AMS Transl., Ser. 2, vol. 180 (1997), pp. 147–155

To Vladimir Igorevich Arnold on the occasion of his 60-th birthday

INFORMAL COMPLEXIFICATION AND POISSON STRUCTURES ON MODULI SPACES

Boris A. Khesin

Abstract. We show how the Cauchy (or, more generally, the Leray) residue formula can be understood as an informal complex analog of the Stokes formula. It allows one to treat the Poisson (and symplectic) structures on the moduli spaces of flat connections on real manifolds and those structures on the moduli spaces of holomorphic bundles on complex manifolds in a parallel way.

Introduction

The formal "real-complex" correspondence extends to many mathematical notions in a rather "straightforward" way: for instance, the parallelism between the groups of orthogonal and unitary matrices, or, say, between the Stiefel–Whitney and Chern classes of vector bundles. In [A1, A2] V. Arnold presented many less formal "dual pairs" in this $\mathbb{R} - \mathbb{C}$ correspondence of notions, such as, e.g.,

Real version:	Complex version:
$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$
π_0	π_1
Morse theory	Picard-Lefschetz theory
a manifold with boundary	a 2-ramified covering
manifold's orientation	homotopy class of Hermitian framings

and posed a question about an informal analog of the DeRham cohomology theory.

In this paper we suggest an answer to this question, extending the "informal complexification" table as follows:

DeRham theory of smooth diff. forms	Leray theory of merom. forms
manifold's boundary	polar set of a form
restriction to the boundary	taking residue
Stokes formula	Cauchy (Leray) formula.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

We develop here the idea from [FK], where this parallelism was used for an explicit realization of the central extensions of current groups on Riemann surfaces as a "complex analog" of affine groups. We show how this heuristic principle can be applied to constructions of symplectic and Poisson structures on the moduli spaces of flat connections and of integrable (0,1)-connections (or holomorphic bundles) on real and complex two- and three-dimensional manifolds. The real setting is classical, see [AB, FR, Ad]. The complex counterpart is the result of our discussions with V. Fock and A. Rosly (cf. also [K]). Our goal in this paper is mostly expository, and we refer for all details, proofs, and further applications to [FKR], as well as to a relevant treatment in [FKT].

The statements are presented in "real-complex" pairs with the indexes **R** or **C** respectively. We emphasize that the complexification considered is not "formal" in any way: we study *smooth* objects on complex manifolds, thus doubling the domain dimension. The *holomorphic* objects arise as certain equivalence classes (orbits, symplectic leaves) of these smooth ones (cf. [AHS, EF, FKR]).

Acknowledgements. I am sincerely grateful to V. Fock, A. Rosly, I. Frenkel, H. Fan, R. Lee, A. Levin for many fruitful discussions. I am deeply obliged to V.I. Arnold, who intrigued me with the question of informal complexification. This paper was conceived in the creative atmosphere of the MPI in Bonn, and mostly completed in that of the FIM in Zurich. The research was partially supported by NSF grant DMS-9627782.

§1. Complex analogs of the boundary operator and the Stokes-Leray formula.

1.1. The Leray residue. The Leray residue formula is a higher-dimensional generalization of the Cauchy formula, which gives the value of a contour integral of a meromorphic 1-form via form's residue at the pole.

Let ω be a closed meromorphic k-form on a compact complex n-dimensional manifold M with poles on a nonsingular complex hypersurface $N \subset M$. All poles here and below are supposed to be of the first order. Let ψ be a function defining N in a neighborhood of some point $p \in N$. Then locally, in a certain neighborhood

$U(p)$ the k-form ω can be decomposed into the sum

$$\omega = \frac{d\psi}{\psi} \wedge \alpha + \beta,$$

where α and β are holomorphic in $U(p)$. One can show, that the restriction $\alpha\big|_N$ is a well-defined (i.e. independent of ψ) holomorphic $(k-1)$-form, see [L].

Definition 1.1. The *form-residue* res ω of the form ω is the holomorphic $(k-1)$-form on N, such that in any neighborhood $U(p)$ of an arbitrary point $p \in N$ it coincides with the form $\alpha\ |_N$ of the decomposition above: res $\omega = \alpha\ |_N$. The *coboundary Leray operator* δ associates to every point $p \in N$ a topological circumference $\delta p \subset M \setminus N$ satisfying certain conditions of continuity in p. Any $(k-1)$-cycle Γ on N gives rise to a k-cycle $\delta\Gamma$ in the complement $M \setminus N$.

