Commun. Math. Phys.
Digital Object Identifier (DOI) 10.1007/s00220-007-0234-2

Communications in
**Mathematical
Physics**

Pseudodifferential Symbols on Riemann Surfaces and Krichever–Novikov Algebras

Dmitry Donin, Boris Khesin

Dept. of Mathematics, University of Toronto, Toronto, Ont M5S 2E4, Canada.
E-mail: donin@math.toronto.edu; khesin@math.toronto.edu

Received: 6 April 2006 / Accepted: 23 October 2006

Abstract: We define the Krichever-Novikov-type Lie algebras of differential operators and pseudodifferential symbols on Riemann surfaces, along with their outer derivations and central extensions. We show that the corresponding algebras of meromorphic operators and symbols have many invariant traces and central extensions, given by the logarithms of meromorphic vector fields. Very few of these extensions survive after passing to the algebras of operators and symbols holomorphic away from several fixed points. We also describe the associated Manin triples and KdV-type hierarchies, emphasizing the similarities and differences with the case of smooth symbols on the circle.

1. Introduction

The Krichever-Novikov algebras are the (centrally extended) Lie algebras of meromorphic vector fields on a Riemann surface Σ, which are holomorphic away from several fixed points [7, 8], see also [11, 15]. They are natural generalizations of the Virasoro algebra, which corresponds to the case of $\Sigma = \mathbb{C}P^1$ with two punctures. Central extensions of the corresponding algebras of vector fields on a given Riemann surface are defined by fixing a projective structure (that is a class of coordinates related by projective transformations) and the corresponding Gelfand-Fuchs cocycle, along with the change-of-coordinate rule.

In this paper we deal with two generalizations of the Krichever-Novikov (KN) algebras. The first one is the Lie algebras of all *meromorphic* differential operators and *pseudodifferential symbols* on a Riemann surface, while the second one is the Lie algebras of meromorphic differential operators and pseudodifferential symbols which are *holomorphic* away from several fixed points. The main tool which we employ is fixing a reference meromorphic vector field instead of a projective structure on Σ. It turns out that such a choice allows one to write more explicit formulas for the corresponding cocycles, both for the Krichever-Novikov algebra of vector fields and for its generalizations.

Several features of these algebras of *meromorphic* symbols make them different from their *smooth* analogue, the algebra of pseudodifferential symbols with smooth coefficients on the circle. First of all, this is the existence of many invariant traces on the former algebras: one can associate such a trace to every point on the surface. Furthermore, we show that the logarithm $\log X$ of any meromorphic pseudodifferential symbol X defines an outer derivation of the Lie algebra of meromorphic symbols. In turn, the combination of invariant traces and outer derivations produces a variety of independent non-trivial 2-cocycles on the Lie algebras of meromorphic pseudodifferential symbols and differential operators, as well as it gives rise to Lie bialgebra structures (see Sect. 2). Note that the above mentioned scheme of generating numerous 2-cocycles in the meromorphic case, which involve $\log X$ for any meromorphic pseudodifferential symbol X, provides a natural unifying framework for the existence of two independent cocycles (generated by $\log \partial/\partial x$ and $\log x$) in the smooth case, cf. [6, 5].

The second type of algebras under consideration, those of *holomorphic* differential operators and pseudodifferential symbols, are more direct generalizations of the Krichever-Novikov algebra of holomorphic vector fields on a punctured Riemann surface. For them we prove the density and filtered generalized grading properties, similarly to the corresponding properties of the KN algebras [7, 8]. Furthermore, one can adapt the notion of a local cocycle proposed in [7] to the filtered algebras of (pseudo)differential symbols. It turns out that all logarithmic cocycles become linearly dependent when we confine to local cocycles on holomorphic differential operators. On the other hand, for holomorphic pseudodifferential symbols the local cocycles are shown to form a two-dimensional space (see Sect. 3).

Finally, for meromorphic differential operators, as well as for holomorphic differential operators on surfaces with trivialized tangent bundle, there exist Lie bialgebra structures and integrable hierarchies mimicking the structures in the smooth case.

We deliberately put the exposition in a form which emphasizes the similarities with and differences from the algebras of (pseudo)differential symbols with smooth coefficients on the circle, developed in [3, 5]. In many respects the algebras of holomorphic symbols extended by local 2-cocycles turn out to be similar to their smooth counterparts on the circle. On the other hand, by giving up the condition of locality, one obtains higher-dimensional extensions of the Lie algebras of holomorphic symbols by means of the 2-cocycles related to different paths on the surface. This way one naturally comes to holomorphic analogues of the algebras of "smooth symbols on graphs," which also have central extensions given by 2-cocycles on different loops in the graphs.

2. Meromorphic Pseudodifferential Symbols on Riemann Surfaces

2.1. The algebras of meromorphic differential and pseudodifferential symbols. Let Σ be a compact Riemann surface and $\mathcal{M}$ be the space of meromorphic functions on Σ. Fix a meromorphic vector field v on the surface and denote by D (or D_v) the operator of Lie derivative L_v along the field v. Then D sends the space $\mathcal{M}$ to itself, and one can consider the operator algebras generated by it.

Definition 2.1. *The associative algebras of* ***meromorphic differential operators***

$$MDO := \left\{ A = \sum_{k=0}^{n} a_k D^k \mid a_k \in \mathcal{M} \right\}$$

and ***meromorphic pseudodifferential symbols***

$$M\Psi DS := \left\{ A = \sum_{k=-\infty}^{n} a_k D^k \mid a_k \in \mathcal{M} \right\}$$

are the above spaces of formal polynomials and series in D which are equipped with the multiplication $\circ$ defined by

$$D^k \circ a = \sum_{\ell \geq 0} \binom{k}{\ell} (D^\ell a) D^{k-\ell}, \tag{2.1}$$

where the binomial coefficient $\binom{k}{\ell} = \frac{k(k-1)...(k-\ell+1)}{\ell!}$ makes sense for both positive and negative k. This multiplication law naturally extends the Leibnitz rule $D \circ a = aD + (Da)$. The algebras MDO and $M\Psi DS$ are also Lie algebras with respect to the bracket

$$[A, B] = A \circ B - B \circ A.$$

Note that the algebra MDO is both an associative and Lie subalgebra in $M\Psi DS$. (In the sequel, we will write simply XY instead of $X \circ Y$ whenever this does not cause an ambiguity.)

For different choices of the meromorphic vector field v the corresponding algebras of (pseudo) differential symbols are isomorphic: any other meromorphic field w on Σ can be presented as $w = fv$ for $f \in \mathcal{M}$, and then the relation $D_w = fD_v$ delivers the (both associative and Lie) algebra isomorphism. Meromorphic vector fields on Σ embed both into MDO and $M\Psi DS$ as Lie subalgebras.

Remark 2.2. Equivalently, one can define the product of two pseudodifferential symbols by the following formula: if $A(D) = \sum_{i=-\infty}^{m} a_i D^i$ and $B(D) = \sum_{j=-\infty}^{n} b_j D^j$, for $D := L_v$ then

$$A \circ B := \left(\sum_{k \geq 0} \frac{1}{k!} \partial_\xi^k A(\xi)\, \partial_v^k B(\xi) \right)_{\xi = D}. \tag{2.2}$$

Here ∂_v is the operator of taking the Lie derivative of coefficients of a symbol (i.e. of functions b_j) along v. Note that the right-hand side of this formula expresses the commutative multiplication of functions $A(z, \xi)$ and $B(z, \xi)$. Of course, this formula also extends the usual composition of differential operators.

2.2. Outer derivations of pseudodifferential symbols. It turns out that both the associative and Lie algebras of meromorphic pseudodifferential symbols have many outer derivations.

Definition 2.3. (cf. [6]) *Let v be a meromorphic vector field on Σ and set $D := L_v$. Define the operator* $\log D$ *or, rather,* $[\log D, \cdot] : M\Psi DS \to M\Psi DS$ *by*

$$[\log D, aD^n] := \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} (D^k a) D^{n-k}. \tag{2.3}$$

The above formula is consistent with the Leibnitz formula (2.1): it can be obtained from the latter by regarding k as a complex parameter, say, λ and differentiating in λ at $\lambda = 0$: $d/d\lambda|_{\lambda=0} D^\lambda = \log D$. (Below we will be using the notation $\log D$ for the derivation and $[\log D, \cdot]$ for explicit formulas.)

Proposition 2.4. *The operator* $\log D : M\Psi DS \to M\Psi DS$ *defines a derivation of the (both associative and Lie) algebra* $M\Psi DS$ *of meromorphic pseudodifferential symbols for any choice of the meromorphic vector field* v.

