for a proper holomorphic analogue of the linking number (see Khesin & Rosly 2001). The holomorphic linking can be naturally defined in the language of polar cycles and their intersections, as described below. Another approach can be found in Thomas (1997) and Frenkel & Todorov (2002).

(*a*) *Motivation and holomorphic linking*

The classical linking number arises in the study of quantum theory associated with the abelian Chern–Simons functional

$$\mathrm{CS}(A) = \int_M A \wedge \mathrm{d}A,$$

defined on smooth 1-forms $\{A\}$ on a real threefold M (Schwarz 1977/78; Polyakov 1988). (The non-abelian Chern–Simons theory leads to higher link invariants, such as Jones polynomials (Witten 1989).)

A holomorphic analogue of the Chern–Simons theory on a Calabi–Yau three-dimensional manifold X (with a holomorphic 3-form μ) was suggested in Witten (1995). In the simplest abelian case, the action functional is

$$\mathrm{CS}_\mu(A) = \int_X A^{01} \wedge \bar{\partial} A^{01} \wedge \mu,$$

where A^{01} is a smooth $(0,1)$-forms on X. In the latter case, one can consider the following natural observables, analogues of Wilson loops: $\int_C \alpha \wedge A^{01}$, where C is a *complex* curve in the threefold X and α is a holomorphic 1-form on C. The correlator of two such observables in this quadratic field theory is given by the following expression

$$\left\langle \int_{C_1} \alpha_1 \wedge A^{01} \cdot \int_{C_2} \alpha_2 \wedge A^{01} \right\rangle = \sum_{P \in C_1 \cap S_2} \frac{\alpha_1(P) \wedge \beta_2(P)}{\mu(P)}.$$

Here, S_2 is a complex surface containing the curve C_2 and equipped with a meromorphic 2-form β_2, such that $\mathrm{div}_\infty \beta_2 = C_2$ and $\mathrm{res}_{C_2} \beta_2 = \alpha_2$. Such a sum over the intersection points of C_1 and S_2 can be thought of as a holomorphic linking number of two non-intersecting complex curves C_1 and C_2, equipped with holomorphic 1-forms α_i in the Calabi–Yau manifold X

$$\ell k_\mu((C_1, \alpha_1), (C_2, \alpha_2)) := \sum_{P \in C_1 \cap S_2} \frac{\alpha_1(P) \wedge \beta_2(P)}{\mu(P)}.$$

Recall that the topological linking number of two closed curves in a *real* threefold is defined as the intersection of the first curve with a surface spanning the other curve. One can show that the expression above does not depend on the choice of (S_2, β_2), a holomorphic analogue of a spanning surface, and it has certain invariance properties copying those of the classical linking number in this holomorphic situation.

(*b*) *Holomorphic intersections*

We have implicitly defined a complex analogue of the intersection number in topology. To make it explicit, let (X, μ) be a complex manifold equipped with a meromorphic volume form μ without zeros. Consider two complex submanifolds A and B of complimentary dimensions that intersect transversely in X and are endowed with holomorphic volume forms α and β on the corresponding submanifolds. Then, the holomorphic intersection number is defined by the formula

$$\langle (A, \alpha) \cdot (B, \beta) \rangle = \sum_{P \in A \cap B} \frac{\alpha(P) \wedge \beta(P)}{\mu(P)}.$$

At every intersection point P, the ratio in the right-hand side is the 'comparison' of the orientations of the 'cycles' (A, α) and (B, β) at that point with the orientation of the ambient manifold. This is a straightforward analogue of the use of mutual orientation of cycles in the definition of the topological intersection number. Note, that in the holomorphic case, the intersection number does not have to be an integer. Rather, it is a holomorphic function of the 'parameters' (A, α), (B, β) and (X, μ).

(*c*) *Holomorphic orientation and boundary operator*

The above consideration prompts us to consider a top degree *holomorphic* form on a complex manifold as manifold's 'holomorphic orientation'. Furthermore, a pair (W, ω), which consists of a k-dimensional submanifold W equipped with a *meromorphic* top degree form ω (with first-order poles on a smooth hypersurface V), will be thought of as an analogue of a compact oriented submanifold *with boundary.*

Below, we start by defining a homology theory in which the pairs (W, ω) will play the role of k-chains. The corresponding boundary operator assumes the form $\partial(W, \omega) = (V, 2\pi \mathrm{i}\ \mathrm{res}\ \omega)$, where V is the polar set of the k-form ω, while $\mathrm{res}\ \omega$ is the $(k-1)$-form on V, the Poincaré residue of ω. Note, that in the situation under consideration, when the polar set V of the form ω is a smooth $(k-1)$-dimensional submanifold in a smooth k-dimensional W, the induced 'orientation' on V is given by a regular $(k-1)$-form $\mathrm{res}\ \omega$. This means that $\partial(V, \mathrm{res}\ \omega) = 0$, or the boundary of a boundary is zero. The latter will be the source of the identity $\partial^2 = 0$ in the homology theory discussed below. We shall call it the *polar homology.*

(*d*) *Pairing to smooth forms and the Cauchy–Stokes formula*

There is a pairing between polar chains and smooth differential forms on a manifold: for a polar k-chain (W, ω) and any $(0, k)$-form u such a pairing is given by the integral

$$\langle (W,\omega),u\rangle = \int_W \omega\wedge u.$$

In other words, the polar chain (W,ω) defines a current on X of degree $(n, n-k)$, where $n=\dim X$. This pairing descends to (co)homology classes by virtue of the following Cauchy–Stokes formula.

Consider a meromorphic k-form ω on W having first-order poles on a smooth hypersurface $V\subset W$. Let the smooth $(0,k)$-form u on X be $\bar\partial$-exact; that is, $u=\bar\partial v$ for some $(0,k-1)$-form v on X. Then,

$$\int_W \omega\wedge\bar\partial v = 2\pi \mathrm{i}\int_V \mathrm{res}\,\omega\wedge v.$$

We shall exploit this straightforward generalization of the Cauchy formula as a complexified analogue of the Stokes theorem. Note that the Cauchy–Stokes formula defines the pairing between the polar homology groups of a complex manifold X and the Dolbeault cohomology groups $H^{0,k}_{\bar\partial}(X)$. In a separate paper (Khesin *et al.* 2004) it was shown that this pairing defines an isomorphism of the corresponding groups, see the polar de Rham theorem in §3.

This allowed one to give below the definition of the polar intersection for *non-transversal* varieties (see §4); namely, the polar de Rham theorem allowed to express the polar intersection in terms of the product in (Dolbeault) cohomology. The new consequences of this are as follows:

(i) the polar intersection pairing is non-degenerate;
(ii) there exists a polar intersection product extending the polar intersection pairing;
(iii) there exist smooth transverse representatives in the polar classes;
(iv) one can give explicit expressions for the intersection product in terms of the latter. (We note that before the polar de Rham was proved in Khesin *et al.* (2004), such propositions were only possible 'in one direction'.)