Theorem 1.2 ([L]). *Let $\Gamma \subset N$ be a $(k-1)$-dimensional cycle in N and let ω be a meromorphic k-form with a polar set N of order 1. Then*

$$\int_{\delta\Gamma} \omega = 2\pi i \int_{\Gamma} res\ \omega.$$

The theory above admits natural generalizations to the case of closed C^∞-smooth forms ω, as well as to the case of polar sets which consist of several complex hypersurfaces in a general position in M.

Remark 1.3. According to the table in Introduction we will regard the poles of meromorphic forms as the "complex analogs of the boundary", while the boundary orientation (being $\mathbb{Z}_2$-valued) is replaced by an (integral valued) index of a contour with respect to the polar set.

For several problems in algebraic geometry V. Arnold used the following counterpart of the boundary operator (see, e.g. [A2, A3]). Let $f(x)$ be a function defined on the manifold with boundary, say $\{x \mid x \geq 0\}$. The latter is not an algebraic constraint, and hence it cannot be straightforwardly understood over $\mathbb{C}$. To overcome this difficulty, introduce an additional variable y, and replace the condition $x \geq 0$ by the algebraic condition $x = y^2$. The latter relation over $\mathbb{C}$ defines a two-fold covering of the x-axis ramified at $\{x = 0\}$. Thus, the *passage to the ramified covering* defines a would-be complexification of the *boundary operator.*

Note, that this version of the boundary operator is consistent with the complexification table above once we apply it to the set of *meromorphic* forms. If we start with a meromorphic form $\omega(x)$ of the variable x with a pole (of first order) at $x = 0$ (say, dx/x), then after the change of variable $x = y^2$ we obtain the form $\tilde{\omega}(y)$ (respectively, equal to $2dy/y$) with the pole of first order at $y = 0$.

1.2. The Stokes–Leray Theorem and its applications. The theorems of Stokes and Leray admit the following "joint" version. Let M be a complex compact manifold of complex dimension n, and N a complex hypersurface.

Theorem 1.4 (see e.g. [GS, FK]**).** *Assume that a $2n$-form $\mu = d\alpha \wedge \rho$ on $M \setminus N$ is a wedge product of an exact smooth n-form $d\alpha$ with $\alpha \in \Omega^{n-1}(M)$ and a meromorphic n-form ρ on M with a polar set N of order* 1*. Then the form μ is an integrable form on M, and*

$$\int_M d\alpha \wedge \rho = -2\pi i \int_N \alpha \wedge \ res\ \rho. \tag{1}$$

The integrability of the form μ follows from the fact that μ, as well as ρ has a pole of first order on N, while the (real) codimension of N equals 2. The equality is proved by successive application of the Stokes and Leray formulas to the integral $\int_M \mu$, where M is split into the sum of a tubular neighborhood of $N \subset M$ and the neighborhood's complement. Theorem 1.4 remains true if ρ is allowed to have poles of the first order along a union of hypersurfaces.

Note, that the dimension of the manifold of integration in (1) drops by 2 at once. Moreover, both the smooth objects and the meromorphic ones appear together in the relation (1), and the meromorphic form ρ can be thought of as "measure-like", while smooth α as variable. The formula (1), as well as that of Cauchy, will further be regarded as a complexification of the Stokes formula, and it replaces the latter in complex versions of proofs of the statements below, cf. [FK].

Remark 1.5. We denote by $\mathcal{H}^n(M \setminus N)$ the space of meromorphic n-forms on M with poles on N (of the first order). For Kähler M this is a finite-dimensional space, and it turns out to be isomorphic to the direct sum of the space $\mathcal{H}^{n-1}(N)$ of holomorphic $(n-1)$-forms on N and the space $\mathcal{H}^n(M)$ of holomorphic n-forms on M, provided that $h^{n,1}(M,\mathbb{C}) = 0$ (see e.g. [Ch]). In other words, under this assumption every meromorphic n-form on M is defined by its residue on N uniquely up to a holomorphic n-form on M.

