Geometry of fluid motion

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Abstract

We survey two problems illustrating geometric-topological and Hamiltonian methods in fluid mechanics: energy relaxation of a magnetic field and conservation laws for ideal fluid motion. More details and results, as well as a guide to the literature on these topics can be found in [3].

1 Energy relaxation.

The first problem we are going to discuss is related to topological obstructions to energy relaxation of a magnetic field in a perfectly conducting medium. A motivation for this problem is the following model of a star. The magnetic field is supposed to be *frozen in* the perfectly conducting medium (plasma) filling the star, i.e. the topology of the field's trajectories does not change under the fluid flow. On the other hand, the magnetic energy can and does change, and the conducting fluid keeps moving (due to Maxwell's equations) until the excess of magnetic energy over its possible minimum is fully dissipated (this process is called "energy relaxation"). It turns out that mutual linking of magnetic lines may prevent complete dissipation of the magnetic energy. The problem is to describe the energy lower bounds of the magnetic field in terms of topological characteristics of its trajectories.

More precisely, consider a divergence-free (magnetic) vector field ξ in a (simply connected) bounded domain $M \subset \mathbb{R}^3$ tangent to the boundary ∂M. The *energy* of the field ξ is the square of its L_2-norm, i.e., the integral

$$E(\xi) = \int\limits_M (\xi, \xi)\, d^3x.$$

Let us act on the field ξ by a volume-preserving diffeomorphism h. Given divergence-free field ξ, the main problem is to give a good lower bound for the energy $\inf_h E(h_*\xi)$ of the

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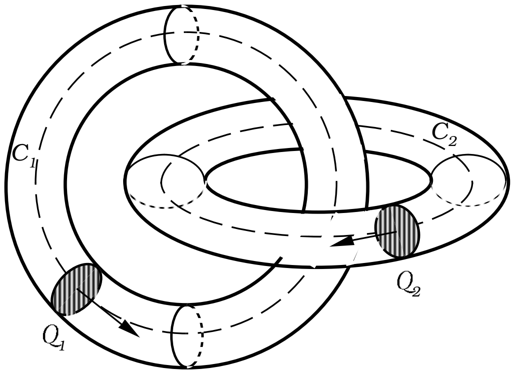

Figure 1: C_1, C_2 are axes of the tubes; Q_1, Q_2 are the corresponding fluxes.

push-forward field $h_*\xi$ in terms of topology of the field ξ. (Here we minimize the energy over the action of all volume-preserving diffeomorphisms h of M.)

1.A. Helicity bounds the energy. A topological obstruction to the energy relaxation can be seen in the example of a magnetic field confined to two linked solitori. Assume that the field vanishes outside those tubes and the field trajectories are all closed and oriented along the tube axes inside. To minimize the energy of a vector field with closed orbits by acting on the field by a volume-preserving diffeomorphism, one has to shorten the length of most trajectories. In turn, the shortening of the trajectories implies a fattening of the solitori (since the acting diffeomorphisms are volume-preserving).

For a linked configuration, as in Fig.1, the solitori prevent each other from endless fattening and therefore from further shrinking of the orbits. Therefore, heuristically, in the volume-preserving relaxation process the magnetic energy of the field supported on a pair of linked tubes is bounded from below and cannot attain too small values.

To describe the first obstruction to the energy minimization we need the following notion.

Definition 1.1 [8] *The helicity of the field ξ in the domain $M \subset \mathbb{R}^3$ is*

$$\mathcal{H}(\xi) = \int_M (\xi, \operatorname{curl}^{-1}\xi)\, d^3x,$$

where (,) is the Euclidean inner product, and $A = \operatorname{curl}^{-1}\xi$ is a divergence-free vector potential of the field ξ, i.e., $\nabla \times A = \xi$, $\operatorname{div} A = 0$.