Furthermore, most of our consideration extends to polar chains where the meromorphic forms are not necessarily of top degree. Another interesting case is that of degree zero. The corresponding polar homology groups turn out to be isomorphic to the groups of cycles modulo algebraic equivalence. Along with the polar de Rham theorem, this gives that the polar homology groups interpolate between the Dolbeault groups and the groups of algebraic cycles (modulo the algebraic equivalence).

Finally, it turned out that the polar linking is closely related to the Weil pairing and Parshin symbols (see §5). Roughly speaking, the holomorphic linking is the logarithmic 'rate of change' of the Weil (or Parshin) pairing.

2. Polar homology

Here, we define a homological complex based on the notion of the polar boundary. The construction is analogous to the definition of homology of a topological space with replacement of continuous maps by complex analytic ones. The notion of the boundary (of a simplex or a cell) is replaced by the Poincaré residue of a meromorphic differential form.

(*a*) *Preliminaries: residue and push-forward*

The Poincaré residue is a higher-dimensional generalization of the classical Cauchy residue, where the residue at a point in a domain of one complex variable is generalized to the residue at a hypersurface.

Let M be an n-dimensional complex manifold and ω be a meromorphic n-form on M, which is allowed to have first-order poles on a smooth hypersurface V. Then, the form ω can be locally expressed as

$$\omega = \frac{\varrho \wedge \mathrm{d}z}{z} + \varepsilon,$$

where $z=0$ is a local equation of V and ϱ (respectively, ε) is a holomorphic $(n-1)$-form (respectively, n-form). Then, the restriction $\varrho|_V$ is a well-defined holomorphic $(n-1)$-form on V.

Definition 2.1. The Poincaré residue of the n-form ω is the following $(n-1)$-form on V

$$\mathrm{res}\,\omega := \varrho|_V.$$

Definition 2.2. The above can be readily extended to the case of normal crossing divisors. Let V be a normal crossing divisor in M; i.e. $V=\cup_i V_i$ has only smooth components V_i (each entering with multiplicity one) that intersect generically. Suppose that the meromorphic n-form ω in M has the first order poles on V. Analogous to definition 2.1, one can define a residue at each component V_i. The resulting $(n-1)$-forms $\mathrm{res}_{V_j}\omega$ are then meromorphic and have first-order poles at the pairwise intersections $V_{ij}=V_i \cap V_j$. One can now consider the repeated Poincaré residue at V_{ij}. Representing ω as $\omega=\rho\wedge(\mathrm{d}z_i/z_i)\wedge(\mathrm{d}z_j/z_j)$, where $z_i=0$ and $z_j=0$ are local equations of the components V_i and V_j, respectively, one finds that

$$\mathrm{res}_{i,j}\omega := \mathrm{res}_{V_{ij}}(\mathrm{res}_{V_j}\omega) = \mathrm{res}_{z_i=0}\left(\mathrm{res}_{z_j=0}\rho \wedge \frac{\mathrm{d}z_i}{z_i} \wedge \frac{\mathrm{d}z_j}{z_j}\right) = \rho|_{V_{ij}}.$$

Note that the repeated residue differs by sign for different order of the components: $\mathrm{res}_{i,j}\omega = -\mathrm{res}_{j,i}\omega$.

Definition 2.3. For a finite covering $f: X \to Y$ and a function φ on X one can define its push-forward, or the trace, $f_*\varphi$ as a function on Y whose value at a point is calculated by summing over the preimages taken with multiplicities. The operation f_* can be generalized to p-forms and to the maps f, which are only generically finite.

Suppose that $f: X \to Y$ is a proper, surjective holomorphic mapping where both X and Y are smooth complex manifolds of the same dimension n. The *push-forward* (or, *trace*) map is a mapping

$$f_* : \Gamma(X, \Omega_X^p) \to \Gamma(Y, \Omega_Y^p).$$

To define it, one notes that f is finite unramified covering away from a hypersurface in Y. Thus, f is locally an isomorphism of neighbourhoods, and the push-forward form is defined by summing the image forms from all the leaves. The form obtained extends to the whole of Y.

The push-forward map is also defined for meromorphic forms. Furthermore, the operations of push-forward and residue commute.

(*b*) *Polar chains and boundary*

In this section, we deal with complex projective varieties—closed subvarieties of a complex projective space. (In this setting, the complex analytic considerations are equivalent to algebraic ones.) By a smooth projective variety, we always understand a smooth and connected one. For a smooth variety M of dimension n, we denote by K_M the sheaf of regular n-forms on M.

The space of polar k-chains for a complex projective variety X, dim $X=n$, will be defined as a $\mathbb{C}$-vector space with certain generators and relations.

Definition 2.4. The space of polar k-chains $\mathcal{C}_k(X)$ is a vector space over $\mathbb{C}$ defined as the quotient $\mathcal{C}_k(X)=\hat{\mathcal{C}}_k(X)/\mathcal{R}_k$, where the vector space $\hat{\mathcal{C}}_k(X)$ is freely generated by the triples (A, f, α) described in (i), (ii) and (iii) below and $\mathcal{R}_k$ is defined as relations (R1), (R2) and (R3) imposed on the triples.

- (i) A is a smooth complex projective variety, dim $A=k$;
- (ii) $f\colon A\to X$ is a holomorphic map of projective varieties;
- (iii) α is a rational k-form on A with first-order poles on $V\subset A$, where V is a normal crossing divisor in A; that is $\alpha\in\Gamma(A,\ K_A(V))$.

The relations are:

- (R1) $\lambda(A, f, \alpha)=(A, f, \lambda\alpha)$;
- (R2) $\sum_i(A_i, f_i, \alpha_i)=0$ provided that $\sum_i f_{i*}\alpha_i\equiv 0$, where $\dim f_i(A_i)=k$ for all i and the push-forwards $f_{i*}\alpha_i$ are considered on the smooth part of $\cup_i f_i(A_i)$;
- (R3) $(A, f, \alpha)=0$ if $\dim f(A)<k$.

Remark 2.5. By definition, $\mathcal{C}_k(X)=0$ for $k<0$ and $k>\dim X$.

The relation (R2), in particular, represents additivity with respect to α, that is

$$(A, f, \alpha_1) + (A, f, \alpha_2) = (A, f, \alpha_1 + \alpha_2).$$

Here, we make no difference between a triple and its equivalence class. In particular, if the polar divisor $\mathrm{div}_\infty(\alpha_1+\alpha_2)$ is not normal crossing, then one can replace A by an appropriate blow-up, by the Hironaka theorem, where the pull-back of $\alpha_1+\alpha_2$ is already admissible.

