BOOK REVIEWS

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Topological methods in hydrodynamics, Second edition, by V. Arnold and B. Khesin, Applied Mathematical Sciences, Vol. 125, Springer, Cham, 2021, xx+455 pp., ISBN 978-3-030-74277-5

Hydrodynamics is a vast field with long history and enormous literature, including excellent treatises and monographs devoted to this beautiful and challenging subject; see, e.g., [Bir50], [CM93], [LL59], [Lic68], [MB02], [MP94], and also the more recent surveys [Con07], [DE], [KMS] and references therein. And yet the book of Arnold and Khesin is truly unique because of its deliberate, consistently geometric approach and because of the way in which it manages to combine the best qualities of an original research monograph and an inspiring graduate textbook. In fact, the first edition of this book has already become a classic right after its publication in 1998, and the second, extended, edition under review here will surely strengthen its landmark position in the area of geometric hydrodynamics.

Perhaps the opening paragraph of the Preface to the first edition describes best how the authors view their subject:

> Hydrodynamics is one of those fundamental areas in mathematics where progress at any moment may be regarded as a standard to measure the real success of mathematical science.

From its very beginnings (thanks to the pioneering work of Euler), hydrodynamics developed two parallel but distinct perspectives on the motion of an ideal, that is, incompressible and inviscid, fluid in a fixed domain M (typically, $\mathbb{R}^n$ or $\mathbb{T}^n$ or a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂M and $n = 2, 3$). In the first of these, referred to as Eulerian, the fluid is described from the viewpoint of a fixed observer. Its main object of study is the familiar system of nonlinear partial differential equations, the incompressible Euler equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \qquad \text{in } M, \tag{0.1}$$

where $u = u(t, x)$ is the velocity of the fluid particle which at time t is at $x \in M$ and $p = p(t, x)$ is the pressure function at the same time and position. In the boundary case the fluid is not allowed to penetrate the boundary, which introduces an additional condition

$$u \cdot n = 0 \qquad \text{on } \partial M, \tag{0.2}$$

where n is the outward unit normal to ∂M. The first equation in (0.1) corresponds to conservation of momentum and the second represents incompressibility. Both equations make perfect sense on a general n-dimensional Riemannian manifold, in which case $u \cdot \nabla$ is interpreted as the covariant derivative in the direction of the vector field u. Although not immediately apparent, the equations are also nonlocal in the sense that the gradient term ∇p at time t cannot be determined from the information around x alone. In fact, applying divergence to the first equation in (0.1) leads to the Neumann problem $\Delta p = -\operatorname{div}(u \cdot \nabla u)$ with $\partial p/\partial n = -(u \cdot \nabla u) \cdot n$.

The second approach is called Lagrangian.[1] In this case the fluid is described by an observer who follows the trajectories of fluid particles. Given a velocity field $u(t, x)$ the solution of the Cauchy problem

$$\frac{d\gamma}{dt}(t,x) = u(t, \gamma(t,x)), \quad \gamma(0,x) = x, \tag{0.3}$$

defines the particle flow map with $\gamma(t,x)$ representing the position at time t of that fluid particle which at time $t = 0$ was at x. Assuming that particles do not fuse or split, a natural configuration space of an ideal fluid is thus the group of diffeomorphisms $S\mathrm{Diff}(M)$ preserving the volume form μ of the domain M. One can regard $S\mathrm{Diff}(M)$ as a smooth manifold modeled on a Fréchet space and an infinite-dimensional Lie group under composition and inversion of diffeomorphisms.[2] Its tangent space at the identity map e serves both as the modeling space of $S\mathrm{Diff}(M)$ and its Lie algebra $T_e S\mathrm{Diff}(M) = S\mathrm{Vect}(M)$. It consists of smooth divergence-free vector fields on M and is equipped with the commutator given by (minus) the Lie–Poisson bracket of vector fields

$$[v, w] = -\{v, w\}, \qquad v, w \in S\mathrm{Vect}(M), \tag{0.4}$$

and defined using the Lie derivative by $L_{\{v,w\}} = L_v L_w - L_w L_v$.