1.2 Example. Consider a magnetic (that is, divergence-free) field ξ which is identically zero except in two narrow linked flux tubes whose axes are closed curves C_1 and C_2. The magnetic fluxes of the field in the tubes are Q_1 and Q_2 (Fig.1). Suppose further that there is no net twist within each tube or, more precisely, that the field trajectories foliate each of the tubes into pairwise unlinked circles. One can show that the helicity invariant of such a field is given by

$$\mathcal{H}(\xi) = 2\ lk(C_1, C_2) \cdot Q_1 \cdot Q_2,$$

where $lk(C_1, C_2)$ is the linking number of C_1 and C_2 [8]. Recall, that the (Gauss) *linking number* $lk(C_1, C_2)$ of two oriented closed curves C_1, C_2 in $\mathbb{R}^3$ is the signed number of the intersection points of one curve with an arbitrary (oriented) surface spanning the other curve.

Theorem 1.3 [2] *For a divergence-free vector field* ξ,

$$E(\xi) \geq C \cdot |\mathcal{H}(\xi)|,$$

where C *is a positive constant dependent of the shape and size of the compact domain* M.

The proof is a composition of the Schwarz inequality and the Poincaré inequality, applied to the potential vector field $A = \mathrm{curl}^{-1}\xi$.

1.4 REMARK. One can give a metric-free definition of helicity as follows. Let M be a simply connected manifold with a volume form μ, and ξ a divergence-free vector field on M. The latter means that the Lie derivative of μ along ξ vanishes: $L_\xi \mu = 0$, or, which is the same, the substitution $i_\xi \mu =: \omega_\xi$ of the field ξ to the volume form μ is a closed 2-form: $d\omega_\xi = 0$. On a simply connected manifold M the latter means that ω_ξ is exact: $\omega_\xi = d\alpha$ for some 1-form α (called a potential). (If M is not simply connected, that we have to require that the field ξ is null-homologous, i.e., that the 2-form ω_ξ is exact. If M has boundary, we require that ξ is tangent to it.)

Definition 1.5 [2] *The helicity* $\mathcal{H}(\xi)$ *of a null-homologous field* ξ *on a three-dimensional manifold* M *(possibly with boundary) equipped with a volume element* μ *is the integral of the wedge product of the form* $\omega_\xi := i_\xi \mu$ *and its potential:*

$$\mathcal{H}(\xi) = \int_M d\alpha \wedge \alpha, \ \text{where } d\alpha = \omega_\xi.$$

An immediate consequence of this pure topological (metric-free) definition is the following

Theorem 1.6 [2] *The helicity* $\mathcal{H}(\xi)$ *is preserved under the action on* ξ *of a volume-preserving diffeomorphism of* M.

In this sense $\mathcal{H}(\xi)$ is a topological invariant: it was defined without coordinates or a choice of metric, and hence every volume-preserving diffeomorphism carries a field ξ into a field with the same helicity.

1.B. What is helicity? V. Arnold proposed the following ergodic interpretation of helicity in the general case of any divergence-free field (when the trajectories are not necessarily closed or confined to invariant tori) as the average linking number of the field's trajectories. Let ξ be a divergence-free field on M and $\{g^t : M \to M\}$ its phase flow. We will associate to each pair of points in M a number that characterizes the "asymptotic linking" of the trajectories of the flow $\{g^t\}$ passing through these points. Given any

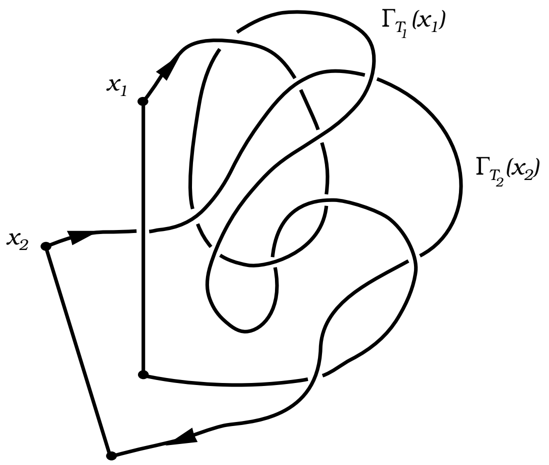

Figure 2: The long segments of the trajectories are closed by the "short paths."

two points x_1, x_2 in M and two large numbers T_1 and T_2, we consider "long segments" $g^t x_1 (0 \le t \le T_1)$ and $g^t x_2 (0 \le t \le T_2)$ of the trajectories of ξ issuing from x_1 and x_2. Close these long pieces by the shortest geodesics between $g^{T_k} x_k$ and x_k. We obtain two closed curves, $\Gamma_1 = \Gamma_{T_1}(x_1)$ and $\Gamma_2 = \Gamma_{T_2}(x_2)$; see Fig.2. Assume that these curves do not intersect (which is true for almost all pairs x_1, x_2 and for almost all T_1, T_2). Then the linking number $lk_\xi(x_1, x_2; T_1, T_2) := lk(\Gamma_1, \Gamma_2)$ of the curves Γ_1 and Γ_2 is well-defined.

