Arch. Rational Mech. Anal. 234 (2019) 549–573
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-019-01397-2

Check for updates

Geometry of the Madelung Transform

Boris Khesin, Gerard Misiołek & Klas Modin

Communicated by V. Šverák

Abstract

The Madelung transform is known to relate Schrödinger-type equations in quantum mechanics and the Euler equations for barotropic-type fluids. We prove that, more generally, the Madelung transform is a Kähler map (that is a symplectomorphism and an isometry) between the space of wave functions and the cotangent bundle to the density space equipped with the Fubini-Study metric and the Fisher–Rao information metric, respectively. We also show that Fusca's momentum map property of the Madelung transform is a manifestation of the general approach via reduction for semi-direct product groups. Furthermore, the Hasimoto transform for the binormal equation turns out to be the 1D case of the Madelung transform, while its higher-dimensional version is related to the Willmore energy in binormal flows.

Contents

1. Introduction

In 1927 Madelung [14] introduced a transformation, which now bears his name, in order to give an alternative formulation of the linear Schrödinger equation for a single particle moving in an electric field as a system of equations describing the motion of a compressible inviscid fluid. Since then other derivations have been proposed in the physics literature primarily in connection with various models in quantum hydrodynamics and optimal transport, cf. [7,15,17,21].

In this paper we focus on the geometric aspects of Madelung's construction and prove that the Madelung transform possesses a number of surprising properties. It turns out that in the right setting it can be viewed as a symplectomorphism, an isometry, a Kähler morphism or a generalized Hasimoto map. Furthermore, geometric properties of the Madelung transform are best understood not in the setting of the L^2-Wasserstein geometry but (an infinite-dimensional analogue of) the Fisher–Rao information geometry—the canonical Riemannian geometry of the space of probability densities. These results can be summarized in the following theorem (a joint version of Theorems 2.4 and 3.3 below):

Main Theorem. *The Madelung transform is a Kähler morphism between the cotangent bundle of the space of smooth probability densities, equipped with the (Sasaki)–Fisher–Rao metric, and an open subset of the infinite-dimensional complex projective space of smooth wave functions, equipped with the Fubini-Study metric.*

The above statement is valid in both the Sobolev topology of H^s-smooth functions and Fréchet topology of C^∞-smooth functions. In a sense, the Madelung transform resembles the passage from Euclidean to polar coordinates in the infinite-dimensional space of wave functions, where the modulus is a probability density and the phase corresponds to the fluid's velocity field. The main theorem shows that, after projectivization, this transform relates not only equations of hydrodynamics and those of quantum physics, but the corresponding symplectic structures underlying them as well. Surprisingly, it also turns out to be an isometry between two well-known Riemannian metrics in geometry and statistics.

Our first motivation comes from hydrodynamics where groups of diffeomorphisms arise as configuration spaces for flows of compressible and incompressible fluids in a domain M (typically, a compact connected Riemannian manifold with a volume form μ). When equipped with a metric given at the identity diffeomorphism by the L^2 inner product (corresponding essentially to the kinetic energy) the geodesics of the group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of M describe motions of a gas of noninteracting particles in M whose velocity field v satisfies the inviscid Burgers equation

$$\dot{v} + \nabla_v v = 0.$$

On the other hand, when restricted to the subgroup $\mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms, the L^2-metric becomes right-invariant, and, as was discovered by Arnold [1,2], its geodesics can be viewed as motions of an ideal (that is, incompressible and inviscid) fluid in M whose velocity field satisfies the incompressible

Euler equations

$$\begin{cases} \dot{v} + \nabla_v v = -\nabla p \\ \operatorname{div} v = 0. \end{cases}$$

Here the pressure gradient ∇p is defined uniquely by the divergence-free condition on the velocity field v and can be viewed as a constraining force on the fluid. What we describe below can be regarded as an extension of this framework to various equations of compressible fluids, where the evolution of density becomes foremost important.

Our second motivation is to study the geometry of the space of densities. Namely, consider the projection $\pi : \mathrm{Diff}(M) \to \mathrm{Dens}(M)$ of the full diffeomorphism group $\mathrm{Diff}(M)$ onto the space $\mathrm{Dens}(M)$ of normalized smooth densities on M. The fiber over a density ν consists of all diffeomorphisms ϕ that push forward the Riemannian volume form μ to ν, that is, $\phi_* \mu = \nu$. It was shown by Otto [18] that π is a Riemannian submersion between $\mathrm{Diff}(M)$ equipped with the L^2-metric and $\mathrm{Dens}(M)$ equipped with the (Kantorovich-) Wasserstein metric used in the optimal mass transport. More interesting for our purposes is that a Riemannian submersion arises also when $\mathrm{Diff}(M)$ is equipped with a right-invariant homogeneous Sobolev $\dot{H}^1$-metric and $\mathrm{Dens}(M)$ with the Fisher–Rao metric which plays an important role in geometric statistics, see [10].

In the present paper we prove the Kähler property of the Madelung transform thus establishing a close relation of the cotangent space of the space of densities and the projective space of wave functions on M. Furthermore, this transform also identifies many Newton-type equations on these spaces that are naturally related to equations of fluid dynamics.

As an additional perspective, the connection between equations of quantum mechanics and hydrodynamics described below might shed some light on the hydrodynamical quantum analogues studied in [4,5]: the motion of bouncing droplets in certain vibrating liquids manifests many properties of quantum mechanical particles. While bouncing droplets have a dynamical boundary condition with changing topology of the domain every period, apparently a more precise description of the phenomenon should involve a certain averaging procedure for the hydrodynamical system in a periodically changing domain. Then the droplet–quantum particle correspondence could be a combination of the averaging and the Madelung transform.

2. Madelung Transform as a Symplectomorphism

In this section we show that the Madelung transform induces a symplectomorphism between the cotangent bundle of smooth probability densities and the projective space of smooth non-vanishing complex-valued wave functions.

Definition 2.1. Let μ be a (reference) volume form on M such that $\int_M \mu = 1$. The space of probability densities on a compact connected oriented n-manifold M is

$$\mathrm{Dens}^s(M) = \left\{ \rho \in H^s(M) \mid \rho > 0,\ \int_M \rho\, \mu = 1 \right\}, \tag{1}$$

where $H^s(M)$ denotes the space of real-valued functions on M of Sobolev class H^s with $s > n/2$ (including the case $s = \infty$ corresponding to C^∞ functions).[1]

The space $\mathrm{Dens}^s(M)$ can be equipped in the standard manner with the structure of a smooth infinite-dimensional manifold (Hilbert, if $s < \infty$ or Fréchet, if $s = \infty$, cf. Appendix A). It is an open subset of an affine hyperplane in $H^s(M)$. Its tangent bundle is trivial:

$$T\mathrm{Dens}^s(M) = \mathrm{Dens}^s(M) \times H^s_0(M),$$

where $H^s_0(M) = \left\{c \in H^s(M) \mid \int_M c\,\mu = 0\right\}$. Likewise, the (regular part of the) co-tangent bundle is

$$T^*\mathrm{Dens}^s(M) = \mathrm{Dens}^s(M) \times H^s(M)/\mathbb{R},$$

where $H^s(M)/\mathbb{R}$ is the space of cosets $[\theta]$ of functions θ modulo additive constants $[\theta] = \{\theta + c \mid c \in \mathbb{R}\}$. The pairing is given by

$$T_\rho\mathrm{Dens}^s(M) \times T^*_\rho\mathrm{Dens}^s(M) \ni (\dot\rho, [\theta]) \mapsto \int_M \theta\,\dot\rho\,\mu.$$

It is independent of the choice of θ in the coset $[\theta]$ since $\int_M \dot\rho\,\mu = 0$.

Definition 2.2. The *Madelung transform* is a map $\mathbf{\Phi}$ which to any pair of functions $\rho\colon M \to \mathbb{R}_+$ and $\theta\colon M \to \mathbb{R}$ associates a complex-valued function

$$\mathbf{\Phi}\colon (\rho, \theta) \mapsto \psi := \sqrt{\rho \mathrm{e}^{\mathrm{i}\theta}} = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\theta/2}. \tag{2}$$

Remark 2.3. The latter expression defines a particular branch of the square root $\sqrt{\rho\mathrm{e}^{\mathrm{i}\theta}}$. The map $\mathbf{\Phi}$ is unramified, since ρ is strictly positive. Note that this map is not injective because θ and $\theta + 4\pi k$ have the same image. Despite this fact, there is, as we shall see next, a natural geometric setting in which the Madelung transform (2) becomes invertible.

2.1. Symplectic Properties

Let $H^s(M, \mathbb{C})$ denote the space of complex-valued functions of Sobolev class on a compact connected manifold M and let $\mathbb{P}H^s(M, \mathbb{C})$ denote the corresponding complex projective space. Its elements can be represented as cosets of the unit L^2-sphere $S^\infty(M, \mathbb{C})$ of complex functions (Fig. 1)

$$\mathbb{P}H^s(M, \mathbb{C}) = \left\{[\psi] \mid \psi \in H^s(M, \mathbb{C}),\ \|\psi\|_{L^2} = 1\right\},$$

where the cosets are

$$[\psi] = \left\{\mathrm{e}^{\mathrm{i}\tau}\psi \mid \tau \in \mathbb{R}\right\}.$$

[1] From a geometric point of view it is more natural to define densities as volume forms instead of functions. This way, they become independent of the reference volume form μ. However, since some of the equations studied in this paper depend on the reference volume form μ anyway, it is easier to define densities as functions to avoid notational overload.