§2. Spaces of flat connections on real and complex surfaces.

In this Section we recall several results on Poisson structures on the space of *smooth* connections on *real* 1-, 2-, and 3-dimensional manifolds. Their "complexification" according to the table in Introduction allows one to describe the (holomorphic) Poisson structures on the spaces of *smooth* connections on *complex* 1-, 2-, and 3-dimensional manifolds.

We will use the following notations throughout the paper.

Definition 2.1R. In the real case G stands for a simple simply connected *compact* Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ is its Lie algebra. Let S (or C, D) be a real manifold (possibly with boundary), and E be a principle G-bundle over S. Let $\mathcal{A}^S$ be the affine space of all *smooth* connections in E: $\mathcal{A}^S = \{d + A \mid A \in \Omega^1(S,\mathfrak{g})\}$. For a nontrivial bundle E, d stands for the covariant differential d_B for some flat ($d_B^2 = 0$) connection B in E, and the identification of $\mathcal{A}^S$ with $\Omega^1(S,\mathfrak{g})$, as well as the most structures below, depends on the choice of B.

Definition 2.1C. In the complex framework, G is a simple simply connected *complex* Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. All manifolds N (or Σ, M) are supposed to be Kähler and, moreover, equipped with some holomorphic or meromorphic form of highest degree. Let E be a complex principle G-bundle over the (complex) manifold N. Denote by $\mathcal{A}^N_{\bar\partial}$ the space of all *smooth* $(0,1)$-connections in E: $\mathcal{A}^N_{\bar\partial} = \{\bar\partial + A \mid A = A^{01} \in \Omega^{0,1}(N,\mathfrak{g})\}$. For a topologically nontrivial E one replaces $\bar\partial$ by $\bar\partial_B$ for some integrable $(0,1)$-connection B (i.e. $\bar\partial_B^2 = 0$).

For an (either real or complex) manifold S and its submanifold $C \subset S$ denote by G^S the gauge group of *smooth* currents $G^S = \{f \in C^\infty(S,G)\}$ with the pointwise product, and by G^S_C the group of currents on S "based on C": $G^S_C = \{f \in C^\infty(S,G) \mid f|_C = id\}$. The corresponding Lie algebras are denoted by $\mathfrak{g}^S$ and $\mathfrak{g}^S_C$.

2.1. Symplectic structures on the spaces of connections. Let S be a compact Riemann surface (with no complex structure fixed).

Proposition 2.2R. *The space $\mathcal{A}^S$ of all smooth G-connections in a bundle E over S is an affine symplectic space with the symplectic structure*

$$W(\delta A_1, \delta A_2) := \int_S tr(\delta A_1 \wedge \delta A_2),$$

where $\delta A_i \in \Omega^1(S, \mathfrak{g})$, and $tr(\delta A_1 \wedge \delta A_2)$ is the real-valued 2-form, being the wedge product of the 1-forms δA_1 and δA_2 on the surface S and the Killing form in the Lie algebra $\mathfrak{g}$. The symplectic structure W is invariant with respect to the gauge transformations $A \mapsto g^{-1}Ag + g^{-1}dg$, where $g \in G^S$.

Now, let N be a compact complex surface, and ω a holomorphic 2-form on N.

Proposition 2.2C. *Given a holomorphic 2-form ω, the affine space $\mathcal{A}^N_{\bar\partial}$ of all smooth $(0,1)$-connections in the bundle E over N has a natural holomorphic symplectic structure:*

$$W_{\mathbb{C},\omega}(\delta A_1, \delta A_2) := \int_N tr(\delta A_1 \wedge \delta A_2) \wedge \omega.$$

The symplectic structure $W_{\mathbb{C},\omega}$ is invariant with respect to the gauge transformations $A \mapsto g^{-1}Ag + g^{-1}\bar\partial g$, where $g \in G^N$.