Definition 1.7 [2] *The* asymptotic linking number *of the pair of trajectories $g^t x_1$ and $g^t x_2$ $(x_1, x_2 \in M)$ of the field ξ is defined as the limit*

$$\lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{lk_\xi(x_1, x_2; T_1, T_2)}{T_1 \cdot T_2},$$

where T_1 and T_2 are to vary so that Γ_1 and Γ_2 do not intersect.

It turns out that this limit exists (as an element of the space $L_1(M \times M)$ of the Lebesgue-integrable functions on $M \times M$) and is independent of the system of geodesics (i.e., of the Riemannian metric), see [12].

Theorem 1.8 [2] *The helicity of a divergence-free vector field ξ on a simply connected manifold M with a volume element μ is equal to the average self-linking of trajectories of this field, i.e., to the asymptotic linking number $\lambda_\xi(x_1, x_2)$ of trajectory pairs integrated over $M \times M$:*

$$\mathcal{H}(\xi) = \int_M \int_M \lambda_\xi(x_1, x_2) \; \mu_1 \mu_2.$$

1.C. Energy estimates. As we have seen above, a nonzero helicity (or average linking of the trajectories) of a field ξ provides a lower bound for the energy. However, heuristically, there should be a lower bound for the energy for a field which has at least one linked pair of solitori as in the example above, even if the total helicity vanishes. One of the the best results in this direction is as follows.

Theorem 1.9 [6] *Suppose a vector field ξ in $\mathbb{R}^3$ has an invariant torus T forming an nontrivial knot of type K. Then*

$$E(\xi) \geq \left(\frac{16}{\pi \cdot \mathrm{Vol}(T)}\right)^{1/3} \cdot Q^2 \cdot (2 \cdot \mathrm{genus}(K) - 1),$$

where $Q = \mathrm{Flux}\,\xi$ is the flux of ξ through a crossection of T, $\mathrm{Vol}(T)$ is the volume of the solid torus, and $\mathrm{genus}(K)$ is the genus of the knot K.

Recall, that for any knot its *genus* is the minimal number of handles of a spanning (oriented) surface for this knot. For an unknot the genus is 0, since one can take a disk as a spanning surface. For a nontrivial knot one has $\mathrm{genus}(K) \geq 1$ and, therefore, the above energy is bounded away from zero: $E(\xi) > 0$.

1.10 Remark. Note that there are no restrictions on the behavior of the divergence-free field inside this invariant torus, and hence this result has a wide range of applicability. In particular, it is sufficient for the field to have at least one closed linked trajectory of the *elliptic* type. The latter means that its Poincaré map has two eigenvalues of modulus 1. Then the KAM theory implies that a generic elliptic orbit is confined to a set of nested invariant tori. Hence any such orbit ensures that the energy of the corresponding field has a non-zero lower bound. The question remains whether the presence of any nontrivially linked closed trajectory (of any type: hyperbolic, non-generic, etc.) or the presence of chaotic behavior of trajectories for a field could provide a positive lower bound for the energy (even if the averaged linking of all trajectories totals zero) and therefore could prevent a relaxation of the field to arbitrarily small energies.

The rotation field in the three-dimensional ball is an example of an opposite type: all its trajectories are *pairwise unlinked.* It was suggested by A. Sakharov and Ya. Zeldovich, and proved by M. Freedman (see [5]), that this field can be transformed by a volume-preserving diffeomorphism to a field whose energy is less than any given ϵ.

2 Euler equations and geodesics.

2.A. Geodesics on Lie groups. In [1] V. Arnold suggested a general framework for the Euler equations on an arbitrary group, which we recall below. In this framework the Euler equation describes a geodesic flow with respect to a suitable one-sided invariant Riemannian metric on the given group.