Proof. One readily verifies that for any two symbols A and B,

$$[\log D, AB] = [\log D, A]B + A[\log D, B],$$

i.e. $\log D$ is a derivation of the associative algebra $M\Psi DS$. This also implies that $\log D$ is a derivation of the Lie algebra structure. □

It turns out that one can describe a whole class of derivations in a similar way.

Definition 2.5. *Associate to any meromorphic pseudodifferential symbol* X *the derivation* $\log X : M\Psi DS \to M\Psi DS$, *where the commutator* $[\log X, A]$ *with a symbol* A *is defined by means of the formula* (2.2).

Namely, recall that $\log(v(z)\xi)$ can be regarded as a (multivalued) symbol for $\log D_v$, i.e. a multivalued function on $T^*\Sigma$. Indeed, only the derivatives of this function in ξ or along the field v appear in the formula for the commutator $[\log D, A] = \log D \circ A - A \circ \log D$, where the products in the right-hand-side are defined by formula (2.2). Similarly, one can regard $\log X(\xi)$ as a function on $T^*\Sigma$ and only its derivatives appear in the commutators $[\log X, A]$ with any meromorphic symbol $A \in M\Psi DS$.

Remark 2.6. We note that the formula for $[\log X, A]$ involves the inverse X^{-1}, which is a well-defined element of $M\Psi DS$. Indeed, to find, say, the inverse of a pseudodifferential symbol X we have to solve $X \circ A = 1$ with unknown coefficients. If

$$X = f_n D^n + f_{n-1} D^{n-1} + \dots, \quad A = a_n D^{-n} + a_{n-1} D^{-n-1} + \dots,$$

we solve recursively the equations

$$f_n a_n = 1, \quad f_n a_{n-1} + f_{n-1} a_n + n f_n (D a_n) = 0, \quad \dots.$$

Each equation involves only one new unknown a_j as compared to preceding ones and hence the series for $X^{-1} = A$ can be obtained term by term, i.e. its coefficients are meromorphic functions.

Example 2.7. To any meromorphic function $f \in \mathcal{M}$ on Σ we associate the operator $\log f : M\Psi DS \to M\Psi DS$ given by

$$[\log f, aD^n] := na\frac{Df}{f} D^{n-1} + n(n-1)a\frac{f(D^2 f) - (Df)^2}{f^2} D^{n-2} + \dots.$$

Note that, while the function $\log f$ is not meromorphic and branches at poles and zeros of f, all its derivatives $D^k(\log f)$ with $k \geq 1$ are meromorphic, and the right-hand side of the above expression is a meromorphic pseudodifferential symbol.

We shall show that $\log X$ for any symbol X is an *outer* derivation of the Lie algebra $M\Psi DS$, i.e. it represents a nontrivial element of $H^1(M\Psi DS, M\Psi DS)$. The latter space is by definition the space of equivalence classes of all derivations modulo inner ones.

Theorem 2.8. *All derivations defined by* $\log X$ *for any meromorphic pseudodifferential symbol* X *are outer and equivalent to a linear combination of derivations given by logarithms* $\log D_{v_i}$ *of meromorphic vector fields* v_i *on* Σ.

This theorem is implied by the following two properties of the log-map.

Theorem 2.8′. *The map* $X \mapsto \log X \in H^1(M\Psi DS, M\Psi DS)$ *is nonzero and satisfies the properties:*

a) the derivation $\log(X \circ Y)$ *is equivalent to the sum of derivations* $\log X + \log Y$*, and*

b) the derivation $\log(X + Y)$ *is equivalent to the derivation* $\log X$ *if the degree of the symbol* X *is greater than the degree of* Y.

One can see that any derivation $\log X$ is equivalent to a linear combination of $\log f$ for some meromorphic function and $\log D_v$ for one fixed field v. The above properties of derivations $\log X$ modulo inner ones are similar to those of tropical calculus.[1] We prove this theorem in Sect. 2.5.

Conjecture 2.9. *All outer derivations of the Lie algebra* $M\Psi DS$ *are equivalent to those given by* $\log X$ *for pseudodifferential symbols* X.

2.3. The traces. The Lie algebra $M\Psi DS$ has a trace attached to any choice of the "special" points on Σ. All the constructions below will be relying on this choice of the points and we fix such a point (or a collection of points) $P \in \Sigma$ from now on.

Definition 2.10. *Define the* ***residue*** *map* $\operatorname*{res}_D$ *from* $M\Psi DS$ *to meromorphic 1-forms on* Σ *by setting*

$$\operatorname*{res}_D\left(\sum_{k=-\infty}^{n} a_k D^k\right) := a_{-1}\tilde{D}^{-1}.$$

Here $\tilde{D}^{-1}$ in the right-hand side is understood as a (globally defined) meromorphic differential on Σ, the pointwise inverse of the meromorphic vector field v.

On the algebra $M\Psi DS$ we define the *trace* associated to the point $P \in \Sigma$ by

$$\operatorname{Tr} A := \operatorname*{res}_P \operatorname*{res}_D (A).$$

Here

$$\operatorname*{res}_P f D^{-1} = \operatorname*{res}_P \frac{f}{v} = \operatorname*{res}_P \frac{f}{h} dz = \frac{1}{2\pi i}\int_\gamma \frac{f}{h} dz,$$

where $v = h(z)\partial/\partial z$ is a local representation of the vector field v at a neighborhood of the point P, while γ is a sufficiently small contour on Σ around P which does not contain poles of f/h other than P. (Here and below we omit the index P in the notation of the trace Tr_P.)

[1] We are grateful to A. Rosly for drawing our attention to this analogy.

Proposition 2.11. *Both the residue and trace are well-defined operations on the algebra $M\Psi DS$, i.e. they do not depend on the choice of the field v. Furthermore, for any choice of the point(s) P,* Tr *is an algebraic trace, i.e.* $\mathrm{Tr}\,[A, B] = 0$ *for any two pseudo-differential symbols $A, B \in M\Psi DS$.*

In particular, this property allows us to use the notation $\underset{D}{\mathrm{res}}$ or Tr without mentioning a specific field v.

Proof. Under the change of a vector field $v \mapsto w = gv$ only the terms D_v^{-1} contribute to D_w^{-1}, which implies that the corresponding 1-form $a_{-1}\tilde{D}_v^{-1}$ and hence the residue operator are well-defined.

The algebraic property of the trace is of local nature, since Tr is defined locally near P. One can show that for any two $X, Y \in M\Psi DS$ the residue of the commutator is a full derivative, i.e.

$$\underset{D}{\mathrm{res}}\,[X, Y] = (Df)D^{-1}$$

for some function f defined in a neighborhood of P (see [1], p.11). Then the proposition follows from the fact that a complete derivative has zero residue:

$$\int_\gamma (Df)D^{-1} = \int_\gamma \frac{L_v f}{v} = \int_\gamma df = 0$$

for a contour γ around P. □

This proposition allows one to define the *pairing* (,) on $M\Psi DS$, associated with the chosen point $P \in \Sigma$:

$$(A, B) := \mathrm{Tr}\,(AB)\,. \tag{2.4}$$

This pairing is symmetric, non-degenerate, and invariant due to the proposition above. The pairings associated to different choices of the point(s) $P \in \Sigma$ are in general not related by an algebra automorphism (unless there exists a holomorphic automorphism of the surface Σ sending one choice to the other).

Remark 2.12. The existence of the invariant trace(s) on $M\Psi DS$ allows one to identify this Lie algebra with (the regular part of) its dual space. This identification relies on the choice of the point P.

We also note that both $\underset{D}{\mathrm{res}}$ and Tr vanish on the subalgebra MDO of meromorphic purely differential operators. In particular, this subalgebra is isotropic with respect to the above pairing, i.e. $(\ ,\ )|_{MDO} = 0$. The complementary subalgebra to MDO is the Lie algebra MIS of meromorphic integral symbols $\{\sum_{k=-\infty}^{-1} a_k D^k\}$, which is isotropic with respect to this pairing as well.

2.4. The logarithmic 2-cocycles. Being in the possession of a variety of outer derivations, as well as of the invariant trace(s), we can now construct many central extensions of the Lie algebra ΨDS. The simple form of the invariant trace allows us to follow the analogous formalism for pseudodifferential symbols on the circle [3, 5, 6]. We start by defining a logarithmic 2-cocycle attached to the given choice of the point P and a meromorphic field v on Σ:

Theorem 2.13. (cf. [6]) *The bilinear functional*

$$c_v(A, B) := \mathrm{Tr}\,([\log D_v, A] \circ B) \tag{2.5}$$

is a nontrivial 2-cocycle on $M\Psi DS$ and on its subalgebra MDO for any meromorphic field v and any choice of the point P on Σ, where the trace is taken.

In particular, the skew-symmetry property of this cocycle follows from the fact that the derivation $\log D_v$ preserves the trace functional: $\mathrm{Tr}\,([\log D_v, A]) = 0$ for all symbols $A \in M\Psi DS$.