For different 2-forms ω the spaces $\mathcal{A}^N_{\bar\partial}$ are, generally speaking, not symplectomorphic. One can join all these symplectic structures $W_{\mathbb{C},\omega}$ into the "symplectic structure" $W_{\mathbb{C},*}$ on $\mathcal{A}^N_{\bar\partial}$ with values in $(\mathcal{H}^2(N))^*$: given a holomorphic 2-form $\omega \in \mathcal{H}^2(N)$ one obtains a complex valued symplectic form $W_{\mathbb{C},\omega} =: \langle\, W_{\mathbb{C},*}\ ,\ \omega\,\rangle$.

The gauge group G^S acts on $\mathcal{A}^S$ in a Hamiltonian way ([AB]). Respectively, the complex group G^N acts on $\mathcal{A}^N_{\bar\partial}$ in a Hamiltonian way, provided that the holomorphic 2-form ω does not have zeroes on N (i.e. for a K3 surface or a two-dimensional complex torus N), [K]. The moment map of this action takes a connection into its curvature: $A \mapsto F(A) = dA + \frac{1}{2}[A,A]$ (resp., $A \mapsto F_{\bar\partial}(A) \wedge \omega := (\bar\partial A + \frac{1}{2}[A,A]) \wedge \omega$). Preimage of zero value of the moment map consists of flat connections (resp., integrable $(0,1)$-connections). The (nonsingular part of the) result of its

Hamiltonian reduction carries a natural symplectic structure. Below we always mean the nonsingular parts of the moduli spaces when describing the symplectic (or Poisson) structures on them.

Theorem 2.3R ([AB]). *The moduli space* $\mathcal{M}^S = \mathcal{A}^S_{\mathrm{fl}}/G^S$ *of flat* G*-connections carries a natural symplectic structure as the result of the Hamiltonian reduction.*

Theorem 2.3C ([K, M]). *The points of the moduli space* $\mathcal{M}^N = \mathcal{A}^N_{\mathrm{int}}/G^N$ *of integrable* $(0,1)$*-connections are in 1-1 correspondence with the equivalence classes of holomorphic* G*-bundles over* N*. If* N *is a K3 surface or a two-dimensional complex torus then the moduli space carries a holomorphic symplectic structure (depending on the 2-form* ω*) as the result of the reduction.*

The first part of Theorem 2.3C is the classics of four-dimensional Riemannian geometry, see e.g. [AHS, K]. The $(0,1)$-connections $\bar\partial + A$ can be regarded as the $(0,1)$-part of Hermitian connections in E, and the integrability condition $\bar\partial_A^2 = 0$ is equivalent to the condition that the curvature of the Hermitian connection is of $(1,1)$-type. The latter implies the existence of the holomorphic structure in the complex bundle E. The existence of the (holomorphic) symplectic structure on the moduli space $\mathcal{M}^N$ is implied by the Hamiltonian reduction from the symplectic space $\mathcal{A}^N_{\bar\partial}$.

2.2. Poisson structures on the spaces of connections on surfaces with boundaries. Let S be a Riemann surface with a boundary $C = \partial S$ consisting of several curves $C = \cup_1^k C_j$.

Theorem 2.4R ([FR]). *The moduli space* $\mathcal{M}^S = \mathcal{A}^S_{\mathrm{fl}}/G^S$ *of flat* G*-connections on* S *with boundary admits a natural Poisson structure. The symplectic leaves are determined by fixing the conjugacy classes of the connection holonomies along the components* C_j *of the boundary of* S*.*

A complex analog of a real surface with boundary is a complex surface N equipped with a *meromorphic* 2-form ω. Assume that ω does not have zeroes on N. Let $\Sigma = \cup_1^k \Sigma_j$ be the polar set of ω, consisting of complex curves Σ_j.

Theorem 2.4C ([FKR]). *Given a nonvanishing meromorphic 2-form ω on N, the moduli space $\mathcal{M}^N = \mathcal{A}^N_{\text{int}}/G^N$ of integrable $(0,1)$-connections on N admits a holomorphic Poisson structure. The symplectic leaves are determined by fixing the equivalence classes of the holomorphic G-bundles on all the components Σ_j of the polar set Σ of ω.*

Note that if all curves Σ_j are nonsingular and disjoint, then they are elliptic curves. Indeed, the residue of the nonvanishing meromorphic 2-form ω is a holomorphic 1-form on Σ_j without zeroes.