More precisely, consider a (possibly infinite-dimensional) Lie group G, which can be thought of as the configuration space of some physical system. (Examples from [1]: $SO(3)$ for a rigid body or the group $\mathrm{SDiff}(M)$ of volume-preserving diffeomorphisms for an ideal fluid filling a domain M.) The tangent space at the identity of the Lie group G is the corresponding Lie algebra $\mathfrak{g}$. Fix some (positive definite) quadratic form, the "energy," on $\mathfrak{g}$. We consider right translations of this quadratic form to the tangent space at any point of the group (the "translational symmetry" of the energy). This way the energy defines a right-invariant Riemannian metric on the group G. The geodesic flow on G with respect

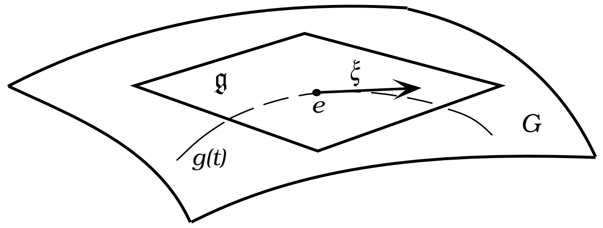

Figure 3: The vector ξ in the Lie algebra $\mathfrak{g}$ is the velocity at the identity e of a geodesic $g(t)$ on the Lie group G.

to this energy metric represents the extremals of the least action principle, i.e., the actual motions of our physical system.[1]

To describe a geodesic on the *Lie group* with an initial velocity $v(0) = \xi$, we transport its velocity vector at any moment t to the identity of the group (by using the right translation). This way we obtain the evolution law for $v(t)$, given by a (non-linear) dynamical system $dv/dt = F(v)$ on the *Lie algebra* $\mathfrak{g}$ (Fig.3).

Definition 2.1 *The system on the Lie algebra $\mathfrak{g}$, describing the evolution of the velocity vector along a geodesic in a right-invariant metric on the Lie group G, is called the* Euler equation *corresponding to this metric on G.*

2.B. Example: fluid motion. The main example is the Euler equation for incompressible fluid filling some domain M in $\mathbb{R}^n$. The fluid motion is described by a velocity field $v(t,x)$ and a pressure field $p(t,x)$ which satisfy the classical Euler equation:

$$\partial_t v + (v \cdot \nabla) v = -\nabla p, \tag{2.1}$$

where $\mathrm{div} v = 0$ and the field v is tangent to the boundary of M.

The flow $(t,x) \to g(t,x)$ describing the motion of fluid particles is defined by its velocity field $v(t,x)$:

$$\partial_t g(t,x) = v(t, g(t,x)), \;\; g(0,x) = x.$$

The chain rule immediately gives $\partial_{tt}^2 g(t,x) = \partial_t v + (v \cdot \nabla) v$, and hence the Euler equation is equivalent to

$$\partial_{tt}^2 g(t,x) = -(\nabla p)(t, g(t,x)),$$

while the incompressibility condition is $\det(\partial_x g(t,x)) = 1$. The latter form of the Euler equation (for a smooth $g(t,x)$) exactly means that it describes a geodesic on the set of volume-preserving diffeomorphisms. Indeed, the acceleration of the flow $(\partial_{tt}^2 g)$, being given by a gradient $(-\nabla p)$, is orthogonal to all divergence-free fields, the tangent space to this set.

A similar equation describes an ideal incompressible fluid filling an arbitrary manifold M equipped with a volume form μ. It turns out that the group-geodesic point of view,

[1]For a rigid body one has to consider left translations, but in our exposition we stick to the right-invariant case in view of its applications to the groups of diffeomorphisms.

developed in [1] is quite fruitful for topological and qualitative understanding of the fluid motion, as well as for obtaining various quantitative results related to stability.

As an illustration we consider here the question of first integrals of the Euler equation on manifolds of different dimension.