Absent any external forces, the dynamics of an ideal fluid is determined by the incompressibility constraint and, according to the least action principle, any motion of an ideal fluid in M traces out a geodesic path $t \mapsto \gamma(t)$ in $S\mathrm{Diff}(M)$ of the Riemannian metric defined by the fluid's kinetic energy. This metric is right-invariant as is easily verified by the change of variables formula. Furthermore, at the identity it coincides with the L^2 inner product

$$\langle v, w \rangle_{L^2} = \int_M v \cdot w \, d\mu, \qquad v, w \in S\mathrm{Vect}(M), \tag{0.5}$$

and establishes a correspondence between $S\mathrm{Vect}(M)$ and (the "smooth part" of) the dual $S\mathrm{Vect}(M)^* \simeq \Omega^1(M)/d\Omega^0(M)$ of 1-forms modulo differentials of functions

$$A : S\mathrm{Vect}(M) \to S\mathrm{Vect}(M)^*, \qquad (Av, w) = \langle v, w \rangle_{L^2}, \tag{0.6}$$

called the inertia operator.

The geodesic equations of the L^2 metric in $S\mathrm{Diff}(M)$ can be derived explicitly from the first Riemannian principles (e.g., by computing the first variation formula).

[1] An unfortunate though convenient misnomer since both descriptions are due to Euler; see [Tru54].

[2] See, e.g., [EM70], [Ham82].

Alternatively, differentiating the flow equation (0.3) and using (0.1), we find

$$\frac{d^2\gamma}{dt^2}(t) = -\nabla p \circ \gamma(t) \tag{0.7}$$

(suppressing the dependence on x) subject to the initial conditions

$$\frac{d\gamma}{dt}(0) = u_0 \quad \text{and} \quad \gamma(0) = e. \tag{0.8}$$

In particular, (0.7) shows that the acceleration vector of γ must be L^2 orthogonal to the space of divergence-free vector fields on M.

While some early ideas go back to the XIX century work of Helmholtz, Kelvin, and Poincare, the above picture was first explained and presented in a clear group theoretic and differential geometric language by Arnold [Arn66] in the 1960s. This beautiful picture lies at the heart of the book under review.

The motivation for its development can be traced back to Kolmogorov's Seminar at Moscow State University in 1958/59 and the early efforts to understand the phenomenon of turbulence; see [AM]. Kolmogorov proposed to study turbulence in low-viscosity regimes hoping to find hydrodynamic attractors by means of ideal fluids. In the Lagrangian setting turbulence could be then rationalized if these attractors were chaotic, in particular, highly sensitive to changes in initial conditions. This provides at least one important reason for the appeal of the geometric approach: since fluid flows correspond to geodesics in $S\mathrm{Diff}(M)$, curvature calculations should help identify regions where such attractors may be located.

But even a closer examination of the geometry and the group structure of $S\mathrm{Diff}(M)$ already reveals valuable information about ideal fluids. Thus, at the infinitesimal level the Lie algebra $S\mathrm{Vect}(M)$ and its dual provide the setting for a Hamiltonian reformulation of (0.1) as the so-called Euler–Arnold equations

$$\frac{\partial[\alpha]}{\partial t} = -L_{A^{-1}[\alpha]}[\alpha], \qquad [\alpha] \in \Omega^1(M)/d\Omega^0(M), \tag{0.9}$$

while their structure determines the well-known conservation laws of ideal hydrodynamics involving enstrophy $\int_{M^2} f(\operatorname{curl} u)\, d\mu$ and helicity $\int_{M^3} \operatorname{curl} u \cdot u \, d\mu$. Here, $\operatorname{curl} u$ is obtained from the vorticity 2-form associated with u by identification with a function if $n = 2$ and with a vector field if $n = 3$ and f is any smooth function. The coadjoint orbits of $S\mathrm{Diff}(M)$ have an elegant description in terms of isovorticity, even though when $n = 3$ their geometry is known to be extremely complicated and its precise description remains an intriguing open problem (n.b., a pair of velocity fields v, w is said to be isovortical if there is an axisymmetric diffeomorphism η on M such that the circulation of v around any closed contour c in M and the circulation of w around $\eta \circ c$ are equal).

At the local level, investigating the equations of geodesic deviation in $S\mathrm{Diff}(M)$ leads naturally to problems of hydrodynamic stability under small perturbations of initial data. Here, the situation turned out to be somewhat more challenging than had been originally hoped. Although $S\mathrm{Diff}(M)$ is negatively curved in "most directions" it also has regions of strictly positive sectional curvature which contain conjugate points. The latter are the singular values of the exponential map $\exp_e(tu_0) = \gamma(t)$ of the L^2 metric (0.5), where γ is the unique solution of (0.7)–(0.8) defined at least for small t. Roughly speaking, they indicate the presence of configurations of "relative" Lagrangian stability in the sense that a family of fluid flows starting at e with different velocities, after initially diverging, will eventually come together due to positive curvature near a conjugate point. Such behaviour is of course very different from that typical of solutions on chaotic hyperbolic attractors (think of geodesic flows on negatively curved spaces).