Remark 2.5R. To prove the real version one notices that the Hamiltonian functions of the infinitesimal G^S-gauge action on $\mathcal{A}^S$ form the Lie algebra $\mathfrak{g}^S$ extended by the 2-cocycle

$$c(X,Y) = \int_S tr(dX \wedge dY) = \int_{C=\partial S} tr(XdY) \qquad \text{for } X, Y \in \mathfrak{g}^S, \tag{2}$$

see e.g. [Ad].

The space of smooth G-connections $\{d + A \mid A \in \Omega^1(C, \mathfrak{g})\}$ on every boundary curve C can be regarded as the dual space to the affine Lie algebra, i.e. the Lie algebra $\mathfrak{g}^C$ extended by the two-cocycle (2). This dual space carries the natural linear Lie-Poisson structure. Coadjoint orbits of the affine group, or the symplectic leaves of this Lie–Poisson structure, consist of gauge-equivalent connections and differ by the holonomy around C, see e.g. [PS].

Remark 2.5C. In the complex case, the Hamiltonians of the infinitesimal G^N-action on $\mathcal{A}^N$ form the Lie algebra $\mathfrak{g}^N$ extended by the two-cocycle

$$c_\omega(X,Y) = \int_N tr(dX \wedge dY) \wedge \omega = -2\pi i \int_{\Sigma = \text{poles of } \omega} tr(XdY) \wedge \text{res } \omega$$

for $X, Y \in \mathfrak{g}^N$ (according to the Stokes–Leray formula (1), see [FK]).

Denote by $\alpha = \text{res } \omega|_\Sigma$ the residue 1-form of the 2-form ω on the curve Σ. Let $\hat{\mathfrak{g}}^\Sigma$ be the one-dimensional extension of the Lie algebra $\mathfrak{g}^\Sigma$ by the two-cocycle

$$c_\alpha(X,Y) = \int_\Sigma tr(X\bar{\partial}Y) \wedge \alpha,$$

where $X, Y \in \mathfrak{g}^\Sigma$. The space of the connections $\{\bar{\partial} + A(z,\bar{z}) \mid A \in \Omega^{0,1}(\Sigma, \mathfrak{g})\}$ with complex Lie algebra $\mathfrak{g}$ on a Riemann surface Σ can be regarded as (a hyperplane in) the dual space $(\hat{\mathfrak{g}}^\Sigma)^*$, see [EF].

The symplectic leaves of the Lie–Poisson structure in this dual space $(\hat{\mathfrak{g}}^{\Sigma})^*$ are enumerated by the equivalence classes of holomorphic G-bundles on the curve Σ, [EF]. Thus, Theorems 2.4**R-C** can be regarded as globalizations of considering the coadjoint orbits in the affine Lie algebra, or in the current Lie algebra $\hat{\mathfrak{g}}^{\Sigma}$, respectively.

Question 2.6. For a given graph on a surface S, the Poisson manifold $\mathcal{M}^S$ admits a combinatorial description as the quotient of the space of graph connections by the graph gauge group, see [FR]. What would be an analogous (complex) combinatorial description of the Poisson manifold $\mathcal{M}^N$?

§3. Complex Chern–Simons functional.

3.1. CS functional and flat connections. First, let D be a real compact three-dimensional manifold, E a trivial vector G-bundle on D, and $S = \partial D$ the boundary of D. Define the *Chern–Simons functional* $\mathrm{CS} : \mathcal{A}^D \to S^1$ on the space of connections $\mathcal{A}^D$ by the formula

$$\mathrm{CS}(A) = \frac{1}{4\pi}\int_D tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \qquad \mod 2\pi.$$

Theorem 3.1R (see e.g. [W1]). *A) The Chern–Simons functional* $\mathrm{CS} : \mathcal{A}^D \to \mathbb{R}/2\pi\mathbb{Z}$ *is gauge invariant with respect to the group G^D_S of currents based on the boundary $S = \partial D$. In particular, for a closed manifold D ($\partial D = \emptyset$) the* CS *functional is gauge invariant with respect to G^D.*

B) If $S = \partial D$ is nonempty, the variation of the CS *functional is a 1-form on the space $\mathcal{A}^S$ of boundary values of the connections. The differential of this 1-form is proportional to the symplectic 2-form of Proposition 2.2R on the space $\mathcal{A}^S$ of smooth connections on the two-dimensional surface S.*

Furthermore, as we discussed above, the moduli space $\mathcal{M}^S = \mathcal{A}^S_{\mathrm{fl}}/G^S$ of flat connections on S possesses a natural symplectic structure.