This way, the relation (R2) allows us, in particular, to refer to polar chains as pairs replacing a triple (A, f, α) by a pair $(\hat{A}, \hat{\alpha})$, where $\hat{A}=f(A)\subset X$, $\hat{\alpha}$

is defined only on the smooth part of $\hat{A}$ and $\hat{\alpha}=f_*\alpha$ there. Owing to the relation (R2), such a pair $(\hat{A},\hat{\alpha})$ carries precisely the same information as (A,f,α).[1] (The only point of concern is that such pairs cannot be arbitrary. In fact, by the Hironaka theorem on resolution of singularities, any subvariety $\hat{A}\subset X$ can be the image of some regular A, but the form $\hat{\alpha}$ on the smooth part of $\hat{A}$ cannot be arbitrary.)

Definition 2.6. The boundary operator $\partial : \mathcal{C}_k(X)\to\mathcal{C}_{k-1}(X)$ is defined by

$$\partial(A,f,\alpha) = 2\pi \mathrm{i}\sum_i (V_i, f_i, \mathrm{res}_{V_i}\alpha),$$

(and by linearity), where V_i are the components of the polar divisor of α, $\mathrm{div}_\infty\alpha=\cup_i V_i$, and the maps $f_i=f|_{V_i}$ are restrictions of the map f to each component of the divisor.

Theorem 2.7. *The boundary operator ∂ is well defined, i.e. is compatible with the relations* (R1)*,* (R2) *and* (R3).

Proof. We have to show that ∂ maps equivalent sums of triples to equivalent ones.

It is trivial with (R1). For (R2), this follows from the commutativity of taking residue and push-forward. To prove the compatibility of ∂ with (R3), consider first the case of a polar 1-chain, a complex curve with a meromorphic 1-form, which is mapped to a point.

Then, the image of the boundary of this 1-chain is zero. Indeed, this image must be the same point, whose coefficient is equal to the sum of all residues of the meromorphic 1-form on the curve (i.e. zero). The general case is similar: the same phenomenon occurs along one of the coordinates. ■

Theorem 2.8. $\partial^2=0$.

Proof. We need to prove this for triples $(A,f,\alpha)\in\mathcal{C}_k(X)$; that is, for forms α with normal crossing divisors of poles. The repeated residue at pairwise intersections differs by a sign according to the order in which the residues are taken (see definition 2.2). Thus, the contributions to the repeated residue from different components cancel out (or, the residue of a residue is zero).[2] ■

[1] Note that the consideration of triples (A,f,α) instead of pairs $(\hat{A},\hat{\alpha})$, which we used in §1, is similar to the definition of chains in the singular homology theory. In the latter case, although one considers the mappings of abstract simplices into the manifold, morally, it is only 'images of simplices' that matter. Here lies an important distinction; unlike the topological homology, where in each dimension k, one uses all continuous maps of one standard object (the standard k-simplex or the standard k-cell) to a given topological space, in polar homology, we deal with complex analytic maps of a large class of k-dimensional varieties to a given one.

[2] An example of the polar divisor $\{xy=0\}$ for the form $\mathrm{d}x\wedge \mathrm{d}y/xy$ in $\mathbb{C}^2$ should be viewed as a complexification of a polygon vertex in $\mathbb{R}^2$. Indeed, the cancellation of the repeated residues on different components of the divisor is mimicking the calculation of the boundary of a boundary of a polygon; every polygon vertex appears twice with different signs as a boundary point of two sides.

(c) Polar homology of projective varieties

Definition 2.9. For a complex projective variety X, $\dim X = n$, the chain complex

$$0 \to \mathcal{C}_n(X) \xrightarrow{\partial} \mathcal{C}_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}_0(X) \to 0$$

is called the polar chain complex of X. Its homology groups, $\mathrm{HP}_k(X)$, $k=0, \ldots, n$, are called the polar homology groups of X.

Functoriality of polar homology is standard; a regular morphism of projective varieties $h: X \to Y$ defines a homomorphism $h_*: \mathrm{HP}_k(X) \to \mathrm{HP}_k(Y)$.[3]

Example 2.10. For a projective curve of genus g, the polar homology groups are as follows: $\mathrm{HP}_0 = \mathbb{C}$, $\mathrm{HP}_1 = \mathbb{C}^g$and $\mathrm{HP}_k=0$ for $k\geq 2$. Indeed, in this (and in any) case, all the 0-chains are cycles. Let (P, a) and (Q, b) be two 0-cycles, where P, Q are points on X and $a, b \in \mathbb{C}$. (For simplicity, we adopt the consideration of pairs, the images of the triples in the curve, as discussed in footnote 1.) They are polar homologically equivalent if, and only if, $a=b$. Indeed, $a=b$ is necessary and sufficient for the existence of a meromorphic 1-form α on X, such that $\mathrm{div}_\infty \alpha = P+Q$ and $\mathrm{res}_P \alpha = 2\pi \mathrm{i} a$, $\mathrm{res}_Q \alpha = -2\pi \mathrm{i} b$. (The sum of all residues of a meromorphic differential on a projective curve is zero by the Cauchy theorem.) Then, we can write in terms of polar chain complex that $(P, a) - (Q, a) = \partial(X, \alpha)$. Thus, $\mathrm{HP}_0(X) = \mathbb{C}$.

As to polar 1-cycles, these correspond to all possible holomorphic 1-forms on. On the other hand, there are no 1-boundaries, since there are no polar 2-chains in X. Hence, $\mathrm{HP}_1(X) = \mathbb{C}^g$, where g is the genus of the curve X. (In particular, the polar Euler characteristic of X equals $1-g$ and coincides with its holomorphic Euler characteristic.)

Similar considerations show that for any n-dimensional X, we have $\mathrm{HP}_n(X) = H^0(X, \Omega_X^n)$ and, if X is connected, also $\mathrm{HP}_0(X) = \mathbb{C}$.

Definition 2.11 (Relative polar homology). Let Z be a projective subvariety in a projective X. Analogously to the topological relative homology, we can define the polar relative homology of the pair $Z \subset X$.

The relative polar homology $\mathrm{HP}_k(X, Z)$ is the homology of the following quotient complex of chains

$$\mathcal{C}_k(X, Z) = \mathcal{C}_k(X)/\mathcal{C}_k(Z).$$

Here, we use the natural embedding of the chain groups $\mathcal{C}_k(Z) \hookrightarrow \mathcal{C}_k(X)$. This leads to the long exact sequence in polar homology

$$\ldots \to \mathrm{HP}_k(X) \to \mathrm{HP}_k(X, Z) \xrightarrow{\partial} \mathrm{HP}_{k-1}(Z) \to \mathrm{HP}_{k-1}(X) \to \ldots .$$

[3] Note that the polar homology is an analogue of singular homology with coefficients in $\mathbb{R}$ or $\mathbb{C}$.