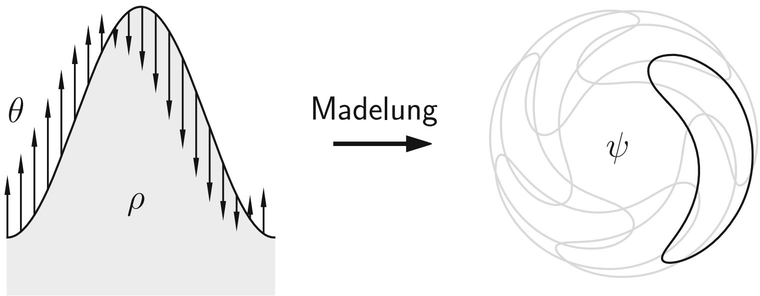

Fig. 1. Illustration of the Madelung transform $\mathbf{\Phi}$ on S^1. For $x \in S^1$, a probability density $\rho(x) > 0$ and a dual infinitesimal probability density $\theta(x)$ are mapped to a wave function $\psi(x) = \sqrt{\rho e^{i\theta}} \in \mathbb{C}$, which is defined up to rigid rotations of the complex plane

The complex projective space $\mathbb{P}H^s(M, \mathbb{C})$ carries a natural symplectic structure, inherited from $H^s(M, \mathbb{C})$:

$$\Omega^{\mathbb{P}H^s}_{[\psi]}\big([\![\dot\psi_1]\!], [\![\dot\psi_2]\!]\big) = \int_M \mathrm{Im}\big(\dot\psi_1 \overline{\dot\psi_2}\big)\mu, \tag{3}$$

where $[\![\dot\psi_k]\!] \in T_{[\psi]}\mathbb{P}H^s(M, \mathbb{C})$, $k = 1, 2$.

If $\tilde\psi \in [\psi]$ is nowhere vanishing then every other representative in the coset $[\psi]$ is nowhere vanishing as well. In particular, the space $\mathbb{P}H^s(M, \mathbb{C}\backslash\{0\})$, the projectivization of nowhere zero complex-valued H^s function on M, is an open subset and hence a symplectic submanifold of $\mathbb{P}H^s(M, \mathbb{C})$.

Theorem 2.4. *The Madelung transform* (2) *induces a map*

$$\mathbf{\Phi}\colon T^*\mathrm{Dens}^s(M) \to \mathbb{P}H^s(M, \mathbb{C}\backslash\{0\}), \tag{4}$$

which, up to scaling by 4*, is a symplectomorphism*[2] *with respect to the canonical symplectic structure of* $T^*\mathrm{Dens}^s(M)$ *and the symplectic structure* $\Omega^{\mathbb{P}H^s}$ *on* $\mathbb{P}H^s(M, \mathbb{C})$.

Proof. We need to establish the following three steps: (i) $\mathbf{\Phi}$ is well-defined, (ii) $\mathbf{\Phi}$ is smooth, surjective and injective and (iii) $\mathbf{\Phi}$ is symplectic.

(i) Let $\rho \in \mathrm{Dens}^s(M)$. Recall that the elements of $T^*_\rho\mathrm{Dens}^s(M)$ are cosets of H^s functions on M modulo constants and given any $\theta \in H^s(M, \mathbb{R})$ and any $\tau \in \mathbb{R}$ the Madelung transform maps $(\rho, \theta + \tau)$ to $\sqrt{\rho} e^{i(\theta+\tau)/2}$. If $s > n/2$ then standard results on products and compositions of Sobolev functions (cf. for example [19]) show that it is smooth as a map to $H^s(M, \mathbb{C})$. Furthermore, we have

$$\big\|\sqrt{\rho} e^{i(\theta+\tau)/2}\big\|_{L^2} = \|\sqrt{\rho} e^{i\theta/2}\|_{L^2} = \|\sqrt{\rho}\|_{L^2} = 1,$$

so that cosets $(\rho, [\theta])$ are mapped to cosets $[\psi]$, that is the map is well-defined.

[2] In the Fréchet topology of smooth functions if $s = \infty$.

(ii) Surjectivity and smoothness of $\mathbf{\Phi}$ are evident. To prove injectivity for the cosets recall that inverting the Madelung map amounts essentially to rewriting of a non-vanishing complex-valued function in polar coordinates. Since preimages for a given ψ differ by a constant polar argument $\tilde{\theta} = \theta + 2\pi k$, they define the same coset $[\theta]$. Similarly, changing ψ by a constant phase does not affect the argument coset $[\theta]$, which implies injectivity of the map between the cosets $(\rho, [\theta])$ and $[\psi]$.[3]

(iii) The canonical symplectic form on $T^*\mathrm{Dens}^s(M)$ is given by

$$\Omega^{T^*\mathrm{Dens}}_{(\rho,[\theta])}\big((\dot\rho_1, [\dot\theta_1]), (\dot\rho_2, [\dot\theta_2])\big)\mu = \int_M \big(\dot\theta_1\dot\rho_2 - \dot\theta_2\dot\rho_1\big)\,\mu. \tag{5}$$

Since $\int_M \dot\rho_k\,\mu = 0$ it follows that it is well-defined on the cosets $[\dot\theta_i]$.

Let us now compare it with the symplectic form $\Omega^{\mathbb{P}H^s}$ on $\mathbb{P}H^s(M, \mathbb{C})$ given by (3). By identifying the tangent space $T_{[\psi]}\mathbb{P}H^s(M, \mathbb{C}) \simeq T_\psi S^\infty(M, \mathbb{C})/T_\psi[\psi]$ we describe a tangent vector $[\![\dot\psi]\!]$ at a representative $\psi \in [\psi]$ as a coset

$$[\![\dot\psi]\!] := \{\dot\psi + \mathrm{i}c\psi \mid c \in \mathbb{R}\}.$$

It is straightforward to verify its independence of a representative.

For $\psi = \mathbf{\Phi}(\rho, [\theta])$ the tangent vector is $T_{(\rho,[\theta])}\mathbf{\Phi}(\dot\rho, [\dot\theta]) = 1/2(\dot\rho/\rho + \mathrm{i}\dot\theta)\mathbf{\Phi}(\rho, [\theta])$. Then (3) gives

$$\begin{aligned}
&\Omega^{\mathbb{P}H^s}_{\mathbf{\Phi}(\rho,[\theta])}\big(T_{(\rho,[\theta])}\mathbf{\Phi}(\dot\rho_1, [\dot\theta_1]), T_{(\rho,[\theta])}\mathbf{\Phi}(\dot\rho_2, [\dot\theta_2])\big)\\
&\quad= \frac{1}{4}\int_M \mathrm{Im}\left(\Big(\frac{\dot\rho_1}{\rho} + \mathrm{i}\dot\theta_1\Big)\Big(\frac{\dot\rho_2}{\rho} - \mathrm{i}\dot\theta_2\Big)\psi\overline{\psi}\right)\mu\\
&\quad= \frac{1}{4}\int_M \left(\dot\theta_1\frac{\dot\rho_2}{\rho} - \dot\theta_2\frac{\dot\rho_1}{\rho}\right)\rho\,\mu = \frac{1}{4}\int_M \big(\dot\theta_1\dot\rho_2 - \dot\theta_2\dot\rho_1\big)\,\mu\\
&\quad= \frac{1}{4}\Omega^{T^*\mathrm{Dens}}_{(\rho,[\theta])}\big((\dot\rho_1, [\dot\theta_1]), (\dot\rho_2, [\dot\theta_2])\big),
\end{aligned}$$

which completes the proof. □

Remark 2.5. In Section 4 the inverse Madelung transform is defined for any C^1 function with no restriction on strict positivity of $|\psi|^2$. It can be defined similarly in a Sobolev setting. Furthermore, extending the result of Fusca [8], we will also show that it can be understood as a momentum map for a natural action of a certain semi-direct product group. Thus the Madelung transform relates the standard symplectic structure on the space of wave functions and the linear Lie-Poisson structure on the corresponding dual Lie algebra.

[3] Note that the injectivity would not hold for L^2 functions, or even for smooth functions if M were not connected. Indeed, the arguments of the preimages could then have incompatible integer jumps at different points of M. For continuous functions on a connected M it suffices to fix the argument at one point only.

Remark 2.6. The fact that the Madelung transform is a symplectic submersion between the cotangent bundle of the space of densities and the unit sphere $S^\infty(M, \mathbb{C}\backslash\{0\}) \subset H^s(M, \mathbb{C}\backslash\{0\})$ of non-vanishing wave functions was proved by von Renesse [21] (with a different choice of rescaling constants). The stronger symplectomorphism property proved in Theorem 2.4 is achieved by considering projectivization $\mathbb{P}H^s(M, \mathbb{C}\backslash\{0\})$.

2.2. Example: Linear and Nonlinear Schrödinger Equations

Let ψ be a wave function on a Riemannian manifold M and consider the family of Schrödinger (or Gross–Pitaevsky) equations with Planck's constant $\hbar = 1$ and mass $m = 1/2$

$$\mathrm{i}\dot{\psi} = -\Delta\psi + V\psi + f(|\psi|^2)\psi, \tag{6}$$

where $V\colon M \to \mathbb{R}$ and $f\colon (0, \infty) \to \mathbb{R}$. If $f \equiv 0$ we obtain the linear Schrödinger equation with potential V. If $V \equiv 0$ we obtain the family of non-linear Schrödinger equations (NLS); two typical choices are $f(a) = \kappa a$ and $f(a) = \frac{1}{2}(a-1)^2$.