Remark 2.14. The restriction of this 2-cocycle to the algebra of vector fields is the *Gelfand-Fuchs 2-cocycle*

$$c_v(aD_v, bD_v) = \frac{1}{6}\operatorname*{res}_P \frac{(D_v^2 a)(D_v b)}{D_v}$$

on the Lie algebra of meromorphic vector fields on Σ. The restriction of the cocycle (2.5) to the algebra MDO gives the *Kac-Peterson 2-cocycle*

$$c_v(aD_v^m, bD_v^n) = \frac{m!n!}{(m+n+1)!}\operatorname*{res}_P \frac{(D_v^{n+1}a)(D_v^m b)}{D_v}, \qquad m, n \geq 0,$$

on meromorphic differential operators on Σ (see [10]).

The above construction can be generalized in the following way.

Definition 2.15. *Associate the* ***logarithmic 2-cocycle***

$$c_X(A, B) := \mathrm{Tr}\,([\log X, A] \circ B)$$

to a meromorphic pseudodifferential symbol X and a point $P \in \Sigma$ (where the trace Tr *is taken).*

Theorem 2.8″. *For any meromorphic pseudodifferential symbols X and Y*

a) the logarithmic 2-cocycle c_{XY} is equivalent to the sum of the 2-cocycles $c_X + c_Y$, and

b) the logarithmic 2-cocycle c_{X+Y} is equivalent to the logarithmic 2-cocycle c_X provided the degree of the symbol X is greater than the degree of Y.

Proof. This follows from Theorem 2.8′ thanks to the following claim (see e.g. [2]). Let $\mathfrak{g}$ be a Lie algebra with a symmetric invariant nondegenerate pairing $(\ ,\ )$. Consider a derivation $\phi : \mathfrak{g} \to \mathfrak{g}$ preserving the pairing, i.e. satisfying $(\phi(a), b) + (\phi(b), a) = 0$ for any $a, b \in \mathfrak{g}$, and associate to it the 2-cocycle $c(a, b) := (\phi(a), b) \in H^2(\mathfrak{g})$ on $\mathfrak{g}$. Then the subspace in $H^1(\mathfrak{g}, \mathfrak{g})$ consisting of invariant derivations (and understood up to coboundary) is isomorphic to the space $H^2(\mathfrak{g})$: the 2-cocycle c is cohomologically nontrivial if and only if the derivation ϕ is outer.

Since the outer derivation $\log X$ preserves the pairing (2.4), it defines a nontrivial 2-cocycle. The properties of the outer derivations in Theorem 2.8′ are equivalent to those of the logarithmic 2-cocycles in Theorem 2.8″. □

Corollary 2.16. *(i) For any meromorphic pseudodifferential symbol X the logarithmic 2-cocycle c_X is equivalent to a linear combination of the 2-cocycles $c_{v_i}(A, B) := \mathrm{Tr}\,([\log D_{v_i}, A] \circ B)$ associated to meromorphic vector fields v_i.*

(ii) For two meromorphic vector fields v and w related by $w = fv$ the cocycles c_v and c_w are related by

$$c_w = c_v + c_f\,,$$

where $c_f(A, B) := \mathrm{Tr}\,([\log f, A] \circ B)$ is the 2-cocycle associated to the meromorphic function $f \in \mathcal{M}$, and the equality is understood in $H^2(M\Psi DS, \mathbb{C})$, i.e. modulo a 2-coboundary.

Theorem 2.17. *All the 2-cocycles c_v are nontrivial and non-cohomologous to each other on the algebra $M\Psi DS$ for different choices of meromorphic fields $v \neq 0$. Equivalently, cocycles c_f are all nontrivial for non-constant functions f.*

Proof. Note that the 2-cocycle c_v is nontrivial, since its restriction to the subalgebra of vector fields holomorphic in a punctured neighborhood of the point P already gives the nontrivial Gelfand-Fuchs 2-cocycle. (In other words, the nontriviality of the cocycle c_v for a meromorphic vector field v follows from its nontriviality on the Krichever-Novikov subalgebra $\mathcal{L}$ of holomorphic vector fields on $\Sigma \setminus \{P, Q\}$, where Q is any other point on Σ, see the next section.)

To show the nontriviality of the cocycle c_f for a non-constant function f we use the existence of many traces on $M\Psi DS$. First we choose the point P (and the corresponding trace Tr_P) at a zero of the function f. The corresponding 2-cocycle c_f is non-trivial, since so is its restriction to pseudodifferential symbols holomorphic in a punctured neighborhood of P. The latter is obtained by exploiting the nontriviality of the 2-cocycle $c'(A, B) = \mathrm{Tr}\,([\log z, A] \circ B)$ on holomorphic pseudodifferential symbols on $\mathbb{C}^*$, see [5, 2].

Now, by applying the above-mentioned equivalence between derivations and cocycles, we conclude that $\log f$ defines an outer derivation of the algebra $M\Psi DS$. Once we know that the derivation is outer, we can use the same equivalence “in the opposite direction” for the point P anywhere on Σ to obtain a nontrivial 2-cocycle from any other invariant trace. □

Remark 2.18. Note that the cocycle c_f for a meromorphic function f vanishes on the subalgebra MDO of meromorphic differential operators: for any purely differential operators X and Y, the expression $[\log f, X] \circ Y$ is also a meromorphic differential operator (see Example 2.7) and hence its coefficient at D_v^{-1} is 0. This shows that all the 2-cocycles c_v for the same point $P \in \Sigma$, but for different choices of the meromorphic field v are cohomologous when restricted to the algebra MDO. The choice of a different point P to define the trace may lead to a non-cohomologous 2-cocycle c_v.

2.5. Proof of the theorem on outer derivations. In this section we will prove Theorem 2.8′ on properties of the derivation $\log X : M\Psi DS \to M\Psi DS$.

Proof. For the part a) we rewrite the product XY of two symbols as $XY = \exp(\log X) \circ \exp(\log Y)$ and use the Campbell-Hausdorff formula:

$$XY = \exp(\log X + \log Y + \mathfrak{R}(\log X, \log Y))\,.$$

Note that the remainder term $\mathfrak{R}(\log X, \log Y)$ is a pseudodifferential symbol, since only (iterated) commutators of $\log X$ and $\log Y$ appear in it, but not the logarithms themselves. (In particular, the commutator $[\log X, \log Y]$ defined by the formula (2.2) is a pseudodifferential symbol from $M\Psi DS$.) Hence

$$\log XY = \log X + \log Y + \mathfrak{R}(\log X, \log Y)\,.$$

Thus the derivation $\log XY$ is cohomological to the sum of derivations $\log X + \log Y$, since the commutation with the symbol $\mathfrak{R}(\log X, \log Y)$ defines an inner derivation of the algebra $M\Psi DS$. This proves a).

To prove the part b) we will show that the derivation $\log X$ for $X = \sum_{i=-\infty}^{n} a_i D_v^i$ is defined by $a_n D_v^n$, the principal term of X. Indeed, rewrite X as

$$X = (a_n D_v^n) \circ (1 + Y),$$

where $Y = \sum_{j=-\infty}^{-1} b_j D_v^j$ is a meromorphic integral symbol. Then due to a), $\log X$ is cohomological to $\log(a_n D_v^n) + \log(1+Y)$. However, the derivation $\log(1+Y)$ is inner, i.e. $\log(1+Y)$ is itself a meromorphic pseudodifferential symbol. Indeed, expand $\log(1+Y)$ in the series: $\log(1+Y) = Y - Y^2/2 + Y^3/3 - \ldots$. The right-hand side is a well-defined meromorphic integral symbol, since so is Y. Thus $\log X$ is cohomological to $\log(a_n D_v^n)$, which proves b). □

2.6. The double extension of the meromorphic symbols and Manin triples.

Definition 2.19. *Consider the following* ***double extension*** *of the Lie algebra $M\Psi DS$ by means of both the central term and the outer derivation for a fixed meromorphic field v:*

$$\widetilde{M\Psi DS} = \mathbb{C} \cdot \log D \oplus M\Psi DS \oplus \mathbb{C} \cdot \mathbb{I} = \left\{ \lambda \log D + \sum_{k=-\infty}^{n} a_k D^k + \mu \cdot \mathbb{I} \right\},$$

where the commutator of a pseudodifferential symbol with another one or with $\log D$ *for $D = D_v$ was defined above, while the cocycle direction $\mathbb{I}$ commutes with everything else.*