3. Dolbeault cohomology as polar de Rham cohomology

As we discussed in §1, the Dolbeault complex of $(0, k)$-forms should be related to the polar homology in the same way as the de Rham complex of smooth forms is related to the topological homology (e.g. singular homology). Now, after the definitions of §2 are given, we are able to make this point more explicit.

Definition 3.1. In a smooth projective variety X, consider a polar k-chain $a=(A, f, \alpha)$. Such a triple can be regarded as a linear functional on the space of smooth $(0, k)$-forms on X. (In other words, the space of polar chains $\mathcal{C}_k(X)$ can be defined as a subspace of currents, functionals on smooth differential forms.) Let u be a smooth $(0, k)$-form on X. Then, the pairing is given by the following integral

$$\langle a, u\rangle := \int_A \alpha \wedge f^* u.$$

The integral is well defined, since α has only first-order poles on a normal crossing divisor. It is now straightforward to show that the pairing $\langle , \rangle$ descends to the space of equivalence classes of triples $\mathcal{C}_k(X)$ and that it is compatible with the relations (R1), (R2) and (R3) of definition 2.4. Indeed, (R1) is obvious, where compatibility with (R3) follows from noticing that $f^* u=0$ if $\dim f(A)<k$, and the compatibility with (R2) follows from the relation $\int_A \alpha \wedge f^* u = \int_{f(A)} f_* \alpha \wedge u$ if $\dim f(A)=k$, where the last integral is taken over the smooth part of $f(A)$.

Proposition 3.2. *The pairing* (3.1) *defines the following homomorphism in* (*co*)*homology*

$$\rho : \mathrm{HP}_k(X) \to H_{\bar\partial}^{n,n-k}(X),$$

where $n=\dim X$.

Proof. By the Serre duality, ρ is the map $\mathrm{HP}_k(X) \to (H_{\bar\partial}^{0,k}(X))^*$ and it is sufficient to verify that the pairing (3.1) vanishes if $\partial a=0$ and $u=\bar\partial v$, or if $\bar\partial u=0$ and $a=\partial b$. This follows immediately from the Cauchy–Stokes formula (§1d),

$$\int_A \alpha \wedge f^*(\bar\partial u) = 2\pi \mathrm{i} \int_{\mathrm{div}_\infty \alpha} (\mathrm{res}\, \alpha) \wedge f^*(u),$$

that is, $\langle a, \bar\partial u\rangle = \langle \partial a, u\rangle$. ■

It turns out that for smooth projective manifolds the homomorphism (3.2) is in fact an isomorphism.

Theorem 3.3. *(Polar de Rham theorem; Khesin et al. 2004)*

(i) *For a smooth projective manifold* X*, the mapping* $\rho : \mathrm{HP}_k(X) \to H_{\bar\partial}^{n,n-k}(X)$ *is an isomorphism of the polar homology and Dolbeault cohomology groups. Equivalently, in terms of dual cohomology groups,*

$$\mathrm{HP}^k(X) \cong H_{\bar\partial}^{0,k}(X).$$

(ii) *Let* V *be a normal crossing divisor in a smooth projective* X*. Then*

$$\mathrm{HP}_k(X, V) \cong H^{n-k}(X, K_X(V)).$$

Example 3.4. If X is a complex curve of genus g, one has $\mathrm{HP}_0(X) \cong \mathbb{C} \cong H^{1,1}_{\bar\partial}(X)$ and $\mathrm{HP}_1(X) \cong \mathbb{C}^g \cong H^{1,0}_{\bar\partial}(X)$ (see example 2.10).

Note that by theorem 3.3, for an ample divisor V represented by a smooth hypersurface in X, $\mathrm{HP}_p(X, V) = H^{n-p}(X, K_X(V)) = 0$. Such a choice of the hypersurface V in the complex manifold X is similar to the choice of an $(n-1)$-skeleton in an n-dimensional real manifold: the corresponding relative homology group of the manifold with respect to the skeleton also vanish.

Remark 3.5. So far, we have considered polar chains with top degree forms. More generally, one could consider polar (k, p)-chains (A, f, α), where α is a meromorphic p-form of not necessarily maximal degree, $p \le k$, on A, which can have only logarithmic singularities on a normal crossing divisor.[4] The requirement of log-singularities is needed to have a convenient definition of the residue and, hence, the boundary operator ∂.

The property $\partial^2 = 0$ and the definition of the polar homology groups can be carried over to this more general situation. The polar homology groups are then enumerated by two indices: $\mathrm{HP}_{k,p}(M)$.

The Cauchy–Stokes formula extends to this case as well. As a consequence, the natural pairing between polar (k, p)-chains and smooth $(k-p, k)$-forms on X gives us as before the homomorphism (cf. (3.2)),

$$\rho : \mathrm{HP}_{k,p}(X) \to H^{n-k+p,n-k}_{\bar\partial}(X).$$

However, unlike the case $p = k$, the map ρ is not generally an isomorphism for other values of p, $0 \le p < k$. For instance, in the case of $p = 0$, the image of ρ belongs to the subspace generated by algebraic cycles, while the full $H^{r,r}_{\bar\partial}(X)$ can be much larger (already for a generic K3 surface).

Corollary 3.6. *The groups* $\mathrm{HP}_{k,p}(X)$ *interpolate between the groups of k-cycles modulo algebraic equivalence in X and the Dolbeault groups* $H^{n,n-k}_{\bar\partial}(X)$ *as p changes from* 0 *to k.*

Indeed, one can see that for $p = 0$ the polar homology groups $\mathrm{HP}_{k,0}(X)$ coincide with the groups of k-cycles in X modulo algebraic equivalence, according to the Severi theorem (Griffiths & Harris 1978).

4. Intersection in polar homology

Here, we define a polar analogue of the topological intersection product. In particular, for polar cycles of complimentary dimensions, one obtains a complex number, called the 'polar intersection number'.

Recall that in topology, one considers a smooth oriented closed manifold M and two oriented closed submanifolds $A, B \subset M$ of complementary dimensions; that is, $\dim A + \dim B = \dim M$. Suppose that A and B intersect transversely at a finite set of points. Then, to each intersection point P, one assigns ± 1 (local intersection index) by comparing the mutual orientations of the tangent vector spaces $T_P A$, $T_P B$ and $T_P M$.

[4] An important property of such forms on projective varieties is that they are closed, see Deligne (1971).

(a) *Polar-oriented manifolds*

Now, let M be a compact complex manifold of dimension n, on which we would like to define a polar intersection theory. It has to be polar oriented (i.e. equipped with a complex volume form). As the discussion below shows, the n-form μ defining its polar orientation has to have no zeros on M, since we are going to consider expressions in which μ, the orientation of the ambient manifold, enters a denominator. Therefore, we adopt the following terminology.