Note that equation (6) is Hamiltonian with respect to the symplectic structure induced by the complex structure of $L^2(M, \mathbb{C})$. Indeed, recall that the real part of a Hermitian inner product defines a Riemannian structure and the imaginary part defines a symplectic structure, so that

$$\Omega(\psi_1, \psi_2) := \mathrm{Im}\langle\!\langle \psi_1, \psi_2 \rangle\!\rangle_{L^2} = \mathrm{Re}\langle\!\langle \mathrm{i}\psi_1, \psi_2 \rangle\!\rangle_{L^2}$$

defines a symplectic form Ω corresponding to the complex structure $J(\psi) = \mathrm{i}\psi$. The Hamiltonian function for the Schrödinger equation (6) is

$$H(\psi) = \frac{1}{2}\|\nabla\psi\|^2_{L^2} + \frac{1}{2}\int_M \Big(V|\psi|^2 + F(|\psi|^2) \Big)\,\mu, \tag{7}$$

where $F\colon (0, \infty) \to \mathbb{R}$ is a primitive function of f, namely $F' = f$.

Observe that the L^2-norm of any solution ψ of (6) is conserved in time. Furthermore, the Schrödinger equation is also equivariant with respect to a constant phase shift $\psi(x) \mapsto e^{\mathrm{i}\tau}\psi(x)$ and therefore descends to the projective space $\mathbb{P}H^s(M, \mathbb{C})$. It can be viewed as an equation on the complex projective space, a point of view first suggested in [12].

Proposition 2.7. (cf. [14,21]). *The Madelung transform* (4) *maps the family of Schrödinger equations* (6) *to the following system on* $T^*\mathrm{Dens}^s(M)$

$$\begin{cases} \dot{\theta} + \dfrac{1}{2}|\nabla\theta|^2 + 2V + 2f(\rho) - \dfrac{2\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0, \\ \dot{\rho} + \mathrm{div}(\rho\nabla\theta) = 0. \end{cases} \tag{8}$$

Equation (8) *has a hydrodynamic formulation as an equation for a barotropic-type fluid*

$$\begin{cases} \dot{v} + \nabla_v v + \nabla\Big(2V + 2f(\rho) - \dfrac{2\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big) = 0, \\ \dot{\rho} + \mathrm{div}(\rho v) = 0 \end{cases} \tag{9}$$

with potential velocity field $v = \nabla\theta$.

Remark 2.8. Note that (8) only makes sense for $\rho > 0$, whereas the NLS equation makes sense even when $\rho \geqq 0$. In particular, the properties of the Madelung transform imply that if one starts with a wave function such that $|\psi|^2 > 0$ everywhere, then it remains strictly positive for all t for which the solution to Equation (8) is defined, since this holds for $\rho = |\psi|^2$ by the continuity equation. Thus, $|\psi|^2$ can become non-positive only if $v = \nabla\theta$ stops being a C^1 vector field (so that the continuity equation breaks).

Note also that in classical barotropic fluids the pressure term depends pointwise on the density ρ. In (9) the pressure term still depends only on ρ (this is what we mean by "barotropic-like") but now also on its derivative: $P := f(\rho) - \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$. The term $\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ is often referred to as "quantum pressure".

Proof. Since the transformation $(\rho, [\theta]) \mapsto \psi$ is symplectic, it is enough to work out the Hamiltonian (7) expressed in $(\rho, [\theta])$. First, notice that

$$\nabla\psi = \mathrm{e}^{\mathrm{i}\theta/2}\Big(\nabla\sqrt{\rho} + \frac{\mathrm{i}}{2}\sqrt{\rho}\,\nabla\theta\Big), \tag{10}$$

so that

$$\begin{aligned} \|\nabla\psi\|_{L^2}^2 &= \Big\langle\!\!\Big\langle \nabla\sqrt{\rho} + \frac{\mathrm{i}}{2}\sqrt{\rho}\nabla\theta, \nabla\sqrt{\rho} + \frac{\mathrm{i}}{2}\sqrt{\rho}\nabla\theta \Big\rangle\!\!\Big\rangle_{L^2} \\ &= \langle\!\langle \nabla\sqrt{\rho}, \nabla\sqrt{\rho}\rangle\!\rangle_{L^2} + \frac{1}{4}\langle\!\langle \rho\nabla\theta, \nabla\theta\rangle\!\rangle_{L^2}. \end{aligned} \tag{11}$$

Thus, the Hamiltonian on $T^*\mathrm{Dens}^s(M)$ corresponding to the Schrödinger Hamiltonian (7) is

$$H(\rho, [\theta]) = \frac{1}{2}\int_M \Big(\frac{1}{4}|\nabla\theta|^2\rho + |\nabla\sqrt{\rho}|^2\Big)\,\mu + \frac{1}{2}\int_M (V\rho + F(\rho))\,\mu.$$

Since

$$\frac{\delta H}{\delta\rho} = \frac{1}{8}|\nabla\theta|^2 - \frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}} + \frac{1}{2}V + \frac{1}{2}f(\rho) \quad \text{and} \quad \frac{\delta H}{\delta\theta} = -\frac{1}{4}\,\mathrm{div}(\rho\nabla\theta),$$

the result now follows from Hamilton's equations:

$$\dot{\theta} = -4\frac{\delta H}{\delta\rho}, \quad \dot{\rho} = 4\frac{\delta H}{\delta\theta},$$

for the canonical symplectic form (5) scaled by $1/4$. □

Corollary 2.9. *The Hamiltonian system* (8) *on* $T^*\mathrm{Dens}^s(M)$ *for potential solutions of the barotropic equation* (9) *is mapped symplectomorphically to the Schrödinger equation* (6).

Example 2.10. Conversely, classical PDE of hydrodynamic type can be expressed as NLS-type equations. For example, potential solutions $v = \nabla\theta$ of the compressible Euler equations of a barotropic fluid are Hamiltonian on $T^*\mathrm{Dens}^s(M)$ with the Hamiltonian given as the sum of the kinetic energy $K = \frac{1}{2}\int_M |\nabla\theta|^2\rho\,\mu$ and the potential energy $U = \int_M e(\rho)\,\rho\,\mu$, where $e(\rho)$ is the fluid internal energy, see [11]. They can be formulated as an NLS equation with the Hamiltonian

$$H(\psi) = \frac{1}{2}\|\nabla\psi\|^2_{L^2} - \frac{1}{2}\|\nabla|\psi|\|^2_{L^2} + \int_M e(|\psi|^2)|\psi|^2\mu. \tag{12}$$

The choice $e = 0$ gives the Schrödinger formulation for potential solutions of Burgers' equation, which describe geodesics in the L^2-type Wasserstein metric on $\mathrm{Dens}^s(M)$. Thus, the geometric framework links the optimal transport for cost functions with potentials with the compressible Euler equations and the NLS-type equations described above.

2.3. Madelung Transform as a Hasimoto Map in 1D

The celebrated *vortex filament equation*

$$\dot\gamma = \gamma' \times \gamma''$$

is an evolution equation on a (closed or open) curve $\gamma \subset \mathbb{R}^3$, where $\gamma = \gamma(x, t)$ and $\gamma' := \partial\gamma/\partial x$ and x is an arc-length parameter along γ. (The equivalent *binormal form* of this equation $\dot\gamma = k(x, t)\mathbf{b}$ is valid in any parametrization, where $\mathbf{b} = \mathbf{t}\times\mathbf{n}$ is the binormal unit vector to the curve at a point x, $\mathbf{t}$ and $\mathbf{n}$ are, respectively, the unit tangent and the normal vectors and $k(x, t)$ is the curvature of the curve at the point x at moment t, see Fig. 2.) This equation describes a localized induction approximation of the 3D Euler equation of an ideal fluid in $\mathbb{R}^3$, where the vorticity of the initial velocity field is supported on a curve γ. (Note that the corresponding evolution of the vorticity is given by the hydrodynamical Euler equation, which becomes nonlocal in terms of vorticity. By considering the ansatz that keeps only local terms, it reduces to the filament equation above.)

The vortex filament equation is known to be Hamiltonian with respect to the Marsden–Weinstein symplectic structure on the space of curves in $\mathbb{R}^3$ and with Hamiltonian given by the length functional, see, for example [2].

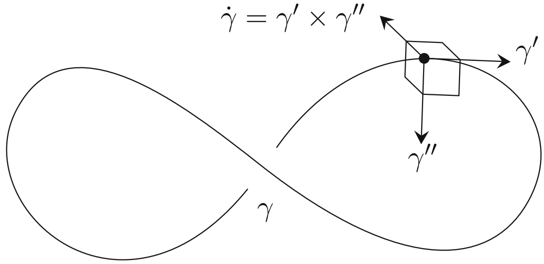

Fig. 2. Vortex filament flow: each point of the curve γ moves in the direction of the binormal. If $k(x)$ and $\tau(x)$ are the curvature and torsion at $\gamma(x)$, then the wave function $\psi(x) = k(x)e^{\mathrm{i}\int_{x_0}^{x}\tau(\tilde x)\mathrm{d}\tilde x}$ satisfies the NLS equation. Moreover, the pair of functions $v = 2\tau$ and $\rho = k^2$ satisfies the equation of the 1D barotropic fluid, which is a manifestation of the 1D Madelung transform

Definition 2.11. The Marsden–Weinstein symplectic structure Ω^{MW} assigns to a pair of variations V, W of a curve γ (understood as vector fields attached at $\gamma \subset \mathbb{R}^3$) the value $\Omega^{MW}(V, W) := \int_\gamma i_V i_W \mu$, where μ is the Euclidean volume form in $\mathbb{R}^3$.

It turns out that the vortex filament equation becomes the equation of the 1D barotropic-type fluid (9) with $\rho = k^2$ and $v = 2\tau$, where k and τ denote curvature and torsion of the curve γ, respectively.