There is a natural invariant pairing on the Lie algebra $\widetilde{M\Psi DS}$, which extends the pairing $\mathrm{Tr}\,(A \circ B)$ on the non-extended algebra $M\Psi DS$. Namely,

$$\langle(\lambda_1 \log D + A_1 + \mu_1 \cdot \mathbb{I}, \lambda_2 \log D + A_2 + \mu_2 \cdot \mathbb{I})\rangle = \mathrm{Tr}\,(A_1 \circ A_2) + \lambda_1 \cdot \mu_2 + \lambda_2 \cdot \mu_1\,.$$

Consider also two subalgebras of the Lie algebra $\widetilde{M\Psi DS}$: the subalgebra of centrally extended meromorphic differential operators

$$\widehat{MDO} = \left\{ \sum_{k=0}^{n} a_k D^k + \mu \cdot \mathbb{I} \right\}$$

and the subalgebra of co-centrally extended meromorphic integral symbols

$$\widetilde{MIS} = \mathbb{C} \cdot \log D \oplus MIS = \left\{ \lambda \log D + \sum_{k=-\infty}^{-1} a_k D^k \right\}.$$

Similarly to the case of smooth coefficients, one proves the following

Theorem 2.20. *Both the triples of algebras* $(M\Psi DS, MDO, MIS)$ *and* $(\widetilde{M\Psi DS}, \widehat{MDO}, \widetilde{MIS})$ *are Manin triples.*

Definition 2.21. *A **Manin triple*** $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *is a Lie algebra* $\mathfrak{g}$ *along with two Lie subalgebras* $\mathfrak{g}_\pm \subset \mathfrak{g}$ *and a nondegenerate invariant symmetric form* $(\ ,\ )$ *on* $\mathfrak{g}$*, such that*

(a) $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *as a vector space and*
(b) $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$ *are isotropic subspaces of* $\mathfrak{g}$ *with respect to the inner product* $(\ ,\ )$.

The existence of a Manin triple means the existence of a Lie bialgebra structure on both $\mathfrak{g}_\pm$ and allows one to regard each of the subalgebras as dual to the other with respect to the pairing (see the Appendix for the definitions).

Corollary 2.22. *Both the Lie algebras* MIS *and* $\widetilde{MIS}$ *are Lie bialgebras, while the groups corresponding to them are Poisson-Lie groups.*

This makes the meromorphic consideration parallel to the case of smooth pseudodifferential symbols developed in [3, 5]. For holomorphic symbols, however, such a Manin triple exists only in special cases, as we discuss below.

3. Holomorphic Pseudodifferential Symbols on Riemann Surfaces

3.1. The Krichever-Novikov algebra. Let Σ be a Riemann surface of genus g. Fix two generic points P_+ and P_- on the surface. Consider the Lie algebra $\mathcal{L}$ of meromorphic vector fields on Σ, holomorphic on $\overset{\circ}{\Sigma} := \Sigma \setminus \{P_\pm\}$. We will call such fields simply *holomorphic* (on $\overset{\circ}{\Sigma}$).

Definition 3.1. *The Lie algebra* $\mathcal{L}$ *of holomorphic on* $\overset{\circ}{\Sigma}$ *vector fields is called the **Krichever-Novikov*** (*KN*) *algebra.*

A special basis in $\mathcal{L}$, called the Krichever-Novikov basis, is formed by vector fields e_k having a pole of order k at P_+ and a pole of order $3g-k-2$ at P_- (as usual we refer to a pole of negative order k as to a zero of order $-k$). (More precisely, this prescription of basis elements works for surfaces Σ of genus $g \geq 2$, while for $g = 1$ one has to alter it for certain small values of k, see [7].) Note that each field e_k has g additional zeros elsewhere on $\Sigma \setminus \{P_\pm\}$, since the degree of the tangent bundle of Σ is $2-2g$.

This algebra was introduced and studied in [7, 8] along with its central extensions. It generalizes the Virasoro algebra, which corresponds to the case $\overset{\circ}{\Sigma} = \mathbb{C}P^1 \setminus \{0, \infty\}$. Below we will be concerned with the case of two punctures $P_\pm$ on Σ, although most of the results below hold for the case of many punctures as well, cf. [11–13, 16, 18].

3.2. Holomorphic differential operators and pseudodifferential symbols. Denote by $\mathcal{O}$ the sheaf of holomorphic functions on $\overset{\circ}{\Sigma}$, which are meromorphic at $P_\pm$.

Definition 3.2. *The **sheaves of holomorphic differential operators** and **pseudodifferential symbols** on* $\Sigma \setminus \{P_\pm\}$ *are defined by assigning to each open set* $U \subset \overset{\circ}{\Sigma}$ *an abelian group* (*a vector space*) *of sections*

$$\mathcal{HDO}(U) := \left\{ X = \sum_{i=0}^{n} u_i D^i \mid u_i \in \mathcal{O}_U \right\} \quad \textit{and}$$
$$\mathcal{H\Psi DS}(U) := \left\{ X = \sum_{i=-\infty}^{n} u_i D^i \mid u_i \in \mathcal{O}_U \right\},$$

respectively, where $D := D_v$ stands for some holomorphic ***non-vanishing*** *vector field v in U. (Another choice of a non-vanishing field v gives the same spaces of operators and symbols.)*

The Lie algebras of global sections of the sheaves $\mathcal{HDO}$ and $\mathcal{H\Psi DS}$ are called the ***Lie algebras of holomorphic differential operators*** *and of* ***pseudodifferential symbols****, respectively. We denote these algebras of global sections by HDO and $H\Psi DS$.*

Note that the definitions of the residue and trace of symbols are local and hence can be defined on holomorphic symbols in the same way as they were defined for meromorphic ones: $\mathrm{res}_D\, X$ is a globally defined holomorphic 1-form on $\overset{\circ}{\Sigma}$, given in local coordinates by $u_{-1}\tilde{D}^{-1}$, while

$$\mathrm{Tr}\; X := \underset{P_+}{\mathrm{res}}\; \underset{D}{\mathrm{res}}\; X = \underset{P_+}{\mathrm{res}}\; \left(u_{-1}\tilde{D}^{-1}\right)$$

for a section X given by $X = \sum_{i=-\infty}^{n} u_i D^i$ in a (punctured) neighborhood of P_+.

The algebra $H\Psi DS$ has two subalgebras: that of holomorphic differential operators (HDO) and of holomorphic integral symbols (HIS). They consist of those symbols whose restriction to any open subset $U \subset \overset{\circ}{\Sigma}$ are, respectively, holomorphic purely differential operators or holomorphic purely integral symbols. As in the meromorphic case, these holomorphic subalgebras are isotropic with respect to the natural pairing.

The KN-algebra $\mathcal{L}$ of holomorphic vector fields on $\overset{\circ}{\Sigma}$ can be naturally viewed as a subalgebra of the algebras of holomorphic differential operators and pseudodifferential symbols ($H\Psi DS$). In turn, the algebra $H\Psi DS$ is a subalgebra in the algebra of meromorphic symbols $M\Psi DS$.

Remark 3.3. The Lie algebra HDO can be alternatively defined as the universal enveloping algebra $HDO = \mathcal{U}(\mathcal{O} \rtimes \mathcal{L})/J$ of the Lie algebra $\mathcal{O} \rtimes \mathcal{L}$ quotiented over the ideal J, generated by the elements $f \circ g - fg$, $f \circ v - fv$, and $\mathbf{1} - 1$, where $\circ$ denotes the multiplication in HDO, $\mathbf{1}$ is the unit of $\mathcal{U}$, while $f, g, 1 \in \mathcal{O}$ and $v \in \mathcal{L}$, see e.g. [12]. It is easy to see that this definition matches the one above.

A convenient way to write some global sections of the above sheaves is by fixing a holomorphic field $v \in \mathcal{L}$. Then the symbols $X = \sum_{i=-\infty}^{n} u_i D_v^i$ with any holomorphic coefficients $u_i \in \mathcal{O}$ define global sections of $\mathcal{H\Psi DS}$, provided that for every $i < 0$ the coefficient u_i has zero of order at least i at zeros of v (this way we compensate all the poles of the negative powers D_v^i by appropriate zeros of the corresponding coefficients).

3.3. Holomorphic pseudodifferential symbols and the spaces of densities. One can think of holomorphic (pseudo)differential symbols as sequences of holomorphic densities on $\overset{\circ}{\Sigma}$. Namely, let $\mathcal{K}$ be the canonical line bundle over Σ and consider the tensor power $\mathcal{K}^n$ of $\mathcal{K}$ for any $n \in \mathbb{Z}$. Denote by $\mathcal{F}^n$ the space of *holomorphic n-densities* on $\overset{\circ}{\Sigma}$, i.e. the space of global meromorphic sections of $\mathcal{K}^n$ which are holomorphic on $\overset{\circ}{\Sigma}$. Note that $\mathcal{F}^0$ is the ring $\mathcal{O}$ of holomorphic functions, $\mathcal{F}^1$ is the space of holomorphic differentials on $\overset{\circ}{\Sigma}$, and $\mathcal{F}^{-1}$ is the space $\mathcal{L}$ of holomorphic vector fields.