Theorem 3.2R (see e.g. [At, CLM]). *If a surface S is the boundary of a three-fold D then the (equivalence classes of all) those flat connections on S that can* *be extended to flat connections on D form an isotropic variety in the symplectic orbifold $\mathcal{M}^S$.*

In particular, a splitting of a three-fold $D = D_1 \bigcup_S D_2$ along a surface S into two handle bodies defines two Lagrangian varieties in the symplectic orbifold $\mathcal{M}^S$. For a rational homology 3-sphere D the intersection number of these varieties can be understood as the *Casson invariant* of D, see [CLM].

3.2. Complex CS setting. Now we consider a complex three-manifold M, a complex hypersurface $N \subset M$, and the space $\mathcal{H}^3(M \setminus N)$ of all meromorphic 3-forms on M with poles on N. To a 3-form $\eta \in \mathcal{H}^3(M \setminus N)$ one associates the *complex-valued Chern–Simons functional* $\mathrm{CS}_{\mathbb{C},\eta}$ on the space $\mathcal{A}^M_{\bar\partial}$ by the formula

$$\mathrm{CS}_{\mathbb{C},\eta}(A) := \frac{1}{8\pi^2}\int_M tr(A \wedge \bar\partial A + \frac{2}{3} A \wedge A \wedge A) \wedge \eta.$$

(This integral converges, since the singularities are of the first order and confine to the set N of real codimension 2.) Thus the *complex* $\mathrm{CS}_{\mathbb{C},*}$ *functional* can be regarded as the $(\mathcal{H}^3(M \setminus N))^*$-valued mapping $\langle \mathrm{CS}_{\mathbb{C},*}\ ,\ \eta\rangle := \mathrm{CS}_{\mathbb{C},\eta}$.

Note, that the homology group $H_3(M \setminus N, \mathbb{Z})$ forms a lattice in $(\mathcal{H}^3(M \setminus N))^*$: one can integrate the meromorphic form η over integral 3-cycles lying off N in M. In the case of Calabi–Yau manifolds M the form η is chosen to be the nonvanishing holomorphic 3-form, see [W2], cf.[FKT].

Theorem 3.1C ([FKR]). *A) The complex* $\mathrm{CS}_{\mathbb{C},*}$ *functional defines the mapping from* $\mathcal{A}^M_{\bar\partial}$ *to the quotient* $(\mathcal{H}^3(M \setminus N))^*/H_3(M \setminus N, \mathbb{Z})$ *gauge invariant with respect to the group* G^M_N *of currents on M based on the polar set N.*

B) Given meromorphic 3-form $\eta \in \mathcal{H}^3(M \setminus N)$*, the differential of the variation 1-form of the complex* $\mathrm{CS}_{\mathbb{C},*}$ *functional defines a holomorphic symplectic structure on the affine space* $\mathcal{A}^N_{\bar\partial}$ *of smooth connections on the polar set N. This symplectic structure coincides with the structure of Proposition 2.2C, where the 2-form ω on N is the residue of the 3-form η:* $\operatorname{res}\eta|_N = \omega$.

Let N be a K3 surface or complex torus of dimension 2 (i.e. N admits a nonvanishing holomorphic 2-form). Similarly to the real case, the moduli space $\mathcal{M}^N$ of holomorphic G-bundles (or the moduli space of the integrable $(0,1)$-connections) on the complex surface N possesses a (holomorphic) symplectic structure.

Theorem 3.2C. *Suppose that the surface N is the polar set of a meromorphic 3-form in a complex three-dimensional manifold M, i.e. N is a complex hypersurface in M and the space $\mathcal{H}^3(M \setminus N)$ is nontrivial. Then those holomorphic G-bundles on the surface N that can be extended to holomorphic G-bundles on the three-fold M form an isotropic variety in the moduli space $\mathcal{M}^N$.*

In other words, the natural restriction $\mathcal{M}^M \to \mathcal{M}^N$ has an isotropic image (see [FKR] for more detail). If the meromorphic 3-form on M has no zeroes, then this restriction map is Lagrangian (cf. [T]).