In 1972 Hasimoto [9] introduced the following surprising transformation:

Definition 2.12. The *Hasimoto transformation* assigns to a curve γ, with curvature k and torsion τ, a wave function ψ according to the formula

$$(k(x), \tau(x)) \mapsto \psi(x) = k(x)e^{\mathrm{i}\int_{x_0}^{x} \tau(\tilde{x})\mathrm{d}\tilde{x}}.$$

This map takes the vortex filament equation to the 1D NLS equation $\mathrm{i}\dot{\psi} + \psi'' + \frac{1}{2}|\psi|^2\psi = 0$. (A change of the initial point x_0 in $\int_{x_0}^{x} \tau(\tilde{x})d\tilde{x}$ leads to a multiplication of $\psi(x)$ by an irrelevant constant phase $e^{\mathrm{i}\alpha}$). In particular, the filament equation turns out to be a completely integrable system whose first integrals are obtained by pulling back those of the NLS equation. The first integrals for the filament equation can be written in terms of the total length $\int dx$, the torsion $\int \tau\, \mathrm{d}x$, the squared curvature $\int k^2\, \mathrm{d}x$, followed by $\int \tau k^2\, \mathrm{d}x$ etc. Furthermore, by introducing variables $v = 2\tau$ and $\rho = k^2$ one arrives at the 1D compressible Euler equation in velocity v and density ρ.

The following proposition relates the Hasimoto transform to the classical Madelung transform.

Proposition 2.13. *The Hasimoto transformation is the Madelung transform in the 1D case.*

This can be seen by comparing Definitions 2.2 and 2.12 which make the Hasimoto transform seem much less surprising. Alternatively, one may note that for $\psi(x) = \sqrt{\rho(x)}e^{\mathrm{i}\theta(x)/2}$ the pair (ρ, v) with $v = \nabla\theta$ satisfies the compressible Euler equation, while in the one-dimensional case these variables are expressed via the curvature $\sqrt{\rho} = \sqrt{k^2} = k$ and the (indefinite) integral of torsion $\theta(x)/2 = \int v(x)\mathrm{d}x = \int \tau(x)\mathrm{d}x$.

Remark 2.14. The filament equation has a higher-dimensional analogue for membranes (which are compact oriented surfaces Σ of co-dimension 2 in $\mathbb{R}^n$) as a skew-mean-curvature flow $\dot{q} = \mathbf{J}(\mathbf{MC}(q))$, where $q \in \Sigma$ is any point of the membrane, $\mathbf{MC}(q)$ is the mean curvature vector to Σ at the point q and $\mathbf{J}$ is the operator of rotation by $\pi/2$ in the positive direction in every normal space to Σ. This equation is again Hamiltonian with respect to the Marsden–Weinstein structure Ω^{MW} on membranes of co-dimension 2 and with a Hamiltonian function given by the $(n-2)$-dimensional volume of the membrane, see for example [20].

An intriguing problem in this area is the following:

Question 2.15. *Find an analogue of the Hasimoto map, which sends a skew-mean-curvature flow to an NLS-type equation for any n.*

The existence of the Madelung transform and its symplectic property in any dimension is a strong indication that such an analogue should exist. Indeed, in any dimension by means of the Madelung transform one can pass from the wave function evolved according to an NLS-type equation to the polar form of ψ, that is to its magnitude $\sqrt{\rho}$ and the phase θ, so that the pair (ρ, v) with $v = \nabla\theta$ will evolve according to the compressible Euler equation. Thus for a surface Σ of co-dimension 2 moving according to the skew-mean-curvature flow, the problem boils down to interpreting the corresponding characteristics $(\rho, \nabla\theta)$ similarly to the one-dimensional curvature and torsion. (Note that both the pair (ρ, θ) and the co-dimension 2 surface Σ in $\mathbb{R}^n$ can be encoded by two functions of $n-2$ variables.)

In any dimension the square of the mean curvature vector can be regarded as a natural analogue of the density, $\rho = \|\mathbf{MC}\|^2$. In this case an analogue of the total mass of the fluid, that is $\int_\Sigma \rho \, \mathrm{d}\sigma$, is the *Willmore energy* $\mathcal{W}(\Sigma) = \int_\Sigma \|\mathbf{MC}\|^2 \, \mathrm{d}\sigma$. An intermediate step in finding a higher-dimensional Hasimoto map is then the following:

Conjecture 2.16. *For a compact co-dimension 2 surface $\Sigma \in \mathbb{R}^n$ moving by the skew-mean curvature flow $\dot{q} = \mathbf{J}(\mathbf{MC}(q))$ the following equivalent properties hold:*

(i) *its Willmore energy $\mathcal{W}(\Sigma)$ is invariant;*
(ii) *its square mean curvature $\rho = \|\mathbf{MC}\|^2$ evolves according to the continuity equation $\dot{\rho} + \operatorname{div}(\rho v) = 0$ for some vector field v on Σ.*

The equivalence of the two statements is a consequence of Moser's theorem: if the total mass on a surface is preserved, the corresponding evolution of density can be realized as a flow of a time-dependent vector field.

Proposition 2.17. *The conjecture is true in dimension 1.*

Proof. In 1D the conservation of the Willmore energy is the time invariance of the integral $\mathcal{W}(\gamma) = \int_\gamma k^2 \, \mathrm{d}x$ or, equivalently, in the arc-length parameterization, of the integral $\int_\gamma |\gamma''|^2 \, \mathrm{d}x$. The latter invariance follows from the following straightforward computation

$$\frac{1}{2}\dot{\mathcal{W}}(\gamma) = \int_\gamma (\dot{\gamma}'', \gamma'') \, \mathrm{d}x = -\int_\gamma (\dot{\gamma}', \gamma''') \, \mathrm{d}x = -\int_\gamma ((\gamma' \times \gamma'')', \gamma''') \, \mathrm{d}x = 0.$$

□

It would be very interesting to find a higher-dimensional analogue of the torsion τ for co-dimension 2 membranes. Note that the integral of the torsion has to play the role of an angular coordinate in the tangent spaces to Σ. The torsion would be the gradient part of the field v transporting the density $\rho = \|\mathbf{MC}\|^2$.

3. Madelung Transform as an Isometry of Kähler Manifolds

3.1. Metric Properties

In this section we prove that the Madelung transform is an isometry and a Kähler map between the lifted Fisher–Rao metric on the cotangent bundle $T^*\mathrm{Dens}^s(M)$ and the Kähler structure corresponding to the Fubini-Study metric on the infinite-dimensional projective space $\mathbb{P}H^s(M,\mathbb{C})$.

Definition 3.1. The Fisher–Rao metric on the density space $\mathrm{Dens}^s(M)$ is given by

$$\mathsf{G}_\rho(\dot\rho,\dot\rho)=\frac{1}{4}\int_M\frac{\dot\rho^2}{\rho}\,\mu. \tag{13}$$

This metric is invariant under the action of the diffeomorphism group. It is, in fact, the *only* Riemannian metric on $\mathrm{Dens}^s(M)$ with this property, cf. for example [3].

Next, observe that an element of $TT^*\mathrm{Dens}^s(M)$ is a 4-tuple $(\rho,[\theta],\dot\rho,\dot\theta)$, where $\rho\in\mathrm{Dens}^s(M)$, $[\theta]\in H^s(M)/\mathbb{R}$, $\dot\rho\in H^s_0(M)$ and $\dot\theta\in H^s(M)$ subject to the constraint

$$\int_M\dot\theta\rho\,\mu=0. \tag{14}$$

Definition 3.2. The lift of the Fisher-Rao metric to the cotangent bundle $T^*\mathrm{Dens}^s(M)$ has the form

$$\mathsf{G}^*_{(\rho,[\theta])}\big((\dot\rho,\dot\theta),(\dot\rho,\dot\theta)\big)=\frac{1}{4}\int_M\left(\frac{\dot\rho^2}{\rho}+\dot\theta^2\rho\right)\mu. \tag{15}$$

We will refer to this metric as the *Sasaki–Fisher–Rao* metric.

Next, recall that the canonical (weak) *Fubini-Study metric* on the complex projective space $\mathbb{P}H^s(M,\mathbb{C})\subset\mathbb{P}L^2(M,\mathbb{C})$ is given by

$$\mathsf{FS}_\psi(\dot\psi,\dot\psi)=\frac{\langle\langle\dot\psi,\dot\psi\rangle\rangle_{L^2}}{\langle\langle\psi,\psi\rangle\rangle_{L^2}}-\frac{\langle\langle\psi,\dot\psi\rangle\rangle_{L^2}\langle\langle\dot\psi,\psi\rangle\rangle_{L^2}}{\langle\langle\psi,\psi\rangle\rangle^2_{L^2}}. \tag{16}$$

Theorem 3.3. *The Madelung transform* $\mathbf{\Phi}\colon T^*\mathrm{Dens}^s(M)\to\mathbb{P}H^s(M,\mathbb{C})$ *is an isometry with respect to the Sasaki–Fisher–Rao metric* (15) *on* $T^*\mathrm{Dens}^s(M)$ *and the Fubini-Study metric* (16) *on* $\mathbb{P}H^s(M,\mathbb{C}\backslash\{0\})$.