Holomorphic vector fields act on holomorphic n-densities by the Lie derivative: to $v \in \mathcal{L}$ and $\omega \in \mathcal{F}^n$ one associates $L_v\omega \in \mathcal{F}^n$. Explicitly, in local coordinates for $v = h(z)\partial/\partial z$ and $\omega = f(z)(dz)^n$ one has

$$L_v\omega = \left(h(z)\frac{\partial f}{\partial z} + nf(z)\frac{\partial h}{\partial z}\right)(dz)^n.$$

This action turns the space $\mathcal{F}^n$ of n-densities into an $\mathcal{L}$-module. The following proposition is well-known:

Proposition 3.4. *Each graded space for the filtration of pseudodifferential symbols by degree is naturally, as an $\mathcal{L}$-module, isomorphic to the corresponding space of holomorphic densities. Namely,*

$$H\Psi DS^n/H\Psi DS^{n-1} \approx \mathcal{F}^{-n},$$

where $H\Psi DS^n$ is the space of pseudodifferential symbols of degree n and the isomorphism is given by taking the principal symbol of the pseudodifferential operator.

Proof. The action of vector fields from $\mathcal{L}$ on pseudodifferential operators of degree n is explicitly given by:

$$[hD, fD^n] = \Big(h(Df) - nf(Dh)\Big)D^n + \text{ (terms of degree} < n \text{ in } D).$$

Thus the action on their principal symbols coincides with the above $\mathcal{L}$-action on $(-n)$-densities $\mathcal{F}^{-n}$, i.e. they satisfy the same change of coordinate rule.

Furthermore, taking the principal symbols of the operators of a given degree n is a surjective map onto $\mathcal{F}^{-n}$. One can see this first for differential operators, i.e. for $n \geq 0$, where it follows from their description as $HDO = \mathcal{U}(\mathcal{O} \rtimes \mathcal{L})/J$ and the PBW theorem. Indeed, one can form a basis in differential operators of degree n from the products $fD_{e_{i_1}}...D_{e_{i_n}}$, where $f \in \mathcal{F}^0$ and e_i form the KN-basis in $\mathcal{L} = \mathcal{F}^{-1}$. Their principal symbols will be the (commutative) products of the principal symbols of the basis elements, which, by definition, span the space of meromorphic sections of $\mathcal{K}^n$, holomorphic on $\overset{\circ}{\Sigma}$, i.e. the space $\mathcal{F}^{-n}$.

The surjectivity for negative n, i.e. for principal symbols of integral operators, can be derived by considering natural pairing on densities ($\mathcal{F}^n \times \mathcal{F}^{-n-1} \to \mathbb{C}$) and on pseudodifferential symbols ($H\Psi DS^{-n} \times H\Psi DS^{n-1} \to \mathbb{C}$) given by taking at the point P_+ the residue for densities and the trace for symbols, respectively. □

The whole vector space $H\Psi DS$ can be treated as the direct limit of the semi-infinite products of the spaces of holomorphic n-densities: $H\Psi DS \approx \varinjlim \Pi_{n=-\infty}^{k}\mathcal{F}^{-n}$ as $k \to \infty$, on which one has a Lie algebra structure.

3.4. The density of holomorphic symbols in the smooth ones. The algebra $H\Psi DS$ of holomorphic symbols on $\overset{\circ}{\Sigma}$, as well as the KN-algebra $\mathcal{L}$ of holomorphic vector fields, can be regarded as a subalgebra of smooth symbols on the circle S^1 in the following way.

In [7] a family of special contours C_τ, $\tau \in \mathbb{R}$ on Σ was constructed as level sets of some harmonic function on $\overset{\circ}{\Sigma}$. These contours separate the points $P_\pm$, and as $\tau \to \pm\infty$ the contours C_τ become circles shrinking to $P_\pm$. Denote by $S^1 \approx C_\tau$ an arbitrary contour from this family for a sufficiently large τ, thought of as a small circle around P_+. Consider the natural restriction homomorphism from $\overset{\circ}{\Sigma}$ to $S^1 \subset \overset{\circ}{\Sigma}$.

Theorem 3.5 [7, 8]. *The restrictions of holomorphic functions, vector fields, and differentials on $\overset{\circ}{\Sigma}$ to the contour $S^1 \approx C_\tau$ are dense among, respectively, smooth functions, vector fields, and differentials on the circle S^1.*

Now we consider the algebra of all smooth pseudodifferential symbols on the circle (with a coordinate x):

$$\Psi DS(S^1) = \left\{ \sum_{i=-\infty}^{n} f_i(x)\partial^i \mid \partial := \frac{d}{dx},\ f_i \in C^\infty(S^1) \right\}.$$

The latter is a topological space under the natural topology (on the Laurent series) while the sum, multiplication, and taking the inverse (for a nowhere vanishing highest coefficient f_n) are continuous operations.

The following theorem is a natural extension of the one above. Consider the restriction homomorphism $H\Psi DS \to \Psi DS(S^1)$ of holomorphic symbols to the smooth ones for the contour $S^1 \approx C_\tau \subset \Sigma$ and denote by $H\Psi DS\,|_{S^1}$ the corresponding image.

Theorem 3.6. *The restriction $H\Psi DS\,|_{S^1}$ of holomorphic symbols is dense in the smooth ones $\Psi DS(S^1)$.*

Proof. It suffices to prove that the monomials of the form $f(x)\partial^i$, $i \in \mathbb{Z}$, $f \in C^\infty(S^1)$ can be approximated by holomorphic ones. Write out such a monomial as a product of $f(x)$, ∂ and ∂^{-1}. Since the smooth function $f(x)$, the vector field $\partial := \frac{d}{dx}$ and the 1-form $\partial^{-1} := dx$ on the circle can be approximated by the restrictions of holomorphic ones [8], the result follows by the continuity of the multiplication in $\Psi DS(S^1)$. □

Note that the original density result in [7] for a pair of points $P_\pm$ extends to a collection of points by representing functions, fields, etc. with many poles as sums of the ones with two poles only.

3.5. The property of generalized grading.

Definition 3.7. *An associative or Lie algebra A is **generalized graded** (or N-**graded**) if it admits a decomposition $A = \oplus_{n\in\mathbb{Z}} A_n$ into finite-dimensional subspaces, with the property that there is a constant N such that*

$$A_i A_j \subset \bigoplus_{s=-N}^{N} A_{i+j+s},$$

for all i, j.

Similarly, a module M over a generalized graded algebra A is ***generalized graded****, if $M = \oplus_{n\in\mathbb{Z}} M_n$, and there is a constant L such that*

$$A_i\, M_j \subset \bigoplus_{s=-L}^{L} M_{i+j+s},$$

for all i, j.

Theorem 3.8 [7, 8]. *The KN-algebra $\mathcal{L}$ of vector fields holomorphic on $\overset{\circ}{\Sigma}$ is generalized graded. The space $\mathcal{F}^n$ of holomorphic n-densities for any n is a generalized graded module over $\mathcal{L}$.*

The generalized graded components $M_j^{(n)}$ for the module $\mathcal{F}^n$ are the spaces $\mathbb{C}\cdot f_j^{(n)}$, where the forms $f_j^{(n)}$ are uniquely determined by the pole orders at both points $P_\pm$ (as before, we assume the points to be generic): for $n \neq 0$ they have the following expansions

$$f_j^{(n)} := a_j^{\pm} z_{\pm}^{\pm j - g/2 + n(g-1)}(1 + O(z_\pm))(dz_\pm)^n$$

in local coordinates $z_\pm$ of neighborhoods of the points $P_\pm$. Here the index j runs over the integers $\mathbb{Z}$ if g is even and over the half-integers $\mathbb{Z} + 1/2$ if g is odd. (The formulas differ slightly for $\mathcal{F}^0 = \mathcal{O}$, see details in [7]. For the space of vector fields $\mathcal{L} \approx \mathcal{F}^{-1}$ this basis $\{f_j^{(-1)}\}$ differs by an index shift from the fields $\{e_j\}$ discussed in Sect. 3.1.)