2.C. Conservation laws in ideal hydrodynamics. The Euler equation of an ideal fluid (2.1) filling a three-dimensional simply connected manifold has the helicity (or Hopf) invariant, discussed in Section 1.A. This invariant describes the mutual linking of the trajectories of the vorticity field curl$v$, and has the form

$$J(v) = \mathcal{H}(\operatorname{curl} v) = \int_{\mathbb{R}^3} (\operatorname{curl} v,\ v)\ d^3x$$

in the Euclidean space $\mathbb{R}^3$.

For an ideal two-dimensional fluid one has an infinite number of conserved quantities. For example, for the standard metric in $\mathbb{R}^2$ one has the *enstrophy invariants*

$$J_k(v) = \int_{\mathbb{R}^2} (\operatorname{curl}\ v)^k\ d^2x = \int_{\mathbb{R}^2} (\Delta\psi)^k\ d^2x,\ \text{ for } k = 1, 2, \dots,$$

where $\operatorname{curl}\ v = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}$ is a *vorticity function* of a 2D flow.

It turns out that helicity-type integrals do exist for all odd-dimensional ideal fluid flows, and so do enstrophy-type integrals for all even-dimensional flows. (In a sense, the situation here is similar to the dichotomy of contact and symplectic geometry in odd- and even-dimensional spaces.) Let M be a manifold equipped with a volume form μ and Riemannian metric $(.,.)$, and we consider the motion of an ideal fluid filling M. First, define the 1-form u as the pointwise inner product with vectors of the velocity field v with the help of the Riemannian metric on the manifold M:

$$u(\xi) = (v, \xi) \ \text{ for all } \ \xi \in T_xM.$$

Theorem 2.2 ([11] for $\mathbb{R}^n$ and [4] for any M) *The Euler equation of an ideal incompressible fluid on a Riemannian manifold M^n (possibly with boundary) with a measure form μ has*

(i) the first integral

$$I(v) = \int_M u \wedge (du)^m$$

in the case of an arbitrary odd-dimensional manifold M $(n = 2m + 1)$; and

(ii) an infinite number of functionally independent first integrals

$$I_f(v) = \int_M f\left(\frac{(du)^m}{\mu}\right)\mu$$

in the case of an arbitrary even-dimensional manifold M $(n = 2m)$, where the 1-form u and the vector field v are related by means of the metric on M, and $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function of one variable.

The fraction $(du)^m/\mu$ for $n = 2m$ is a ratio of two differential forms of the highest degree n. Since the volume form μ vanishes nowhere, the ratio is a well-defined function on M (which may depend on time t). The integral of the function f evaluated at this ratio gives a generalized momentum (i.e., a weighted volume between different level hypersurfaces) of the function $(du)^m/\mu$.

2.3 Example. For the standard metric in $\mathbb{R}^n$ the integrals assume the following form. Let v be the velocity vector field of the fluid in the domain $M \subset \mathbb{R}^n$. Define the components of the generalized curl (or vorticity 2-form) of v by setting $\omega_{ij} := \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$. Then one has

(i) the first integral

$$I(v) = \int_M \sum_{(i_1 \dots i_{2m+1})} \varepsilon^{i_1 \dots i_{2m+1}} v_{i_1} \omega_{i_2 i_3} \dots \omega_{i_{2m} i_{2m+1}} \, d^n x$$

if the dimension n is odd: $n = 2m + 1$;

(ii) an infinite number of independent first integrals

$$I_k(v) = \int_M \left(\det \|\omega_{ij}\|\right)^k \, d^n x$$

if the dimension n is even: $n = 2m$.

Here $\det \|\omega_{ij}\|$ is the determinant of the skew-symmetric matrix $\|\omega_{ij}\|$, the summation in (i) goes over all permutations of the set $(1 \dots 2m+1)$, and $\varepsilon^{i_1 \dots i_{2m+1}}$ is the Kronecker symbol equal to the parity of the permutation $(i_1 \dots i_{2m+1})$. (The momenta I_k correspond to the choice $f(z) = z^{2k}$ in the theorem above.)

This theorem follows, practically without calculations, from the definition of the coadjoint action of the diffeomorphisms group, when formulated in the invariant and coordinate-free way.

2.D. Hamiltonian framework for the Euler equations.

The differential-geometric description of the Euler equation as a geodesic flow on a Lie group has a Hamiltonian reformulation.