Proof. We have

$$T_{(\rho,[\theta])}\mathbf{\Phi}(\dot\rho,\dot\theta)=\frac{\dot\rho}{2\sqrt{\rho}}\,e^{\mathrm{i}\theta/2}+\frac{\mathrm{i}\dot\theta\sqrt{\rho}}{2}\,e^{\mathrm{i}\theta/2}=\frac{1}{2}\left(\frac{\dot\rho}{\rho}+\mathrm{i}\dot\theta\right)\psi,$$

where $\psi=\mathbf{\Phi}(\rho,[\theta])$. Since $\|\psi\|^2_{L^2}=1$, setting $\dot\psi=T_{(\rho,\theta)}\mathbf{\Phi}(\dot\rho,\dot\theta)$ we obtain

$$\mathsf{FS}_\psi(\dot\psi,\dot\psi)=\langle\langle\dot\psi,\dot\psi\rangle\rangle_{L^2}-\langle\langle\dot\psi,\psi\rangle\rangle_{L^2}\langle\langle\psi,\dot\psi\rangle\rangle_{L^2},$$

where

$$\langle\langle\dot\psi,\dot\psi\rangle\rangle_{L^2}=\frac{1}{4}\int_M\left|\frac{\dot\rho}{\rho}+\mathrm{i}\theta\right|^2\rho\,\mu=\frac{1}{4}\int_M\left(\frac{\dot\rho^2}{\rho^2}+\dot\theta^2\right)\rho=\mathsf{G}^*_{(\rho,\theta)}(\dot\rho,\dot\theta)$$

and

$$\langle\langle\dot\psi,\psi\rangle\rangle_{L^2}=\frac{1}{2}\int_M\left(\frac{\dot\rho}{\rho}+\mathrm{i}\dot\theta\right)\rho\,\mu=\frac{1}{2}\int_M\dot\rho\,\mu+\frac{\mathrm{i}}{2}\int_M\dot\theta\rho\,\mu=0,$$

which proves the theorem. □

The metric property in Theorem 3.3 combined with the symplectic property in Theorem 2.4 yields

Corollary 3.4. *The cotangent bundle* $T^*\mathrm{Dens}^s(M)$ *is a Kähler manifold with the Sasaki–Fisher–Rao metric* (15) *and the canonical symplectic structure* (5) *scaled by 1/4. The corresponding integrable almost complex structure is given by*

$$J_{(\rho,[\theta])}(\dot\rho,\dot\theta)=\left(\dot\theta\rho,-\frac{\dot\rho}{\rho}\right). \tag{17}$$

This result can be compared with the result of Molitor [15] who described a similar construction using (the cotangent lift of) the L^2 Wasserstein metric in optimal transport but obtained an almost complex structure on $T^*\mathrm{Dens}^s(M)$ which is not integrable. It appears that the Fisher–Rao metric is a more natural choice for such constructions: its lift to $T^*\mathrm{Dens}^s(M)$ admits a compatible complex (and Kähler) structure. It would be interesting to write down Kähler potentials for all metrics compatible with (17) and identify which of these are invariant under the action of the diffeomorphism group.

3.2. Geodesics of the Sasaki–Fisher–Rao Metric

As an isometry the Madelung transform maps geodesics of the Sasaki metric to geodesics of the Fubini-Study metric. The latter are projective lines in the projective space of wave functions. To see which submanifolds are mapped to projective lines by the Madelung transform we need to describe geodesics of the Sasaki–Fisher–Rao metric.

Proposition 3.5. *Geodesics of the Sasaki–Fisher–Rao metric* (15) *on the cotangent bundle* $T^*\mathrm{Dens}^s(M)$ *satisfy the system*

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\dot\rho}{\rho}\right)=-\dfrac{1}{2}\left(\dfrac{\dot\rho}{\rho}\right)^2+\dfrac{\dot\theta^2}{2}+\dfrac{\lambda}{2},\\[2mm] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dot\theta\rho\right)=0\end{cases}$$

where $\lambda=4\mathsf{G}^*_{(\rho,\theta)}((\dot\rho,\dot\theta),(\dot\rho,\dot\theta))$ *is a time-independent constant.*

Proof. The Lagrangian is given by the metric $L(\rho, \theta, \dot\rho, \dot\theta) = \mathsf{G}^*_{(\rho,\theta)}((\dot\rho, \dot\theta), (\dot\rho, \dot\theta))$. The variational derivatives are obtained from the formulas

$$\frac{\delta L}{\delta \dot\rho} = \frac{1}{2}\frac{\dot\rho}{\rho}, \quad \frac{\delta L}{\delta \dot\theta} = \frac{1}{2}\dot\theta\rho,$$
$$\frac{\delta L}{\delta \rho} = -\frac{1}{4}\left(\frac{\dot\rho}{\rho}\right)^2 + \frac{1}{4}\dot\theta^2, \quad \frac{\delta L}{\delta \theta} = 0,$$

which yield the equations of motion as stated. □

Remark 3.6. The natural projection $(\rho, [\theta]) \mapsto \rho$ is a Riemannian submersion between $T^*\mathrm{Dens}^s(M)$ equipped with the Sasaki–Fisher–Rao metric (15) and $\mathrm{Dens}^s(M)$ equipped with the Fisher–Rao metric (13). The corresponding horizontal distribution on $T^*\mathrm{Dens}^s(M)$ is given by

$$\mathrm{Hor}_{(\rho,[\theta])} = \left\{(\dot\rho, \dot\theta) \in T_{(\rho,[\theta])}\mathrm{Dens}^s(M) \mid \dot\theta = 0\right\}.$$

Indeed, if $\dot\theta = 0$ then the equations of motion of Proposition 3.5, restricted to $(\rho, \dot\rho)$, yield the geodesic equations for the Fisher–Rao metric. One can think of this as a zero-momentum symplectic reduction corresponding to the abelian gauge symmetry $(\rho, [\theta]) \mapsto (\rho, [\theta + f])$ for any function $f \in H^s(M)$.

3.3. Example: 2-Component Hunter–Saxton Equation

This is a system of two equations

$$\begin{cases} \dot v'' = -2v'v'' - vv''' + \sigma\sigma', \\ \dot\sigma = -(\sigma v)', \end{cases} \tag{18}$$

where $v = v(t, x)$ and $\sigma = \sigma(t, x)$ are time-dependent periodic functions on the line and the prime stands for the x-derivative. It can be viewed as a high-frequency limit of the 2-component Camassa–Holm equation, cf. [22].

It turns out that this system is closely related to the Kähler geometry of the Madelung transform and the Sasaki–Fisher–Rao metric (15). Consider the semi-direct product $\mathcal{G} = \mathrm{Diff}_0^{s+1}(S^1) \ltimes H^s(S^1, S^1)$, where $\mathrm{Diff}_0^{s+1}(S^1)$ is the group of circle diffeomorphisms that fix a prescribed point and $H^s(S^1, S^1)$ is the space of Sobolev S^1-valued maps of the circle. The group multiplication is given by

$$(\varphi, \alpha) \cdot (\eta, \beta) = (\varphi \circ \eta, \beta + \alpha \circ \eta).$$

Define a right-invariant Riemannian metric on $\mathcal{G}$ at the identity element by

$$\langle\!\langle (v, \sigma), (v, \sigma) \rangle\!\rangle_{\dot H^1} = \frac{1}{4}\int_{S^1} \left((v')^2 + \sigma^2\right) \mathrm{d}x. \tag{19}$$

If $t \to (\varphi(t), \alpha(t))$ is a geodesic in $\mathcal{G}$ then $v = \dot\varphi \circ \varphi^{-1}$ and $\sigma = \dot\alpha \circ \varphi^{-1}$ satisfy equations (18). Lenells [13] showed that the map

$$(\varphi, \alpha) \mapsto \sqrt{\varphi' \, \mathrm{e}^{\mathrm{i}\alpha}} \tag{20}$$

is an isometry from $\mathcal{G}$ to a subset of $\{\psi \in H^s(S^1, \mathbb{C}) \mid \|\psi\|_{L^2} = 1\}$. Moreover, solutions to (18) satisfying $\int_{S^1} \sigma \, \mathrm{d}x = 0$ correspond to geodesics in the complex projective space $\mathbb{P}H^s(S^1, \mathbb{C})$ equipped with the Fubini-Study metric. Our results show that this isometry is a particular case of Theorem 3.3.

Proposition 3.7. *The 2-component Hunter–Saxton equation* (18) *with initial data satisfying* $\int_{S^1} \sigma \, \mathrm{d}x = 0$ *is equivalent to the geodesic equation of the Sasaki–Fisher–Rao metric* (15) *on* $T^*\mathrm{Dens}^s(S^1)$.

Proof. First, observe that the mapping (20) can be rewritten as $(\varphi, \alpha) \mapsto \mathbf{\Phi}(\pi(\varphi), \alpha)$, where $\mathbf{\Phi}$ is the Madelung transform and π is the projection $\varphi \mapsto \varphi^*\mu$ specialized to the case $M = S^1$.

Next, observe that the metric (19) in the case $\int_{S^1} \sigma \mathrm{d}x = 0$ is the pullback of the Sasaki metric (15) by the mapping

$$\mathrm{Diff}_0^{s+1}(S^1) \ltimes H^s(S^1, S^1) \ni (\varphi, \alpha) \mapsto (\pi(\varphi), [\theta]) \in T^*\mathrm{Dens}^s(S^1),$$

where $\theta(x) = \int_0^x \alpha'(s)\mathrm{d}s$. Indeed, we have

$$\begin{aligned}
\mathsf{G}^*_{(\pi(\varphi),[\theta])}\left(\frac{\mathrm{d}}{\mathrm{d}t}\pi(\varphi), [\dot{\alpha}]\right) &= \frac{1}{4}\int_{S^1} \left(\left(\frac{\dot{\varphi}'}{\varphi'}\right)^2 + \dot{\alpha}^2\right)\varphi' \, \mathrm{d}x \\
&= \frac{1}{4}\int_{S^1} \left(\left((\dot{\varphi} \circ \varphi^{-1})'\right)^2 + (\dot{\alpha} \circ \varphi^{-1})^2\right) \mathrm{d}x \\
&= \frac{1}{4}\int_{S^1} \left((v')^2 + \sigma^2\right) \mathrm{d}x.
\end{aligned}$$

It follows from the change of variables formula by the diffeomorphism φ that the condition $\int_{S^1} \sigma \mathrm{d}x = 0$ corresponds to $\int_{S^1} \dot{\alpha}\varphi' \mathrm{d}x = 0$. Hence, the description of the 2-component Hunter–Saxton equation as a geodesic equation on the complex projective L^2 space is a special case of that on $T^*\mathrm{Dens}^s(M)$ with respect to the Sasaki–Fisher–Rao metric (15). □

Remark 3.8. Observe that if $\sigma = 0$ at $t = 0$ then $\sigma(t) = 0$ for all t and the 2-component Hunter–Saxton equation (18) reduces to the standard Hunter–Saxton equation. This is a consequence of the fact that horizontal geodesics on $T^*\mathrm{Dens}^s(M)$ with respect to the Sasaki–Fisher–Rao metric descend to geodesics on $\mathrm{Dens}^s(M)$ with respect to the Fisher–Rao metric.