Now we consider the spaces HDO and $H\Psi DS$ of holomorphic (pseudo)differential symbols not only as modules over vector fields $\mathcal{L}$, but as Lie algebras. These Lie algebras are not generalized graded, but naturally filtered by the degree of $D = D_v$. Consider a basis $\{F_j^{(n)}\}$ (which we construct below) in pseudodifferential symbols $H\Psi DS$, which is compatible with the basis in the forms: the principal symbol of the operator $F_j^{(n)}$ of degree n is the $(-n)$-form $f_j^{(-n)}$. It turns out that the algebras of holomorphic (pseudo)differential symbols have the following analogue of the generalized grading:

Theorem 3.9. *The Lie algebras HDO and $H\Psi DS$ are filtered generalized graded: the pseudodifferential symbols of an appropriate basis $\{F_j^{(n)}\}$ in $H\Psi DS$ satisfy*

$$[F_i^{(n)}, F_j^{(m)}] = \sum_{k=1}^{\infty} \sum_{s=-N(k)}^{N(k)} \alpha_{ij}^s F_{i+j+s}^{(n+m-k)},$$

for some constants $\alpha_{ij}^r \in \mathbb{C}$, where $n, m \in \mathbb{Z}$, the indices i, j, and s are either integers or half-integers according to parity of the genus g, and $N(k)$ is a linear function of k.

Proof. First we define a basis for differential operators from $HDO \subset H\Psi DS$ recursively in degree n (cf. [12], where a similar basis was constructed for HDO). Assume that the genus g is even, so that all the indices are integers (the case of an odd g is similar). Consider the above KN-basis $\{F_j^{(0)}\}$ in differential operators of degree 0, which constitute 0-densities $\mathcal{F}^0$, and the KN-basis $\{F_j^{(1)}\}$ in holomorphic vector fields, which are differential operators of degree 1.

For a degree $n \geq 2$ consider a differential operator $\tilde{F}_j^{(n)}$ whose principal symbol is the $(-n)$-density $f_j^{(-n)}$, and which exists due to surjectivity discussed in Proposition 3.4. One can "kill the lower order terms" of the operator $\tilde{F}_j^{(n)}$ by adding a linear combination of the basis elements in HDO^{n-1} constructed at the preceding step. (More precisely, due to the filtered structure of HDO only the cone of lower order terms for $\tilde{F}_j^{(n)}$ is well-defined by the pole orders of the coefficients of the differential operators at the points $P_\pm$. By "killing the terms" above we mean confining $\tilde{F}_j^{(n)}$ to this cone.) We call these adjusted differential operators by $F_j^{(n)}$. Along with $\{F_j^{(k)}\}$ for $0 \leq k \leq n-1$ they constitute a basis in HDO^n, holomorphic differential operators of degree $\leq n$.

Finally, for integral symbols (of negative degrees) we choose the basis dual to the one chosen in differential operators, by using the nondegenerate pairing: HIS can be thought of as the dual space HDO^*. It is easy to see that this basis in integral symbols of degree $-n$ is also given by the orders of zeros and poles at $P_\pm$ and hence is compatible with the KN-basis in n-forms.

Once the basis $\{F_j^{(n)}\}$ is constructed, a straightforward substitution of these symbols into the formula for the symbol commutator and the calculation of orders of poles/zeros at the points $P_\pm$ yield the result. □

This theorem implies the property of generalized grading for modules of holomorphic densities, established in [7, 12, 16], as modules of principal symbols of holomorphic pseudodifferential operators.

3.6. Cocycles and extensions. Recall first the cocycle construction for the KN-algebra $\mathcal{L}$ of holomorphic vector fields on $\overset{\circ}{\Sigma}$. A closed contour γ on Σ, not passing through the marked points $P_\pm$, defines the Gelfand-Fuchs 2-cocycle on the algebra $\mathcal{L}$. Namely, in a fixed projective structure (where admissible coordinates differ by projective transformations) it is defined by the Gelfand-Fuchs integral

$$c(f, g) = \int_\gamma f''(z) g'(z)\, dz$$

for vector fields $f = f(z)\frac{\partial}{\partial z}$ and $g = g(z)\frac{\partial}{\partial z}$ given in such a coordinate system. One can check that this cocycle is well-defined, nontrivial, and represents every cohomology class in the space $H^2(\mathcal{L}, \mathbb{C})$ of 2-cocycles on the algebra $\mathcal{L}$ for various contours γ, see [8, 13]. In this variety of 2-cocycles there is a subset of those satisfying the following property of locality.

Definition 3.10 [7]. *Let $A = \oplus_{n\in\mathbb{Z}} A_n$ be a generalized graded Lie algebra. A 2-cocycle c on A is called* ***local*** *if there is a nonnegative constant $K \in \mathbb{Z}$ such that $c(A_m, A_n) = 0$ for all $|m+n| > K$.*

The central extensions of generalized graded Lie algebras defined by local 2-cocycles are also generalized graded Lie algebras.

Theorem 3.11 [7]. *The cohomology space of local 2-cocycles of the Krichever-Novikov algebra $\mathcal{L}$ is one-dimensional. It is generated by the Gelfand-Fuchs 2-cocycle on any separating contour C_τ on Σ.*

As we discussed above, for large τ one can think of C_τ as a small circle around P_+. Thus the local cocycles on $\mathcal{L}$ are defined by the restrictions of vector fields to a small neighborhood of P_+.

To describe the logarithmic cocycles on the algebras HDO and $H\Psi DS$ of holomorphic (pseudo)differential symbols on $\overset{\circ}{\Sigma}$ we adapt the notion of the cocycle locality to the filtered generalized grading.

First we recall the corresponding results for the algebras $DO(S^1)$ and $\Psi DS(S^1)$ of *smooth* operators and symbols of the circle.

Theorem 3.12. *(i) The cohomology space of 2-cocycles on the algebra $DO(S^1)$ of differential operators on the circle is one-dimensional ([4, 9]). A non-trivial 2-cocycle is defined by the restriction of the logarithmic cocycle* Tr $([\log \partial, A] \circ B)$ *to differential operators, $\partial := \frac{\partial}{\partial x}$, and $A, B \in DO(S^1)$* ([6]).

(ii) The cohomology space of 2-cocycles of the algebra $\Psi DS(S^1)$ of pseudodifferential symbols is two-dimensional ([4, 2]). It is generated by the logarithmic cocycle above and the 2-cocycle Tr $([x, A] \circ B)$, *where x is the coordinate on the universal covering of S^1, and $A, B \in \Psi DS(S^1)$* ([5, 6]).

Here the trace is Tr $A := \int_{S^1} \text{res}_{\,\partial}\, A$ for smooth pseudodifferential symbols, which replaces Tr $A := \text{res}_{\,P_+}\, \text{res}_{\,D}\, A$ for holomorphic ones.

Example 3.13. Consider the Lie algebra of holomorphic symbols on $\mathbb{C}^* = \mathbb{C}P^1 \backslash \{0, \infty\}$, whose elements are allowed to have poles at 0 and ∞ only, and where we take $D_v := z\partial/\partial z$. Two independent outer derivations of the latter algebra are $\log(z\partial/\partial z)$ and $\log z$, the logarithms of a vector field and a function, respectively [6, 5]. The corresponding 2-cocycles are

$$\text{Tr}\left([\log z \frac{\partial}{\partial z}, A] \circ B\right) \quad \text{and} \quad \text{Tr}\,([\log z, A] \circ B)\,.$$

This algebra can be thought of as a graded version of smooth complex-valued symbols $\Psi DS(S^1)$ on the circle $S^1 = \{|z| = 1\}$: the change of variable $z = \exp(ix)$ sends $\partial := \partial/\partial x$ to $i\,D_v$:

$$\partial/\partial x = \partial z/\partial x \cdot \partial/\partial z = i \exp(ix)\,\partial/\partial z = i\,z\partial/\partial z = i\,D_v.$$

Under this change of variables (and upon restricting the symbols to the circle S^1), the derivations $[\log(i D_v), .]$ and $-i\,[\log z, .]$ for holomorphic symbols in $\Psi DS(\mathbb{C}^*)$ become the derivations $[\log \partial, .]$ and $[x, .]$ for smooth symbols in $\Psi DS(S^1)$. The above theorem describes the 2-cocycles on $\Psi DS(S^1)$ constructed with the help of the latter outer derivations and the corresponding change in the notion of trace.

After having described the smooth case, we adapt the definition of the local 2-cocycle to the filtered generalized graded case of the algebra $H\Psi DS$ by allowing the constant K in Definition 3.10 to depend on the filtered component.

Definition 3.14. *A 2-cocycle on the filtered generalized graded algebra $H\Psi DS$ is called **local** if for any integers i, j there is a number $N = N(n+m)$ such that $c(F_i^{(n)}, F_j^{(m)}) = 0$ for the basis pseudodifferential symbols $F_i^{(n)}$ and $F_j^{(m)}$ as soon as $|i + j| > N$.*

Such a cocycle preserves the property of filtered generalized grading when passing to the corresponding central extension.