Definition 4.1. (i) A compact complex manifold M, endowed with a nowhere vanishing holomorphic volume form μ, is said to be a polar-oriented closed manifold. (ii) If the volume form μ on a compact complex manifold M is non-vanishing and meromorphic with only first-order poles on a normal crossing divisor $N \subset M$, then M is called a 'polar-oriented manifold' with boundary. The hypersurface N is then endowed with a polar orientation $\nu := 2\pi \mathrm{i} \operatorname{res} \mu \neq 0$ and (N, ν) is called the polar boundary of (M, μ).

Remark 4.2. By definition, polar-orientable closed manifolds are complex manifolds whose canonical bundle is trivial (Calabi–Yau, Abelian manifolds or any complex tori, if we do not restrict ourselves to algebraic manifolds). In considering the definition of chains, we have defined the notion of the polar orientation in a more restrictive sense than before. In fact, polar chains with their orientations are to be compared with oriented piecewise smooth submanifolds in differential topology, while the ambient space on which we want to have Poincaré duality has to be smooth and oriented all over. Zeros of a volume form could be regarded as a complex analogue of singularities of a real manifold.[5]

(b) *Polar intersection number*

Let (M, μ) be a polar-oriented closed manifold of dimension n. In such a case, we define the following natural pairing between its polar homology groups $\mathrm{HP}_p(M)$ and $\mathrm{HP}_q(M)$ of complimentary dimension ($p+q=n$).

First, we define the polar intersection number for the case of a smooth and transverse pair of cycles a and b. That is, $a=(A, \alpha)$ and $b=(B, \beta)$, where A is a smooth p-dimensional subvariety and α a holomorphic p-form on it (and, similarly, for (B, β) in dimension q), and it is assumed that A and B intersect transversely. Then, we have the following formula for the polar intersection index.

[5] For instance, on a complex curve X of genus g one has $\mathrm{HP}_1(X) = \mathbb{C}^g$, and a holomorphic 1-differential representing a generic element in $\mathrm{HP}_1(X)$ has $2g-2$ zeros. From this point of view, the complex genus g curve is like a graph that has g loops joined by $g-1$ edges and having $2g-2$ trivalent (i.e. 'non-smooth') points. The 'smooth orientable cases' are $\mathbb{CP}^1$, which corresponds to a real segment, and an elliptic curve, which is a complex counterpart of the circle in this precise sense.

Definition 4.3. The polar intersection index of two smooth transverse cycles (A, α) and (B, β) is given by the following sum over the set of points in $A \cap B$,

$$(A, \alpha) \cdot_{\mu} (B, \beta) = \sum_{P \in A \cap B} \frac{\alpha(P) \wedge \beta(P)}{\mu(P)}.$$

Here, $\alpha(P)$ and $\beta(P)$ are understood as exterior forms on $T_P M = T_P A \times T_P B$ obtained by the pull-back from the corresponding factors.

To extend this definition to arbitrary and not necessarily transverse polar cycles, we apply the following.

Proposition 4.4. *For any two polar cycles a and b, there exist polar homologous cycles a' and b', which are represented by smooth transversely intersecting submanifolds.*

Proof. By theorem 3.3, if V is a normal crossing divisor in M, then $\mathrm{HP}_p(M, V) = H^{n-p}(M, K_M(V))$. Choose V to be an ample divisor represented by a smooth hypersurface. Then the right-hand side is 0, and so is $\mathrm{HP}_p(M, V) = 0$. This shows that any polar cycle is homologous to a cycle lying in an ample smooth hypersurface V. Iterating this consideration inside V, we arrive at the fact that any polar cycle is homologous to a smooth one. An analogous reasoning shows that one can achieve a transversal intersection for two such cycles. ■

It turns out that the definition of polar intersection extends to the classes of polar homologous cycles.

Theorem 4.5. *The polar intersection is a well-defined pairing in polar homology.*

To define the polar intersection pairing, one can compose the homomorphism ρ and the product in the Dolbeault cohomology. Namely, according to theorem 3.3, the groups $\mathrm{HP}_q(M)$ and $\mathrm{HP}_p(M)$ can be mapped isomorphically to the Dolbeault cohomology groups $H^{n,n-q}_{\bar\partial}(M)$ and $H^{n,n-p}_{\bar\partial}(M)$, respectively. On a polar-oriented closed manifold, we are given a nowhere vanishing section μ of the line bundle K_M. Hence, we have the isomorphism $H^{n,n-q}_{\bar\partial}(M) \xrightarrow{\mu^{-1}} H^{0,n-q}_{\bar\partial}(M)$. Using this and the product in Dolbeault cohomology, we obtain the following pairing for $p+q=n$

$$H^{n,n-p}_{\bar\partial}(M) \otimes H^{n,n-q}_{\bar\partial}(M) \xrightarrow{\mathrm{id} \otimes \mu^{-1}} H^{n,n-p}_{\bar\partial}(M) \otimes H^{0,n-q}_{\bar\partial}(M) \to H^{n,n}_{\bar\partial}(M) \xrightarrow{\sim} \mathbb{C}.$$

One can see that this definition of pairing coincides with the one given in definition 4.3 for transverse representatives. As a by-product, one has the following non-degeneracy theorem.

Corollary 4.6. *The polar intersection pairing is non-degenerate, as a bilinear form on the polar homology groups*

$$\mathrm{HP}_p(M) \otimes \mathrm{HP}_q(M) \to \mathbb{C},$$

where $p+q=n$.

(*c*) *Polar intersection product*

Now, consider the case when on a polar-oriented closed manifold (M,μ), we have two polar cycles of arbitrary dimensions p and q (not necessarily complimentary ones). For any p and q, one can define an intersection product in polar homology.

Definition 4.7. To describe this intersection product on two smooth transverse cycles, we need the following construction from the linear algebra for $\mathbb{C}$-oriented vector spaces. Let V_A, $V_B \subset W$ be two transverse vector subspaces of dimensions p and q, respectively, in a complex n-dimensional vector space W $(p+q \geq n)$. Suppose we are given '$\mathbb{C}$-orientations' of all these spaces; that is, the (non-zero) exterior forms $\alpha_0 \in \bigwedge^p V_A^*$, $\beta_0 \in \bigwedge^q V_B^*$ and $\mu_0 \in \bigwedge^n W^*$. Then, the intersection $V_A \cap V_B$ can be naturally endowed with a $\mathbb{C}$-orientation, that is with the exterior form

$$\gamma_0 = \frac{\alpha_0 \wedge \beta_0}{\mu_0}.$$

This notation stands for the following. In coordinate form, we can always choose the coordinates $z_1, \ldots, z_n$ in W such that the space V_A is spanned by $z_1, \ldots, z_p$ and V_B is spanned by $z_{n-q+1}, \ldots, z_n$, while $\alpha = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p$ and $\beta = \mathrm{d}z_{n-q+1} \wedge \cdots \wedge \mathrm{d}z_n$. Then, for $\mu_0 = c_0 \cdot \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$, one has

$$\gamma_0 := \frac{\mathrm{d}z_{n-q+1} \wedge \ldots \wedge \mathrm{d}z_p}{c_0}.$$

It can be readily verified that γ_0 does not depend on a coordinate choice (see Khesin & Rosly 2003).