4. Madelung Transform as a Momentum Map

In Section 2 we described the Madelung transform as a symplectomorphism from $T^*\mathrm{Dens}^s(M)$ to $\mathbb{P}H^s(M, \mathbb{C}\backslash\{0\})$ which associates a wave function $\psi = \sqrt{\rho}e^{\mathrm{i}\theta/2}$ (modulo a phase factor $e^{\mathrm{i}\tau}$) to a pair $(\rho, [\theta])$ consisting of a density ρ of unit mass and a function θ (modulo an additive constant). Here, we start by outlining (following [8]) another approach, which shows that it is natural to regard the inverse Madelung transform as a momentum map from the space $\mathbb{P}H^s(M, \mathbb{C})$ of wave functions ψ to the set of pairs $(\rho \, \mathrm{d}\theta, \rho)$ regarded as elements of the dual space of a certain Lie algebra. The latter is a semidirect product Lie algebra $\mathfrak{s} = \mathfrak{X}(M) \ltimes H^s(M)$ corresponding to the Lie group $S = \mathrm{Diff}(M) \ltimes H^s(M)$. (For simplicity, in this section we assume that $s = \infty$ for both diffeomorphisms and functions.)

Furthermore, this construction generalizes to the group-valued case $S_{(G)} = \mathrm{Diff}(M) \ltimes H^s(M, G)$, where $H^s(M, G) = H^s(M) \otimes G$. The case of general G provides a setting for quantum systems with spin degrees of freedom. For example, $G = \mathrm{SU}(2)$-framework (rank-1 spinors) describes fermions with spin $1/2$ (such as electrons, neutrons, and protons). For $G = \mathrm{U}(1)^2$ (or, simply, by setting $G = \mathbb{R}^2$) this group appears naturally in the description of general compressible fluids including transport of both density and entropy.

In Section 4.7 below we present a unifying point of view which explains the origin of the Madelung transform as the momentum map in a semidirect product reduction.

4.1. A Group Action on the Space of Wave Functions

We start by defining a group action on the space of wave functions. First, observe that it is natural to think of $H^s(M, \mathbb{C})$ as a space of complex-valued half-densities on M. Indeed, $\psi \in H^s(M, \mathbb{C})$ is assumed to be square-integrable and $|\psi|^2$ is interpreted as a probability measure. Half-densities are characterized by how they are transformed under diffeomorphisms of the underlying space: the pushforward $\varphi_*\psi$ of a half-density ψ on M by a diffeomorphism φ of M is given by the formula

$$\varphi_*\psi = \sqrt{|\mathrm{Det}(D\varphi^{-1})|}\, \psi \circ \varphi^{-1}.$$

This formula explains the following natural action of a semidirect product group on the vector space of half-densities.

Definition 4.1. [8] The semidirect product group $S = \mathrm{Diff}(M) \ltimes H^s(M)$ *acts on the space* $H^s(M, \mathbb{C})$ as follows: for a group element $(\varphi, a) \in S$ the action on wave functions ψ is

$$(\varphi, a) \circ \psi = \sqrt{|\mathrm{Det}(D\varphi^{-1})|}\, e^{-\mathrm{i}a/2}(\psi \circ \varphi^{-1}). \tag{21}$$

This action descends to the space of cosets $[\psi] \in \mathbb{P}H^s(M, \mathbb{C})$.

Thus, a wave function ψ is pushed forward under the diffeomorphism φ as a complex-valued half-density, followed by a pointwise phase adjustment by $e^{-\mathrm{i}a/2}$.

4.2. The Inverse of the Madelung Transform

Consider the following alternative definition of the inverse Madelung transform, which will be our primary object here. Let $\Omega^1(M)$ denote the space of 1-forms on M of Sobolev class H^s. Recall the definition (2) of the Madelung transform: $(\rho, \theta) \mapsto \psi = \sqrt{\rho e^{i\theta}}$, where $\rho > 0$.

Proposition 4.2. [8] *The map*

$$\mathbf{M}\colon\ H^s(M,\mathbb{C}) \to \Omega^1(M) \times \mathrm{Dens}^s(M) \tag{22}$$

given by

$$\psi \mapsto (m,\rho) = \big(2\,\mathrm{Im}(\bar\psi\, \mathrm{d}\psi),\, \bar\psi\psi\big)$$

is the inverse of the Madelung transform (2) *in the following sense: if* $\psi = \sqrt{\rho e^{\mathrm{i}\theta}}$ *then* $\mathbf{M}(\psi) = (\rho\, \mathrm{d}\theta, \rho)$.

Proof. For $\psi = \sqrt{\rho}e^{\mathrm{i}\theta/2}$ one evidently has $\bar\psi\psi = \rho$. The expression for the other component follows from the observation

$$\mathrm{Im}\,\bar\psi\,\mathrm{d}\psi = \bar\psi\psi\,\mathrm{Im}\,\mathrm{d}\,(\ln\psi) = \rho\,\mathrm{Im}\,\mathrm{d}\,((\ln\sqrt{\rho}) + \mathrm{i}\theta/2) = \rho\,\mathrm{d}\theta/2.$$

These two components allow one to obtain ρ and $\rho\,\mathrm{d}\theta$ and hence, by integration, to recover θ modulo an additive constant. (The ambiguity involving an additive constant in the definition of θ corresponds to recovering the wave function ψ modulo a constant phase factor.) □

For a positive function ρ satisfying $\int_M \rho\,\mu = 1$ the pair $(\rho\,\mathrm{d}\theta, \rho)$ can be identified with $(\rho, [\theta])$ in $T^*\mathrm{Dens}(M)$, where the momentum variable $m = \rho\,\mathrm{d}\theta$ is naturally thought of as an element of $\mathfrak{X}(M)^*$. Note, however, that this definition of the inverse Madelung works in greater generality: the momentum variable m is defined even when ρ is allowed to be zero, although θ cannot be recovered there.

Remark 4.3. So far we have viewed ψ as a function on an n-manifold M. One can also consistently regard ψ as a complex half-density $\varpi = \psi\,\mu^{1/2}$. The set of complex half-densities on M is denoted $\sqrt{\Omega^n}(M)\otimes\mathbb{C}$ indicating that it is "the square root" of the space $\Omega^n(M)$ of n-forms. Then the map $\mathbf{M}$ in (22) can be understood as follows. For a half-density $\varpi \in \sqrt{\Omega^n}(M)\otimes\mathbb{C}$ the second component $\bar\varpi\varpi$ of the map $\mathbf{M}$ is understood as a tensor product $(\bar\psi\psi)\,\mu = \rho\,\mu$ of two half-densities on M, thus yielding the density $\rho\in\mathrm{Dens}^s(M)$. One can show that the first component $\mathrm{Im}\,(\bar\varpi\,\mathrm{d}\varpi)$ of $\mathbf{M}$ can be regarded as an element $m\otimes\mu = \rho\,\mathrm{d}\theta\otimes\mu \in \Omega^1(M)\otimes_{H^s(M)}\Omega^n(M)$. Namely, given a reference density μ, for any half-density $\varpi = f(x)\mu^{1/2}$ define its differential $\mathrm{d}\varpi := \mathrm{d}f(x)\otimes\mu^{1/2}$. While the differential $\mathrm{d}\varpi$ depends on the choice of the reference density, the momentum map does not.

Proposition 4.4. *For any half density* $\varpi = f(x)\mu^{1/2} \in \sqrt{\Omega^n}(M)\otimes\mathbb{C}$ *the momentum* $2\mathrm{Im}\,(\bar\varpi\mathrm{d}\varpi) = 2\mathrm{Im}\,\bar f\mathrm{d}f\otimes\mu$ *is a well-defined element of* $\Omega^1(M)\otimes_{H^s(M)}\Omega^n(M)$ *and does not depend on the choice of the reference density* μ.

Proof. Given a different reference volume form $\nu = h(x)\mu$ with a positive function $h>0$ one has $\varpi = f(x)\mu^{1/2} = g(x)\nu^{1/2} = g(x)(h(x)\mu)^{1/2}$, where $f(x) = g(x)\sqrt{h(x)}$ and

$$\begin{aligned}
\mathrm{Im}\,(\bar\varpi\mathrm{d}\varpi) &= \mathrm{Im}\,\bar f\mathrm{d}f\otimes\mu = \mathrm{Im}\,\bar g\sqrt{h}\,d(g\sqrt{h})\otimes\mu\\
&= \mathrm{Im}\,(\bar g\sqrt{h}\sqrt{h}\,\mathrm{d}g + \bar g\sqrt{h}g\,\mathrm{d}(\sqrt{h}))\otimes\mu\\
&= \mathrm{Im}\,\bar g h\,\mathrm{d}g\otimes\mu = \mathrm{Im}\,\bar g\,\mathrm{d}g\otimes_{H^s(M)}(h\mu)\\
&= \mathrm{Im}\,\bar g\mathrm{d}g\otimes\nu,
\end{aligned}$$

where we dropped the term with $\bar{g}g\sqrt{h}\,\mathrm{d}(\sqrt{h})$ since it is purely real. □

Remark 4.5. The pair $(m,\rho)\otimes\mu = (\rho\,\mathrm{d}\theta\otimes\mu, \rho\,\mu)$ is understood as an element of the space $\mathfrak{s}^* = \Omega^1(M)\otimes_{H^s(M)}\Omega^n(M)\oplus\Omega^n(M)$ dual to the Lie algebra $\mathfrak{s} = \mathfrak{X}(M)\ltimes H^s(M)$, while the inverse Madelung transformation is a map $\mathbf{M}\colon H^s(M,\mathbb{C}) \to \mathfrak{s}^*$. Note that the dual space $\mathfrak{s}^*$ has a natural Lie-Poisson structure (as any dual Lie algebra).