Last but not least, there are results involving global properties of the group of volume-preserving diffeomorphisms, especially those which bring out the difference between two-dimensional and three-dimensional hydrodynamics. For example, it turns out that the L^2 diameter of $S\mathrm{Diff}(M)$, where M is a simply connected 3-manifold, is finite while that of a 2-manifold is infinite. This surprising result seems to depend ultimately upon the fact that in a 3-manifold fluid particles have "more room" to move around each other. A related result states that finding a shortest path connecting a pair of diffeomorphisms in $S\mathrm{Diff}(M)$ may not be possible in general if the fluid domain is three dimensional. Furthermore, the L^2 exponential map on $S\mathrm{Diff}(M)$ is a nonlinear Fredholm map (of index zero) if $n = 2$ but not if $n = 3$. An interesting consequence of this is the noninjectivity of $\exp_e$ near conjugate points. Its hydrodynamic interpretation is that every neighbourhood of a conjugate point in $S\mathrm{Diff}(M)$ (in any "reasonable" topology) must contain fluid configurations that can be reached from e by (at least) two distinct fluid flows in the same time. Another result is that $S\mathrm{Diff}(M)$ has no self-intersecting geodesics unless they are periodic (i.e., smooth images of a circle). Note that this is a property of bi-invariant metrics on Lie groups.

Hydrodynamical implications of results like these are yet to be fully worked out. They may help us to better understand the behaviour of fluids over long time intervals and will provide a strong motivation for future research.

The geometric framework described above is very flexible. It can be developed in the setting of general Lie groups equipped with one-sided invariant metrics and include numerous examples of great interest in mathematical physics. In particular, the Euler top, Kirchhoff's equations of motion of a rigid body in an ideal fluid, the equations of magneto-hydrodynamics, a number of completely integrable PDEs in $1+1$ (space-time) dimensions, or the family of generalized surface quasi-geostrophic equations can all be (with minor adjustments) expressed as geodesic equations on a suitable (infinite-dimensional) Lie group and then reduced to its Lie algebra as the corresponding Euler–Arnold equations.

Most of the topics above are found already in the first edition of the book, which also covers topological obstructions to energy relaxation problems leading to the notion of helicity and its ergodic interpretation as the asymptotic linking number, a detailed discussion of conservation laws in ideal fluid dynamics including higher-dimensional analogues of enstrophy and helicity, applications to the problem of hydrodynamic stability, and the famous Arnold's criterion for Liapunov stability of two-dimensional fluids, various ramifications of the fast dynamo problem, generalized flows of Y. Brenier and their application to the shortest path problem (the whole Section IV.7 was contributed by A. Shnirelman) and bi-Hamiltonian structures of nonlinear evolution equations arising in various hydrodynamical approximations. The second edition retains the structure and all the virtues of the original text (as well as its updated 2007 Russian version) and, in addition, includes a 40-page long Appendix with its own bibliography. It is a real treasure trove of ideas presented clearly and lucidly in the best tradition of the Russian mathematical school. It is also a testament to the authors' lifelong fascination and interest in the challenging mathematics of fluid dynamics, closely intertwined as it is with geometry, topology, and analysis. The book lays down the foundations of a new field of mathematics with important contributions made by both authors as well as other mathematicians, and it includes many open problems at the frontier of current research. It has already helped educate a whole generation of graduate students and scholars, young and old. In light of this, the second author's decision to keep the text of the first edition essentially unchanged, supplementing it with an Appendix structured in roughly the same order as the original text, is entirely justified and should be applauded. The Appendix is written in a similarly clear and engaging manner and does an excellent job of outlining new approaches and developments in the field as well as surveying the enormous literature from the two decades since the publication of the first edition. Among the topics we find here a generalization of the Euler–Arnold equations to the setting of groupoids which includes fluids with moving boundaries as well as vortex sheets and generalized flows, a description of invariant knots and tubes with arbitrary topology in steady Euler flows, and a discussion of the relation between Lagrangian and Eulerian instabilities. The two hundred or so new references make it easy for a reader to follow up on any of these topics.

I have no doubt that this milestone monograph will serve as a classic reference and an invaluable guide for all those interested in group theoretic, differential geometric, or topological aspects of hydrodynamics. It will also surely continue to attract new generations of students and researchers to the field.

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