A transverse intersection of two such three-dimensional complex manifolds M_1 and M_2 along the complex surface $N = M_1 \bigcap M_2$ defines a pair of Lagrangian varieties in the holomorphic symplectic orbifold $\mathcal{M}^N$.

Question 3.3. *What analog of the Casson invariant is associated with the intersection of these two Lagrangian varieties in $\mathcal{M}^N$?*

References

[Ad] M. Audin, *Lectures on integrable systems and gauge theory*, preprint (1995), 1-52.

[A1] V.I. Arnold, *On some problems in singularity theory*, Trudy Seminare S.L. Sobolev, Novosibirsk **1** (1976), 5–15; (in Russian), English transl. in: Singularities, Proc. Symp. Pure Math., AMS **40:1** (1983), 57–69.

[A2] V.I. Arnold, *Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connectins and quantum Hall effect*, Selecta Mathematica, New Series **1:1** (1995), 1–19.

[A3] V.I. Arnold, *Arrangement of ovals of real plane algebraic curves, involutions of smooth four-dimensional manifolds, and on arithmetics of integral-valued quadratic forms*, Func. Anal. and Appl. **5:3** (1971), 169–176.

[At] M. Atiyah, *New invariants of three and four dimensional manifolds*, Symposium on the Math. Heritage of Hermann Weyl (R. Wells et al., eds.) **48** (1988), 288–294.

[AB] M. Atiyah and R. Bott, *The Yang-Mills equations over a Riemann surface*, Philos. Trans. Roy. Soc. London **308** (1982), 523–615.

[AHS] M. Atiyah, N. Hitchin, and I. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London A **362** (1978), 425–461.

[CLM] S.E. Cappell, R. Lee, and E.Y. Miller, *A symplectic geometry approach to generalized Casson's invariants of 3-manifolds*, Bull. of AMS **22:2** (1990), 269–275.

[Ch] S.S. Chern, *Complex manifolds*, Univ. of Chicago, 1955/56.

[EF] P.I. Etingof and I.B. Frenkel, *Central extensions of current groups in two dimensions*, Commun. Math. Phys **165** (1994), 429-444.

[FK] I.B. Frenkel and B.A. Khesin, *Four-dimensional realization of two-dimensional current groups*, Commun. Math. Phys **178** (1996), 541-562.

[FKT] I.B. Frenkel, B.A. Khesin, and A.N. Todorov, *in preparation.*

[FKR] V.V. Fock, B.A. Khesin, and A.A. Rosly, *in preparation.*

[FR] V.V. Fock and A.A. Rosly, *Poisson structures on moduli of flat connections on Riemann surfaces and r-matrices*, preprint (1992); *Flat connections and polyubles*, Teor. and Mat. Fizika **95:2** (1993), 228–238.

[GS] P. Griffiths and W. Schmid, *Recent developments in Hodge theory: a discussion of techniques and results*, in Discrete subgroups of Lie groups and applications to moduli, Bombay Colloquium 1973, Oxford Univ. Press (1975), 31–127.

[K] S. Kobayashi, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan (Iwanami Shoten and Princeton Univ. Press), 1987.

[L] J. Leray, *La calcul différential et intégral sur une variété analytique complexe (Problém de Cauchy, III)*, Bull. Soc. Math. France **87** (1959), 81-180.

[M] S. Mukai, *Symplectic structure of the moduli space of stable sheaves on an abelian or $K3$ surface*, Inv. Math. **77** (1984), 101–116.

[PS] A. Pressley and G. Segal, *Loop groups*, Clarendon Press, Oxford, 1986.

[T] A. Tyurin, *The moduli spaces of vector bundles on threefolds, surfaces, and curves*, Preprint, Math. Inst. Gottingen (1990).

[W1] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351-.

[W2] E. Witten, *Chern–Simons gauge theory as a string theory.*, The Floer memorial volume, Progr. Math. 133, (1995), Birkhäuser, Basel, 637-678.

Department of Mathematics, Yale University, New Haven, CT 06520, USA and
Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3, Canada
e-mail: khesin@math.yale.edu