4.3. A Reminder on Momentum Maps

In the next section we show that the inverse Madelung transform (22) is a momentum map associated with the action (21) of the Lie group $S = \mathrm{Diff}(M)\ltimes H^s(M)$ on $H^s(M,\mathbb{C})$. We start by recalling the definition of a momentum map.

Suppose that a Lie algebra $\mathfrak{g}$ acts on a Poisson manifold P and denote its action by $A\colon \mathfrak{g}\to\mathfrak{X}(P)$ where $A(\xi) = \xi_P$. Let $\langle,\rangle$ denote the pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$.

Definition 4.6. A *momentum map* associated with a Lie algebra action $A(\xi) = \xi_P$ is a map $\mathcal{M}\colon P\to\mathfrak{g}^*$ such that for every $\xi\in\mathfrak{g}$ the function $H_\xi\colon P\to\mathbb{R}$ defined by $H_\xi(p) := \langle\mathcal{M}(p),\xi\rangle$ for any $p\in P$ is a Hamiltonian of the vector field ξ_P on the Poisson manifold P, that is $X_{H_\xi}(p) = \xi_P(p)$.

Thus, Lie algebra actions that admit momentum maps are Hamiltonian actions and the pairing of the momentum map at a point $p\in P$ with an element $\xi\in\mathfrak{g}$ defines a Hamiltonian function associated with the Hamiltonian vector field ξ_P at that point p.

A momentum map $\mathcal{M}\colon P\to\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is *infinitesimally equivariant* if for all $\xi,\eta\in\mathfrak{g}$ one has $H_{[\xi,\eta]} = \{H_\xi, H_\eta\}$, which means that not only any Lie algebra vector defines a Hamiltonian vector field on the manifold, but also the Lie algebra bracket of two such fields corresponds to the Poisson bracket of their Hamiltonians.

4.4. Madelung Transform as a Momentum Map

We now show (following Fusca [8]) that the transformation $\mathbf{M}$ is a momentum map associated with the group action (21).

First note that the vector space $H^s(M,\mathbb{C})\subset L^2(M,\mathbb{C})$ of Sobolev wave functions on M is naturally equipped with the symplectic (and hence Poisson) structure $\{F,G\}(\psi) = \langle\!\langle\nabla F, -\mathrm{i}\nabla G\rangle\!\rangle_{L^2} = \langle\!\langle\mathrm{d}F, J\mathrm{d}G\rangle\!\rangle_{L^2}$. This structure is related to the natural Hermitian inner product on $L^2(M,\mathbb{C})$: $\langle\!\langle f,g\rangle\!\rangle_{L^2} := \int_M f\bar{g}\,\mu$ and the complex structure of multiplication by i. Now define the Hamiltonian function $H_\xi\colon H^s(M,\mathbb{C})\to\mathbb{R}$ by $H_\xi(\psi) := \langle\mathbf{M}(\psi),\xi\rangle$.

Theorem 4.7. [8] *For the Lie algebra $\mathfrak{s} = \mathfrak{X}(M)\ltimes H^s(M,\mathbb{R})$ its action on the Poisson space $H^s(M,\mathbb{C})\subset L^2(M,\mathbb{C})$ admits a momentum map. The inverse Madelung transformation* $\mathbf{M}\colon H^s(M,\mathbb{C})\to\mathfrak{s}^*$ *defined by* (22) *is, up to scaling by* 4*, a momentum map associated with this Lie algebra action.*

Proof. The Lie algebra action corresponding to the group action (21) can be described as follows: an element $\xi = (v, \alpha) \in \mathfrak{s} = \mathfrak{X}(M) \ltimes H^s(M)$ acts on a wave function ψ in $H^s(M, \mathbb{C})$ by the vector field

$$V_\xi(\psi) = -\frac{1}{2}\psi\,\mathrm{div}(v) - \frac{\mathrm{i}}{2}\alpha\psi - \iota_v \mathrm{d}\psi.$$

On the other hand, the Hamiltonian vector field for the function H_ξ is $X_{H_\xi} = -\mathrm{i}\,\mathrm{d}H_\xi$ where the differential is given by

$$\left\langle \mathrm{d}H_\xi(\psi), \phi\right\rangle = \mathrm{Re}\,\langle\!\langle \mathrm{d}H_\xi(\psi), \phi\rangle\!\rangle_{L^2} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\varepsilon=0} H_\xi(\psi + \varepsilon\phi)$$

for any test function ϕ in $H^s(M, \mathbb{C})$. Since the pairing of $\xi = (v, \alpha) \in \mathfrak{s}$ with $(m, \rho) \in \mathfrak{s}^*$ is given by

$$\langle (v, \alpha), (m, \rho)\rangle := \int_M (\rho \cdot \alpha + m \cdot v)\,\mu,$$

we have

$$H_\xi(\psi) = \int_M \big(\mathbf{M}(\psi)^\rho \cdot \alpha + \mathbf{M}(\psi)^m \cdot v\big)\mu = \mathrm{Re}\int_M (\bar{\psi}\psi\alpha - 2\mathrm{i}\,\iota_v\bar{\psi}\,\mathrm{d}\psi)\,\mu.$$

Then $\mathrm{d}H_\xi(\psi) = 2\psi\alpha - 2\mathrm{i}\psi\,\mathrm{div}(v) - 4\mathrm{i}\,\iota_v\,\mathrm{d}\psi$, since

$$\begin{aligned}\frac{d}{d\varepsilon}H_\xi(\psi + \varepsilon\phi)|_{\varepsilon=0} &= \mathrm{Re}\int_M \Big(\bar{\psi}\phi\alpha + \bar{\phi}\psi\alpha - 2\mathrm{i}\,\iota_v\bar{\phi}\,\mathrm{d}\psi - 2\mathrm{i}\,\iota_v\bar{\psi}\,\mathrm{d}\phi\Big)\mu \\ &= \mathrm{Re}\,\langle\!\langle 2\psi\alpha - 4\mathrm{i}\,\iota_v\,\mathrm{d}\psi - 2\mathrm{i}\psi\,\mathrm{div}(v), \phi\rangle\!\rangle_{L^2}.\end{aligned}$$

This implies that $X_{H_\xi}(\psi) = -2\mathrm{i}\alpha\psi - 4\iota_v\,\mathrm{d}\psi - 2\psi\,\mathrm{div}\,v = 4V_\xi(\psi)$. $\square$

Moreover, the Madelung transform turns out to be an infinitesimally equivariant momentum map, as was verified in [8]. (Recall that its equivariance means morphism of the Lie algebras: the Hamiltonian of the Lie bracket of two fields is the Poisson bracket of their Hamiltonians.) In particular, it follows that the Madelung transform is also a Poisson map taking the Poisson structure on P (up to scaling by 4) to the Lie-Poisson structure on $\mathfrak{g}$, that is the map $\mathbf{M}\colon H^s(M, \mathbb{C}) \to \mathfrak{s}^*$ is infinitesimally equivariant for the action on $H^s(M, \mathbb{C})$ of the semidirect product Lie algebra $\mathfrak{s}$. This result is expected from the symplectomorphism result in Theorem 2.4 since $T^*\mathrm{Dens}^s(M)$ is a coadjoint orbit in $\mathfrak{s}^*$ via $(\rho, [\theta]) \mapsto (\rho\,\mathrm{d}\theta, \rho)$.

4.5. Multi-component Madelung Transform as a Momentum Map

There is a natural generalization of the above approach to the space of wave vector-functions $\Psi \in \mathbb{P}H^s(M, \mathbb{C}^\ell)$, notably rank 1 spinors for which $\ell = 2$. Let $G \subset \mathrm{U}(\ell)$ be a Lie subgroup of the unitary group and consider the semi-direct product $S_{(G)} = \mathrm{Diff}(M) \ltimes H^s(M, G)$ with group multiplication given by

$$(\varphi, g) \cdot (\eta, h) = \big(\varphi \circ \eta, g \cdot (h \circ \varphi^{-1})\big). \tag{23}$$

The corresponding Lie algebra is $\mathfrak{s}_{(\mathfrak{g})} = \mathfrak{X}(M) \ltimes H^s(M, \mathfrak{g})$. We need to define an action of the group $S_{(G)}$ on the subspace of smooth vector-functions.

Definition 4.8. The semidirect product group $S_{(G)} = \mathrm{Diff}(M) \ltimes H^s(M, G)$ *acts on the space* $\mathbb{P}H^s(M, \mathbb{C}^\ell)$ as follows: if $(\varphi, g) \in S_{(G)}$ is a group element where φ is a diffeomorphism, $g \in H^s(M, G)$ is a group-valued function and $\Psi = (\psi_1, \ldots, \psi_\ell)$ is a smooth wave vector-function, then

$$(\varphi, g) \cdot \Psi := \sqrt{|\mathrm{Det}\,(D\varphi^{-1})|}\; g \cdot (\Psi \circ \varphi^{-1}). \tag{24}$$

Observe that this action commutes with multiplication by complex scalars and therefore is well defined on the projective space. Furthermore, this action is Kähler as it preserves both the symplectic and Riemannian structures of $\mathbb{P}H^s(M, \mathbb{C}^\ell)$.