Fix the notation $E(v) = \frac{1}{2}\langle v, Av\rangle$ for the energy quadratic form on $\mathfrak{g}$ which we used to define the Riemannian metric. Identify the Lie algebra and its dual with the help of this quadratic form. This identification $A : \mathfrak{g} \to \mathfrak{g}^*$ (called the *inertia operator*) allows one to rewrite the Euler equation on the dual space $\mathfrak{g}^*$.

It turns out that the Euler equation on $\mathfrak{g}^*$ is Hamiltonian with respect to the natural Lie–Poisson structure on the dual space [1]. Moreover, the corresponding Hamiltonian function is minus the energy quadratic form lifted from the Lie algebra to its dual space by the same identification: $-H(m) = -\frac{1}{2}\langle A^{-1}m, m\rangle$, where $m = Av$. Here we are going to take it as the *definition* of the Euler equation on the dual space $\mathfrak{g}^*$. (The minus is related to the consideration of a right-invariant metric on the group. It changes to plus for left-invariant metrics.)

Definition 2.4 (see, e.g., [3]) *The* Euler equation on $\mathfrak{g}^*$, *corresponding to the right-invariant metric $E(m) = \frac{1}{2}\langle Av, v\rangle$ on the group, is given by the following explicit formula:*

$$\frac{dm}{dt} = -\mathrm{ad}^*_{A^{-1}m} m,$$

as an evolution of a point $m \in \mathfrak{g}^*$.

2.5 Remark. For the ideal fluid the Lie algebra $\mathfrak{g} = \mathrm{SVect}(M)$ consists of divergence-free vector field in M. The corresponding dual space $\mathfrak{g}^* = \Omega^1(M)/\Omega^0(M)$ is the space of cosets of 1-forms on M modulo exact 1-forms. The group coadjoint action is the change of coordinates in the 1-form, while the corresponding Lie algebra action is the Lie derivative along a vector field. The Euler equation (2.4) on the dual space has the form

$$\partial_t[u] = -L_v[u],$$

where $[u] \in \Omega^1(M)/\Omega^0(M)$ stands for a coset of 1-forms and the vector field v is related with a 1-form u by means of a Riemannian metric on M. The latter equation for a coset $[u]$ can be rewritten as an equation for a representative 1-form modulo a function differential dp:

$$\partial_t u + L_v u = -dp,$$

where one can recognize the elements of the Euler equation (2.1) for an ideal fluid.

The invariance of the integrals in Theorem 2.2 essentially follows from their coordinate-free definition. The latter means that the integrals are invariant with respect to coordinate changes, and hence, are invariants of the corresponding Euler equations.

2.E. Geodesic description for various equations. A similar Arnold-type description as the geodesic flow on a Lie group can be given to a variety of conservative dynamical systems in mathematical physics. Below we list several examples of such systems to emphasize the range of applications of this approach. (This list is by no means complete. There are plenty of other interesting systems, e.g., super-equations or gas dynamics.) The choice of a group G (column 1) and an energy metric E (column 2) defines the corresponding Euler equations (column 3).

Group	*Metric*	*Equation*
$SO(3)$	$< \omega, A\omega >$	Euler top
$SO(3) \dot{+} \mathbb{R}^3$	quadratic forms	Kirchhoff equations for a body in a fluid
$SO(n)$	Manakov's metrics	$n-$dimensional top
$\mathrm{Diff}(S^1)$	L^2	Hopf (or, inviscid Burgers) equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa – Holm equation
Virasoro	$\dot{H}^1$	Hunter – Saxton (or Dym) equation
$\mathrm{SDiff}(M)$	L^2	Euler ideal fluid
$\mathrm{SDiff}(M) \dot{+} \mathrm{SVect}(M)$	L^2	Magnetohydrodynamics
$\mathrm{Maps}(S^1, SO(3))$	H^{-1}	Landau – Lifschits equation

Note that in some cases these systems turn out to be not only Hamiltonian, but also bihamiltonian, and the geodesic description helps in describing the corresponding Poisson pairs (this is the case, e.g., for the systems related to the Virasoro group, see [10, 7]). More detailed descriptions and references can be found in the book [3].

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