Consider a holomorphic vector field v on $\overset{\circ}{\Sigma}$ and a smooth path γ on $\overset{\circ}{\Sigma} \setminus (div\ v)$, i.e. a smooth path γ avoiding $P_{\pm}$, as well as zeros and poles of v. Associate to v and γ the 2-cocycle $c_{v,\gamma}$ defined as the following bilinear functional on $H\Psi DS$ and HDO:

$$c_{v,\gamma}(A, B) := \int_{\gamma} \operatorname*{res}_{D_v} ([\log D_v, A] \circ B)\,,$$

where we integrate over γ the residue, which is a meromorphic 1-form on $\overset{\circ}{\Sigma}$ (with possible poles at the divisor of v, and hence off γ). We confine ourselves to considering cocycles of the form $c_{v,\gamma}$.

Theorem 3.15. *(i) The cohomology space of local 2-cocycles of the form $c_{v,\gamma}$ on the Lie algebra HDO on $\overset{\circ}{\Sigma}$ is one-dimensional and it is generated by the 2-cocycle $c_{v,P_+}(A, B)$* $:=$ Tr $([\log D_v, A] \circ B)$ *for a holomorphic vector field v. The cocycles c_{v,P_+} are local for any choice of a holomorphic field v on $\overset{\circ}{\Sigma}$.*

(ii) The cohomology space of local 2-cocycles $c_{v,\gamma}$ on the algebra $H\Psi DS$ is two-dimensional. It is generated by the 2-cocycles c_{v_i,P_+} for two holomorphic vector fields v_1 and v_2 with different orders of poles/zeros at P_+.

Remark 3.16. Alternatively, one can generate the 2-dimensional space of local cocycles for $H\Psi DS$ by considering

$$\mathrm{Tr}\ ([\log D_v, A] \circ B) \quad \text{and} \quad \mathrm{Tr}\ ([\log f, A] \circ B),$$

where v is any holomorphic vector field, while f is a function with a zero or pole (of any non-zero order) at P_+. The restriction of the latter 2-cocycle to HDO vanishes.

Proof. We have adapted the definition of grading and locality in such a way that the locality of a 2-cocycle on the filtered algebras HDO and $H\Psi DS$ implies its locality on the subalgebra $\mathcal{L}$. According to the Krichever-Novikov Theorem 3.11 local cocycles on $\mathcal{L}$ are given by the integrals over contours C_τ. In turn, the cocycles $c_{v,\gamma}$ for $\gamma = C_\tau$ for large τ correspond to integration over a simple contour around P_+, and hence reduce to

$$c_{v,P_+}(A, B) := \mathrm{Tr}_{P_+}([\log D_v, A] \circ B) = \operatorname*{res}_{P_+} \operatorname*{res}_{D_v} ([\log D_v, A] \circ B)\,.$$

To find the dimension of the cohomology space of such cocycles for HDO and $H\Psi DS$ we consider the restriction homomorphism to the smooth operators and symbols on the contour. In both cases the image is dense in the latter due to Theorem 3.6.

One can see that the cocycle c_{v,P_+} for any v is nontrivial in both HDO and $H\Psi DS$, since it is nontrivial on the subalgebra $\mathcal{L}$. Indeed, upon restriction to the contour $S^1 \approx C_\tau$ it gives the (nontrivial) Gelfand-Fuchs 2-cocycle on $Vect(S^1)$. Furthermore, the cohomology space of 2-cocycles for smooth differential operators $DO(S^1)$ is 1-dimensional, and hence so is the cohomology space of local 2-cocycles for holomorphic differential operators HDO, due to the density result and continuity of the cocycles. Verification of locality of c_{v,P_+} for any holomorphic field v is a straightforward calculation. This proves part (i).

For part (ii) we note that the algebra $\Psi DS(S^1)$ admits exactly two independent nontrivial 2-cocycles up to equivalence. The density theorem will imply the required statement, once we show that there are two linearly independent cocycles of the type c_{v,P_+}. Take 2-cocycles c_{v,P_+} and c_{w,P_+} for two fields v and w of different orders of pole/zero at P_+. Then $v = fw$ for a meromorphic function f on Σ, which is either zero or infinity at P_+. The same argument as in Corollary 2.16 (ii) gives that $c_{v,P_+} = c_{w,P_+} + c_{f,P_+}$, where $c_{f,P_+}(A, B) := \text{Tr}\ _{P_+}([\log f, A] \circ B)$.

In order to show that the cocycle c_{f,P_+} is nontrivial and independent of c_{v,P_+}, provided that f has a zero or pole at P_+, we again consider the restriction homomorphism to the smooth symbols $\Psi DS(S^1)$ on the contour $S^1 \approx C_\tau$. In a local coordinate system around P_+ we have $f(z) = z^k g(z)$ with holomorphic $g(z)$ such that $g(0) \neq 0$ and $k \neq 0$. Since $\log f(z) = k \log z + \log g(z)$, the corresponding logarithmic cocycles are related in the same way. Note that $\text{Tr}\ _{P_+}([\log g, A] \circ B)$ defines a trivial cocycle (i.e. a 2-coboundary) upon restriction to S^1. Indeed, the function $\log g(z)$ is holomorphic at $P_+ = 0$ since $g(0) \neq 0$, and its restriction to a small contour around $P_+ = 0$ is univalued. Hence, it defines a smooth univalued function on the contour, and therefore $[\log g, A]$ is an inner derivation of the corresponding algebra $\Psi DS(S^1)$.

On the other hand, $\text{Tr}\ _{P_+}([\log z, A] \circ B)$ upon restriction to $S^1 \approx C_\tau$ defines the second non-trivial cocycle of the algebra $\Psi DS(S^1)$, see Example 3.13. Hence the cocycle c_{f,P_+} is nontrivial and defines the same cohomology class as $k \cdot c_{z,P_+}$. This completes the proof of (ii). □

Conjecture 3.17. *Every continuous 2-cocycle on the Lie algebras $H\Psi DS$ and HDO is cohomologous to a linear combination of regular 2-cocycles $c_{v,\gamma}$ for some holomorphic fields v and a contour γ on $\overset{\circ}{\Sigma}$.*

Remark 3.18. The latter is closely related to Conjecture 2.9. Presumably, all the continuous 2-cocycles on $H\Psi DS$ and HDO have the form $c_{X,\gamma}(A, B) := \int_\gamma \text{res}\ _D([\log X, A] \circ B)$ for holomorphic pseudodifferential symbols X and cycles γ on the surface Σ. In turn, one can reduce the cocycles $c_{X,\gamma}$ for an arbitrary symbol X to cocycles $c_{v,\gamma}$ with $X = D_v$, similarly to the proof of Theorem 2.8.

3.7. Manin triples for holomorphic pseudodifferential symbols. Given the point $P_+ \in \Sigma$ and the invariant pairing on $H\Psi DS$ associated to the trace at P_+ it is straightforward to verify the following proposition.

Proposition 3.19. *The (non-extended) algebras $(H\Psi DS, HDO, HIS)$ form a Manin triple.*

Although to any holomorphic field v with poles at $P_\pm$ one can associate the central extension of the Lie algebra $H\Psi DS$ by the local 2-cocycle $c_v(A, B) = \text{Tr}\ ([\log D_v, A] \circ B)$, the double extension of $H\Psi DS$ does not necessarily exist.

Confine first to the special case, in which on $\overset{\circ}{\Sigma}$ there exists a holomorphic field v without zeros, i.e. to $\overset{\circ}{\Sigma}$ with a trivialized tangent bundle. Such a surface $\overset{\circ}{\Sigma}$ can be obtained from any Σ and any field v on it by choosing the collections of points $P_\pm$ to include all zeros and poles of v. (Example: $v = z\,\partial/\partial z$ in $\mathbb{C}^*$.)

In this case the operator $\log D_v$ maps $H\Psi DS$ to itself, i.e it is an outer derivation of the latter. Then the construction of the co-central extension $\widetilde{HIS} = \mathbb{C} \cdot \log D_v \oplus HIS$ and the double extension $\widetilde{H\Psi DS}$ goes through in the same way as for the meromorphic or smooth cases.

Proposition 3.20. *If the holomorphic field v has no zeros on $\overset{\circ}{\Sigma}$, the Lie algebras ($\widetilde{H\Psi DS}$, $\widehat{HDO}$, $\widetilde{HIS}$) form a Manin triple. Equivalently, $\widetilde{HIS}$ is a Lie bialgebra.*

In this case the Lie bialgebra on $\widetilde{HIS}$ defines a Poisson-Lie structure on the corresponding pseudodifferential symbols of complex degree, just like in the case of $\mathbb{C}^*$ or in the smooth case on the circle. Furthermore, the Poisson structure on this group is the Adler-Gelfand-Dickey quadratic Poisson bracket, while the corresponding Hamiltonian equations are given by the n-KdV and KP hierarchies on Riemann surfaces, following the recipe for the smooth case. We recall the latter consideration from [3, 5] in the Appendix.