Now, we can define the polar orientation for intersection of polar cycles.

Definition 4.8. For a pair of two smooth transverse polar cycles $a=(A,\alpha)$ and $b=(B,\beta)$ of dimensions p and q, respectively, in a polar-oriented closed manifold (M,μ), their intersection is a polar $(p+q-n)$-cycle $a \cdot_\mu b = c = (C,\gamma)$, where $C = A \cap B$ and

$$\gamma = \frac{\alpha \wedge \beta}{\mu}.$$

(Here, the definition of the holomorphic form γ on C is given by the linear algebra above at every point of C). If $p+q<n$, then $a \cdot_\mu b = 0$.

The intersection of smooth transverse cycles

$$(A,\alpha) \cdot_\mu (B,\beta) = (C,\gamma)$$

defines the intersection product in polar homology

$$\mathrm{HP}_p(M) \otimes \mathrm{HP}_q(M) \to \mathrm{HP}_{p+q-n}(M),$$

upon finding smooth transverse representatives for every pair of homology classes. By theorem 3.3 this product agrees with the product in Dolbeault cohomology:

$$\mathrm{HP}_p(M)\otimes\mathrm{HP}_q(M)\xrightarrow{\sim}H_{\bar\partial}^{n,n-p}(M)\otimes H_{\bar\partial}^{n,n-q}(M)$$

$$\xrightarrow{id\otimes\mu^{-1}}H_{\bar\partial}^{n,n-p}(M)\otimes H_{\bar\partial}^{0,n-q}(M)\to H_{\bar\partial}^{n,2n-p-q}(M)\xrightarrow{\sim}\mathrm{HP}_{p+q-n}(M).$$

Remark 4.9. We have defined the polar intersection on any complex manifold M that can be equipped with a *holomorphic* non-vanishing volume form μ. This is analogous to the topological intersection theory on a compact smooth oriented manifold without boundary. (Note that the Poincaré duality in this context should correspond to the Serre duality; cf. Thomas 1997; Donaldson & Thomas 1998) Furthermore, the consideration above easily extends to the case of a complex manifold possessing a *meromorphic* non-vanishing form μ (in particular, to a complex projective space), especially to the case of a polar-oriented manifold (M,μ) with boundary $(N, 2\pi\mathrm{i}\,\mathrm{res}\,\mu)$. The latter setting is similar to the topological intersection theory on manifolds with boundary. In this case, the above formulae can be used to define the pairing between polar homology $\mathrm{HP}_k(M)$ and polar homology relative to the boundary $\mathrm{HP}_{n-k}(M,N)$.

5. Polar linking number and Weil pairing

Recall that the Gauss linking number of two *oriented closed* curves in $\mathbb{R}^3$ is an integer topological invariant equal to the algebraic number of crossings of one curve with a two-dimensional oriented surface bounded by the other curve. The linking number does not depend on the choice of the surface. This follows from the fact that the algebraic number of intersections of a closed curve and a closed surface in a simply connected three-dimensional manifold is equal to zero. Note that the Gauss linking number is a homology invariant in that it does not change if one of the curves is replaced by a homologically equivalent cycle in the compliment to the other curve. More generally, the linking number can be defined for two oriented closed submanifolds of linking dimensions in any oriented (but not necessarily simply connected) manifold, provided that both submanifolds are homologous to zero.

(a) *Definition of polar linking number*

Polar linkings mimic the classical definition of linking in the polar language. Let $a=(A,\alpha)$ and $b=(B,\beta)$ be two polar smooth non-intersecting cycles of dimensions p and q in a polar-oriented closed n-manifold (M,μ). Suppose that these cycles are polar boundaries (i.e. they are polar homologous to 0) and are of linking dimensions $p+q=n-1$. Then, one can associate to them the following polar linking number.

Definition 5.1. The polar linking number of cycles a and b in (M,μ) is

$$\ell k_\mu(a,b) := \sum_{P\in A\cap S}\frac{\alpha(P)\wedge\sigma(P)}{\mu(P)},$$

where a chain (S,σ) has the polar boundary (B,β), $\partial(S,\sigma)=(B,\beta)$.

In other words, $\ell k_\mu(a, b)$ is the intersection of the polar cycle a and the polar chain $s=(S,\sigma)$, provided they intersect transversely.

(*b*) *Properties of polar linking number*

It turns out that the polar linking, being defined analogously to the topological one, also mimics the properties of the latter.

(i) The polar linking is (anti-)symmetric

$$\ell k_\mu(a, b) = (-1)^{(n-p)(n-q)} \ell k_\mu(b, a).$$

(ii) It is well defined; that is, ℓk_μ does not depend on the choice of the polar chain $s=(S,\sigma)$, provided that $\partial(S,\sigma)=(B,\beta)$.

(iii) For a polar oriented manifold with (polar) boundary there exists a relative version of ℓk_μ, as in topology.

The simplest curves that can have non-trivial linking are elliptic curves. The linking number of a rational curve with any other curve is zero since any holomorphic differential on a rational curve must vanish. (As discussed above, a rational curve $\mathbb{C}P^1$ equipped with a meromorphic 1-form with two simple poles is a complexification of a segment, while an elliptic curve with a holomorphic 1-form on it is an analogue of a circle.)

Similar to the Gauss formula for the topological linking number, the polar linking can be defined via integrals involving the Green function for $\bar{\partial}$. This approach to a complex analogue of the linking number can be seen in Atiyah (1981), Thomas (1997) and Frenkel & Todorov (2002). As noted by Geramisov (1995, unpublished work), such a Green function plays the role of the propagator in the holomorphic Chern–Simons theory, which should cause the appearance of the complex linking number in that quantum theory (in the very same way as the Gauss linking number appears in the ordinary Chern–Simons theory).

Note also that the polar linking is an invariant of the corresponding polar homology class $[a]\in \mathrm{HP}^p(M\backslash B)$. To formulate this property precisely, one needs a definition of the polar homology of complex quasi-projective manifolds and we describe it elsewhere.