Now let v have zeros in $\overset{\circ}{\Sigma}$. Consider the central extension $\widehat{HDO}$ of the algebra of holomorphic differential operators HDO by the 2-cocycle c_v. One can see that now the "regular dual" space to $\widehat{HDO}$ cannot be naturally identified with the vector space $\mathbb{C} \cdot \log D_v \oplus HIS$. Indeed, the coadjoint action of $\widehat{HDO}$ is uniquely defined by the commutator $[\log D_v, A]$ wherever $v \neq 0$. However, this commutator may have poles at zeros of v, i.e. the space $\mathbb{C} \cdot \log D_v \oplus HIS$ does not form a Lie algebra, as it is not closed under commutation. This does not allow one to define a natural Lie bialgebra structure on $\widehat{HDO}$ or $\widetilde{HIS}$. The same type of obstruction arises for the existence of a formal group of symbols of complex degrees on the surface Σ.

4. Appendix

4.1. Poisson–Lie groups, Lie bialgebras, and Manin triples.

Definition 4.1. *A group G equipped with a Poisson structure η is a* ***Poisson–Lie group*** *if the group product $G \times G \to G$ is a Poisson morphism (i.e., it takes the natural Poisson structure on the product $G \times G$ into the Poisson structure on G itself) and if the map $G \to G$ of taking the group inverse is an anti-Poisson morphism (i.e. it changes the sign of the Poisson bracket).*

Theorem 4.2 [17]. *For any connected and simply connected group G with Lie algebra $\mathfrak{g}$ there is a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}$ and Poisson-Lie structures η on G. This correspondence sends a Poisson-Lie group (G, η) into the Lie bialgebra $\mathfrak{g}$ tangent to (G, η).*

By definition, a Lie algebra $\mathfrak{g}$ is a *Lie bialgebra* if its dual space $\mathfrak{g}^*$ is equipped with a Lie algebra structure such that the map $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ dual to the Lie bracket map $\mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$.

Theorem 4.3 [14]. *Consider a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. Then $\mathfrak{g}_+$ is naturally dual to $\mathfrak{g}_-$ and each of $\mathfrak{g}_-$ and $\mathfrak{g}_+$ is a Lie bialgebra. Conversely, for any Lie bialgebra $\mathfrak{g}$ one can find a unique Lie algebra structure on $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^*$ such that the triple $(\bar{\mathfrak{g}}, \mathfrak{g}, \mathfrak{g}^*)$ is a Manin triple with respect to the natural pairing on $\bar{\mathfrak{g}}$ and the corresponding Lie bialgebra structure on $\bar{\mathfrak{g}}$ is the given one.*

4.2. The Poisson structure and integrable hierarchies on pseudodifferential symbols. Start with the Lie bialgebra of co-centrally extended smooth integral symbols on the circle:

$$\widetilde{IS} = \mathbb{C} \cdot \log \partial \oplus IS = \left\{ \lambda \log \partial + \sum_{k=-\infty}^{-1} u_k(x)\partial^k \right\}.$$

(Alternatively, one can start with holomorphic symbols HIS and $\log D$ corresponding to a non-vanishing holomorphic field v on a punctured Riemann surface.) The corresponding Lie group consists of monic symbols of arbitrary complex degrees:

$$\mathbf{G}_{\widetilde{IS}} = \left\{ L = \partial^\lambda \left(1 + \sum_{k=-\infty}^{-1} u_k(x)\partial^k \right) \mid \lambda \in \mathbb{C} \right\}.$$

The Poisson-Lie structure on this group is given by the generalized quadratic Adler-Gelfand-Dickey bracket. Namely, the degree λ is a Casimir and we can consider the bracket on the hyperplane of symbols $\{L \mid \lambda = const\}$. The cotangent space to such planes can be identified with the symbols of the form $X = \partial^{-\lambda} \circ Y$, where Y is a purely differential operator. Then the bracket on $\{L\}$ is defined by the following Hamiltonian mapping $X \mapsto V_X(L)$ (from the cotangent space $\{X\}$ to the tangent space to symbols $\{L\}$ of fixed degree):

$$V_X(L) = (LX)_+ L - L(XL)_+,$$

see details in [3, 5].

To obtain dynamical systems, consider the following family of Hamiltonian functions $\{H_k\}$ on this Poisson-Lie group $\mathbf{G}_{\widetilde{IS}}$:

$$H_k(L) := \frac{\lambda}{k} Tr(L^{k/\lambda}),$$

where L has degree $\lambda \neq 0$. The corresponding Hamiltonian equations with respect to the quadratic Adler-Gelfand-Dickey Poisson structure form the following universal KdV-KP hierarchy:

$$\frac{\partial L}{\partial t_k} = [(L^{k/\lambda})_+, L], \quad k = 1, 2, \ldots,$$

see [3, 5]. For $\lambda = 1$ this is the standard KP hierarchy of commuting flows. For integer $\lambda = n$ the restriction of this universal hierarchy to the Poisson submanifolds of purely differential operators of degree n gives the n-KdV hierarchy.

Acknowledgements. We are indebted to F. Malikov, I. Krichever and A. Rosly for fruitful discussions. In particular, the definition of the sheaves of holomorphic symbols was proposed to us by F. Malikov. We are also thankful to the anonymous referee for providing us with the reference [12] and useful remarks. B.K. is grateful to the Max-Plank-Institut in Bonn for kind hospitality. The work of B.K. was partially supported by an NSERC research grant.

References

1. Dickey, L.A.: Soliton equations and Hamiltonian systems. Adv. Series in Math. Phys. **12**, Singapore: World Scientific, 1991, 310pp
2. Dzhumadildaev, A.S.: Derivations and central extensions of the Lie algebra of formal pseudodifferential operators. St. Petersburg Math. J. **6**(1), 121–136 (1995)
3. Enriquez, B., Khoroshkin, S., Radul, A., Rosly, A., Rubtsov, V.: Poisson-Lie aspects of classical W-algebras. AMS Translations (2) **167**, Providence, RI: AMS, 1995, pp. 37–59
4. Feigin, B.L.: gl(λ) and cohomology of a Lie algebra of differential operators. Russ. Math. Surv. **43**(2), 169–170 (1988)
5. Khesin, B., Zakharevich, I.: Poisson-Lie group of pseudodifferential symbols. Commun. Math. Phys. **171**, 475–530 (1995)
6. Kravchenko, O.S., Khesin, B.A.: Central extension of the algebra of pseudodifferential symbols. Funct. Anal. Appl. **25**(2), 83–85 (1991)
7. Krichever, I.M., Novikov, S.P.: Algebras of Virasoro type, Riemann surfaces and structure of the theory of solitons. Funct. Anal. Appl. **21**(2), 126–142 (1987)
8. Krichever, I.M., Novikov, S.P.: Virasoro type algebras, Riemann surfaces and strings in Minkowski space. Funct. Anal. Appl. **21**(4), 294–307 (1987)
9. Li, W.L.: 2-cocyles on the algebra of differential operators. J. Algebra **122**(1), 64–80 (1989)
10. Radul, A.O.: A central extension of the Lie algebra of differential operators on a circle and W-algebras. JETP Letters **50**(8), 371–373 (1989)
11. Schlichenmaier, M.: Krichever-Novikov algebras for more than two points. Lett. in Math. Phys. **19**, 151–165 (1990)
12. Schlichenmaier, M.: Verallgemeinerte Krichever-Novikov Algebren und deren Darstellungen, PhD-thesis, Universität Mannheim, Juni 1990; Differential operator algebras on compact Riemann surfaces. In: *Generalized symmetries in physics (Clausthal, 1993)*, River Edge, NJ: World Sci. Publ., 1994, pp. 425–434
13. Schlichenmaier, M.: Local cocycles and central extensions for multipoint algebras of Krichever-Novikov type. J. Reine Angew. Math. **559**, 53–94 (2003)
14. Semenov-Tyan-Shanskii, M.A.: Dressing transformations and Poisson group actions. Kyoto University, RIMS Publications **21**(6), 1237–1260 (1985)
15. Sheinman, O.K.: Affine Lie algebras on Riemann surfaces. Funct. Anal. Appl. **27**(4), 266–272 (1993)
16. Sheinman, O.K.: Affine Krichever-Novikov algebras, their representations and applications. In: *Geometry, Topology and Mathematical Physics*. S. P. Novikov's Seminar 2002–2003, V. M. Buchstaber, I. M. Krichever, eds. AMS Translations (2) **212**, Providence, RI: AMS, 2004, pp. 297–316
17. Lu, J.-H., Weinstein, A.: Poisson Lie groups, dressing transformations, and Bruhat decompositions. J. Diff. Geom. **31**(2), 501–526 (1990)
18. Wagemann, F.: Some remarks on the cohomology of Krichever-Novikov algebras. Lett. in Math. Phys. **47**, 173–177 (1999)

Communicated by L. Takhtajan