While the topological linking has a polar counterpart, it is an interesting open question to find a similar counterpart for the self-linking

Question 5.2. Find a polar analogue of the self-linking number of a framed knot.

A framing of an oriented knot allows one to define its oriented satellite knot and to consider linking of the initial knot with this satellite. It would be very interesting to find analogues of the framings and satellites in the complex algebraic setting. This might be closely related to the Viro self-linking-type invariant of real algebraic knots (Viro 2001).

(*c*) *Intersections with various coefficients*

Now, we would like to extend the notion of polar chains and cycles to include the chains (A,α), where α can be not only a differential form on a subvariety $A\subset M$, but also, for example, a meromorphic section of the normal bundle to A.

With this more general understanding of polar cycles, one does not need to fix the non-vanishing form μ in the ambient manifold M (and hence to confine oneself to the class of polar-oriented closed manifolds) to define the polar intersection theory.

One can define the polar intersection index of two smooth transverse cycles (A, α) and (B, β) in M by

$$(A, \alpha)\cdot(B, \beta) = \sum_{P\in A\cap B} \langle \alpha(P), \beta(P)\rangle,$$

if α and β at points P assume values in dual spaces. In other words, we consider the polar chains with appropriate coefficients and $\langle , \rangle$ is the corresponding pairing. Recall that the polar intersection

$$(A, \tilde{\alpha})\cdot_{\mu}(B, \tilde{\beta}) = \sum_{P\in A\cap B} \frac{\tilde{\alpha}(P)\wedge\tilde{\beta}(P)}{\mu(P)},$$

of polar p- and q-cycles $(A, \tilde{\alpha})$ and $(B, \tilde{\beta})$ in (M, μ) with $p+q=n$ discussed in §5b corresponds to the one just described as follows: $\alpha=\tilde{\alpha}$ is a p-form on A, while $\beta=\tilde{\beta}/\mu$ is a ratio of an $(n-p)$-form on B and an n-form on M; that is, a meromorphic section of $\Lambda^p NB$, the pth power of the normal bundle to B, or, in other words, a p-vector field normal to B in M. The latter can be contracted with the p-form on A at the intersection points $A\cap B$.

(d) *Linkings with other coefficients*

Similarly, one can define the polar linking of more general polar boundaries 'with coefficients'. After passing to a polar chain bounded by one of the cycles, this chain and the other cycle have to be equipped with appropriate sections assuming values in dual spaces.

Example 5.3. Let a and b' be 0-boundaries with certain coefficients on a complex curve X. Namely, $a=\sum(P_i, r_i)$ is a 0-cycle, where $P_i\in X$ are points with complex coefficients $r_i\in\mathbb{C}$, satisfying the exactness condition $\sum r_i=0$, ensuring that a is the boundary of a 1-chain. The cycle $b'=\sum(Q_j, v_j)$ is a set of points $Q_j\in X$ with vectors $v_j\in T_{Q_j}X$ assigned to them. (We discuss below the restriction on b' imposed by the condition of 0-boundary.)

In this setting, the following *polar linking number* of a and b' is defined. Let $a=\partial c$, where a polar 1-chain $c=(X, \alpha)$ is such that the 1-form α has poles of the first order at P_i and satisfies $2\pi\mathrm{i}\,\mathrm{res}_{P_i}\alpha=r_i$. Then,

$$\ell k_{\mathrm{polar}}(b', a) := \sum_j \iota_{v_j}\alpha(Q_j).$$

It turns out that this linking number does not depend (up to a sign) on which of the two 0-boundaries, a or b', we use. The fact that the 0-cycle b' is a polar boundary means that there exists a function ϕ with poles of the first order at Q_j, such that $2\pi\mathrm{i}\,\mathrm{res}_{Q_j}\phi=v_j$. (Note that the Poincaré residue of a *function* is a *vector* attached at its pole). Then, $b'=\partial(X, \phi)$.

One can see that the linking number $\ell k_{\mathrm{polar}}(b', a)$, evaluated with the help of the chain (X, ϕ) bounding b' (rather than a), leads to the same result. Indeed, consider the 1-form $\phi\alpha$ on X. It has poles where so does either the function ϕ, or the form α. Denote by Γ the contour encompassing only the poles of α. Then,

$$\sum_i \phi(P_i)\cdot r_i = 2\pi\mathrm{i}\sum_i \mathrm{res}_{P_i}(\phi\cdot\alpha) = \oint_\Gamma \phi\cdot\alpha$$
$$= -2\pi\mathrm{i}\sum_j \mathrm{res}_{Q_j}(\phi\cdot\alpha) = -\sum_j \iota_{v_j}\alpha(Q_j).$$

Note also that the vector field v_j attached to Q_j can be regarded as an infinitesimal deformation of the divisor Q_j. We are going to exploit this point of view below.

Remark 5.4. The holomorphic linking suggested by Atiyah (1981) can be thought of as the polar linking with appropriate coefficients.

(e) *The Weil pairing and reciprocity law*

Let f and g be two meromorphic functions on X with disjoint divisors: $\mathrm{div}\, f=\sum_i r_i P_i$ and $\mathrm{div}\, g=\sum_j q_j S_j$, where $P_i, S_j\in X$, while $r_i=\deg_{P_i} f$ and $q_j:=\deg_{S_j} g$ are integers.

Definition 5.5. The Weil pairing of functions f and g is $\{f,g\}=\prod_j f(S_j)^{\deg_{S_j} g}$.

The Weil reciprocity law is the symmetry of this bracket

$$\{f,g\} = \{g,f\}.$$

It follows from the identity

$$\oint_\Gamma \log f\cdot d\log g = -\oint_\Gamma \log g\cdot d\log f,$$

applied to a contour Γ embracing div g and leaving div f outside, and using the identity, that $\deg_P g=\mathrm{res}_P(d\log g)$ for all points $P\in X$.

(f) *Relation of the Weil pairing and polar linking*

Let f_t and g_t be one-parameter families of meromorphic functions on X with disjoint divisors $a_t:=2\pi\mathrm{i}$ div f_t and $b_t:=2\pi\mathrm{i}$ div g_t for all t. Note that the divisors a_t and b_t define polar 0-cycles and these are, in fact, polar 0-boundaries: $a_t=\partial(X,\alpha_t)$, where $\alpha_t=\mathrm{d}f_t/f_t$, and similarly for b_t.

Consider the infinitesimal deformation a_t' (respectively, b_t'); that is the derivative in t of a_t (respectively, b_t), which can be defined as follows: $a_t'=\partial(X,\phi_t)$, where $\phi_t:=(\mathrm{d}/\mathrm{d}t)\log f_t$ is a meromorphic function on X. Suppose that ϕ_t has only simple poles, so that (X,ϕ_t) is a polar chain. Note that a_t' and b_t' are polar 0-boundaries 'with coefficients'—a set of points with certain vectors attached.

Proposition 5.6. *The following relation between the Weil pairing and the polar linking holds*

$$2\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\log\{f_t,g_t\} = \ell k_{\mathrm{polar}}(a_t',b_t) + \ell k_{\mathrm{polar}}(b_t',a_t).$$

Proof. Indeed,

$$2\pi \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\log\{f_t, g_t\} = \oint_\Gamma \frac{\mathrm{d}}{\mathrm{d}t}(\log f_t)\cdot\frac{\mathrm{d}g_t}{g_t} - \oint_\Gamma \frac{\mathrm{d}}{\mathrm{d}t}(\log g_t)\cdot\frac{\mathrm{d}f_t}{f_t}$$
$$= \ell k_{\mathrm{polar}}(a_t', b_t) + \ell k_{\mathrm{polar}}(b_t', a_t).$$

Here, we used that

$$\oint_\Gamma \frac{\mathrm{d}}{\mathrm{d}t}(\log f_t)\cdot\frac{\mathrm{d}g_t}{g_t} = \oint_\Gamma \phi_t\cdot\frac{\mathrm{d}g_t}{g_t} = \ell k_{\mathrm{polar}}(a_t', b_t),$$

and, similarly, for $\ell k_{\mathrm{polar}}(b_t', a_t)$. ■

(g) *Higher-dimensional generalization and Parshin symbols*

On an $(n-1)$-dimensional complex manifold X, consider n functions $f_1, \dots, f_n$ whose poles and zeros are of the first-order. Assume also that their poles and zeros divisors are normal crossing and in general position. Define the bracket $\{f_1, \dots, f_n\}$, which has similar symmetry properties to the Weil pairing.

Definition 5.7. For meromorphic functions $f_1, \dots, f_n$ on X, define the following bracket

$$\{f_1, \dots, f_n\} := \prod_P f_1(P)^{d_P(f_2,\dots,f_n)},$$

where the product is taken over all points P belonging to the intersection of the divisors $|\mathrm{div}\, f_2| \cap \cdots \cap |\mathrm{div}\, f_n|$. If $P \in D_2 \cap \cdots \cap D_n$, where D_i is a component of the divisor div f_i entering with multiplicity d_i (which is ± 1 by the assumption), then $d_P(f_2, \dots, f_n) := d_2 \cdot \dots \cdot d_n$.

In fact, here we iterate the following procedure. Take div f_n and restrict to it the remaining functions $f_1, \dots, f_{n-1}$. (For dim $X=1$ above, we considered the restriction of $f=f_1$ to the divisor of $f_2=g$.) Repeat the procedure until we come to the points and can consider the corresponding Weil pairing.

Remark 5.8. Note that the brackets $\{f_1, \dots, f_n\}$ can be thought of as the Parshin symbols defined generally for meromorphic functions in the presence of a flag of subvarieties (see Parshin 1977; Brylinski & McLaughlin 1996). The symmetry properties of the above brackets in $f_1, \dots, f_n$ are a particular case of Parshin's reciprocity laws. For instance, the symmetry in f_1 and f_2 is evident, as it follows from the Weil reciprocity law in dimension 1 after the restriction to a curve.

(h) *Higher-dimensional brackets and higher polar linking*

To discuss the relation of the bracket in definition 5.7 above and the polar linking of cycles, we confine the discussion to the case of dim $X=3$, although what follows can be carried over to n dimensions.

Let $f_{1,t}, \ldots, f_{4,t}$ be meromorphic functions on X as above, depending on a parameter t. Define

$$a_{i,t} := \partial(X, d \log f_{i,t}),$$

an exact 2-cycle on the three-dimensional X with values in integers (0-forms). Its derivative

$$a'_{i,t} := \partial\left(X, \frac{\mathrm{d}}{\mathrm{d}t} \log f_{i,t}\right)$$

is an exact 2-cycle with values in normal vector fields. (Here, $\phi_{i,t} = (\mathrm{d}/\mathrm{d}t) \log f_{i,t}$ is a meromorphic function, assumed to have simple poles. Its residue is a normal vector field on the corresponding divisor of poles.)

Then, the following relation holds:

$$(2\pi \mathrm{i})^3 \frac{\mathrm{d}}{\mathrm{d}t} \log\{f_{1,t}, \ldots, f_{4,t}\} = \sum \ell k_{\mathrm{polar}}(a'_1, a_2 \cdot a_3 \cdot a_4),$$

where the sum is taken over all cyclic permutations. Here, $a_2 \cdot a_3 \cdot a_4$ is the exact polar 0-cycle, obtained by intersecting the corresponding divisors a_i and multiplying the numbers on them, which can be linked with the 2-cycle a'_1 in the three-dimensional X.

By definition of ℓk_{polar}, the right-hand side can also be rewritten as the multiple polar intersection

$$\ell k_{\mathrm{polar}}(a'_1, a_2 \cdot a_3 \cdot a_4) = A_1 \cdot a_2 \cdot a_3 \cdot a_4,$$

where $\partial A_1 = a'_1$. This form can be rewritten via pairwise intersections, for example, as $(A_1 \cdot a_2) \cdot (a_3 \cdot a_4)$. The latter product establishes the relation of the bracket in definition 5.7 with the polar linking of 1-cycles

$$(2\pi \mathrm{i})^3 \frac{\mathrm{d}}{\mathrm{d}t} \log\{f_{1,t}, \ldots, f_{4,t}\} = \ell k_{\mathrm{polar}}(c'_{12}, c_{34}) + \ell k_{\mathrm{polar}}(c'_{34}, c_{12}),$$

where the curves $c_{12} = a_1 \cdot a_2$ and $c_{34} = a_3 \cdot a_4$ are pairwise intersections of the divisors, and, $c'_{12} := a'_1 \cdot a_2 + a'_2 \cdot a_1$, for instance, can be regarded as the t-derivative of c_{12}.

Remark 5.9. The higher polar linkings appear as the answers for the correlators of the abelian holomorphic Chern–Simons functional on higher-dimensional complex manifolds. The most intriguing case is that of a non-abelian holomorphic Chern–Simons theory on a three-dimensional Calabi–Yau manifold, which is expected to produce holomorphic analogs of Vassiliev (finite-order) link invariants.

B.K. and A.R. are grateful for the hospitality of the RiP programme at the Oberwolfach Mathematical Institute, as well as the Max-Planck-Institut für Mathematik in Bonn and the IHES in Bures-sur-Yvette. We are indebted to S. Donaldson and R. Thomas for discussions on various parts of this paper.

The work of B.K. was partially supported by an NSERC research grant. The work of A.R. was supported in part by RFBR-03-02-17554, INTAS-99-590, FPRMIST-40-052-1-1-1112, and NSch-1999.2003.2.

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