Note also that for $\ell = 2$ and $\ell = 4$ the subgroup $G = \mathrm{SU}(\ell)$ acts by rotation of spinors, which may shed light on hydrodynamic formulations of the Pauli and Dirac equations.

Definition 4.9. The (inverse) *multicomponent Madelung transform* is the map $\mathbf{M}^{(G)} \colon H^s(M, \mathbb{C}^\ell) \to \mathfrak{s}^*_{(\mathfrak{g})}$ defined by

$$\mathbf{M}^{(G)}(\Psi) = \big(m, -2\Pi(\mathrm{i}\boldsymbol{\rho})\big), \tag{25}$$

where $m = 2\,\mathrm{tr}(\mathrm{Im}\,(\bar{\Psi}\,\mathrm{d}\Psi^\top)) = 2\sum_{k=1}^{\ell} \mathrm{Im}\,(\bar{\psi}_k\,\mathrm{d}\psi_k)$, $\boldsymbol{\rho} = \bar{\Psi}\Psi^\top$ is the density matrix and $\Pi \colon \mathfrak{u}(\ell) \to \mathfrak{g}$ is the orthogonal projection with respect to the standard inner product which identifies $\mathfrak{u}(\ell)^*$ with $\mathfrak{u}(\ell)$.

It can be viewed as a momentum map $\mathbf{M}^{(G)} \colon \mathbb{P}H^s(M, \mathbb{C}^\ell) \to \mathfrak{s}^*_{(\mathfrak{g})}$ since both m and $\boldsymbol{\rho}$ are independent of the global phase.

We can now prove a multicomponent version of Theorem 4.7.

Theorem 4.10. *For the Lie algebra* $\mathfrak{s}_{(\mathfrak{g})} = \mathfrak{X}(M) \ltimes H^s(M, \mathfrak{g})$*, its action on* $\mathbb{P}H^s(M, \mathbb{C}^\ell)$ *admits a momentum map. The inverse Madelung transformation* $\mathbf{M}^{(G)} \colon \mathbb{P}H^s(M, \mathbb{C}^\ell) \to \mathfrak{s}^*_{(\mathfrak{g})}$ *defined by* (25) *is, up to scaling by* 4*, a momentum map associated with this Lie algebra action.*

Remark 4.11. For a special case of the subgroup $G = \mathrm{U}(1)^\ell \subset \mathrm{U}(\ell)$ of diagonal unitary matrices, one has the surjective group homomorphism

$$\mathbb{R}^\ell \ni (a_1, \ldots, a_\ell) \mapsto \mathrm{diag}(\mathrm{e}^{-\mathrm{i}a_1/2}, \ldots, \mathrm{e}^{-\mathrm{i}a_\ell/2}) \in G.$$

Thus, the group $S_{(G)}$ descends to $S_{(\ell)} = \mathrm{Diff}(M) \ltimes H^s(M, \mathbb{R}^\ell)$, and the corresponding action on $\Psi = (\psi_1, \ldots \psi_\ell) \in \mathbb{P}H^s(M, \mathbb{C}^\ell)$ is

$$(\varphi, (a_1, \ldots, a_\ell)) \cdot \psi_k = \sqrt{|\mathrm{Det}(D\varphi^{-1})|}\, \mathrm{e}^{-\mathrm{i}a_k/2}(\psi_k \circ \varphi^{-1}).$$

This leads to the *diagonal* multicomponent Madelung transform

$$\mathbf{M}^{(\ell)}(\Psi) = (m, \rho_1, \ldots, \rho_\ell),$$

where m is as before and $\rho_k := \bar{\psi}_k \psi_k$.

From the viewpoint of Hamiltonian dynamics, specifying a larger ℓ (and considering the corresponding semi-direct product groups $S_{(\ell)}$) corresponds to "exploring a larger chunk" of the phase space $T^*\mathrm{Diff}(M) \simeq \mathrm{Diff}(M) \times \mathfrak{X}(M)^*$ (cf. next section). Indeed, for $\ell = 1$ the associated equations on $T^*\mathrm{Dens}^s(M)$ only allow for momenta of the form $m = \rho\, \mathrm{d}\theta$ (corresponding to potential-type solutions of the barotropic Euler equations). Choosing $\ell > 1$ allows for momenta of the form $m = \sum_{k=1}^{\ell} \rho_k\, \mathrm{d}\theta_k$ thus filling out a larger portion of $\mathfrak{X}(M)^*$. The next section is an example of this.

4.6. Example: General (Classical) Compressible Fluids

For general compressible (classical, nonbarotropic) inviscid fluids the equation of state describes the pressure as a pointwise function $P\colon \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ of both density ρ and entropy σ. Thus, the corresponding equations of motion include the evolution of all three quantities: the velocity v of the fluid, its density ρ and the entropy σ:

$$\begin{cases} \dot{v} + \nabla_v v + \frac{1}{\rho}\nabla P(\rho, \sigma) = 0, \\ \dot{\rho} + \mathrm{div}(\rho v) = 0, \\ \dot{\sigma} + \mathcal{L}_v \sigma = 0. \end{cases} \tag{26}$$

In the case of constant entropy or the pressure independent of σ, this system describes a classical barotropic flow, see Example 2.10. Note that, while the density evolves as an n-form, the entropy evolves as a function. However, according to the continuity equation, passing to the entropy density $\varsigma = \sigma\rho$ one can regard the corresponding group as the semidirect product $S_{(2)} = \mathrm{Diff}(M) \ltimes H^s(M, \mathbb{R}^2)$, which leads to a Hamiltonian picture on the dual $\mathfrak{s}^*_{(2)}$. The corresponding Hamiltonian function on $\mathfrak{s}^*_{(2)}$ is then given by

$$H(m, \rho, \varsigma) = \frac{1}{2}\int_M \frac{|m|^2}{\rho}\mu + \int_M e(\rho, \varsigma)\rho\mu.$$

By applying the diagonal multicomponent Madelung transform $\mathbf{M}^{(2)}(\psi_1, \psi_2) = (m, \rho, \varsigma)$ one can rewrite and interpret this system on the space of rank-1 spinors $\mathbb{P}H^s(M, \mathbb{C}^2)$. This yields the (non-quadratic) wave function Hamiltonian

$$\bar{H}(\Psi) = \bar{H}(\psi_1, \psi_2) = H\big(2\,\mathrm{Im}(\bar{\psi}_1 \mathrm{d}\psi_1 + \bar{\psi}_2 \mathrm{d}\psi_2), |\psi_1|^2, |\psi_1|^2\big).$$

The corresponding non-linear Schrödinger equation

$$\mathrm{i}\dot{\psi}_i = \frac{\delta \bar{H}}{\delta \psi_i}$$

is thus a quantum formulation for a classical compressible fluid (although only for 'horizontal' momenta of the form $m = 2\,\mathrm{Im}(\bar{\psi}_1 \mathrm{d}\psi_1 + \bar{\psi}_2 \mathrm{d}\psi_2)$ which, in general, does not yield all solutions). Conversely, one can work backwards to obtain fluid formulations of various quantum-mechanical spin Hamiltonians, such as the Pauli equations for spin $1/2$ particles of a given charge.

4.7. Geometry of Semi-direct Product Reduction

In this section we present the geometric structure behind the semi-direct product reduction which reveals the origin of the Madelung transform as the moment map above.

By Moser's Lemma [16], the quotient $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ is identified with $\mathrm{Dens}(M)$ via the projection $\varphi \mapsto \varphi_*\mu$. The space $\mathrm{Dens}(M)$ itself can be thought of as the μ-orbit of the linear left dual action of $\mathrm{Diff}(M)$ on $H^s(M)^*$ by $\varphi\cdot f = \det(D\varphi^{-1}) f\circ\varphi^{-1}$. We thus have an embedding γ of $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M) \simeq \mathrm{Dens}(M)$ as an orbit in $H^s(M)^*$. Since the action of $\mathrm{Diff}(M)$ on $H^s(M)^*$ comes from the linear left action on $H^s(M)$ given by $\varphi\cdot f = f\circ\varphi^{-1}$, we can construct the semi-direct product $S = \mathrm{Diff}(M) \ltimes H^s(M)$. A Poisson reduction then leads to the following result:

Proposition 4.12. *The quotient $T^*\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ is naturally embedded via a Poisson map in the dual space $\mathfrak{s}^* = \mathfrak{X}^*(M)\ltimes H^s(M)$ equipped with the Lie-Poisson structure. This embedding is given by*

$$([\varphi], m) \mapsto (m, \varphi_*\mu), \tag{27}$$

where one uses the right translation to identify

$$T^*\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M) \simeq \Big(\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)\Big) \times \mathfrak{X}^*(M).$$

We now return to the standard symplectic reduction (without semi-direct products). The dual $\mathfrak{X}_\mu(M)^*$ of the subalgebra $\mathfrak{X}_\mu(M) \subset \mathfrak{X}(M)$ is naturally identified with affine cosets of $\mathfrak{X}^*(M)$ such that

$$m \in [m_0] \iff \langle m - m_0, \xi\rangle = 0 \quad \forall\, \xi \in \mathfrak{X}_\mu(M). \tag{28}$$

The momentum map of the subgroup $\mathrm{Diff}_\mu(M)$ acting on $\mathfrak{X}^*(M)$ is then given by $m \mapsto [m]$. If $\big\langle m, \mathfrak{X}_\mu(M)\big\rangle = 0$, that is $m \in (\mathfrak{X}(M)/\mathfrak{X}_\mu(M))^*$, then $m \in [0]$ is in the zero momentum coset. Since we also have

$$T^*(\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)) \simeq \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M) \times \big(\mathfrak{X}(M)/\mathfrak{X}_\mu(M)\big)^*,$$

this gives us, by Moser's Lemma, an embedding of $T^*\mathrm{Dens}(M)$ as a symplectic leaf in $T^*\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M) \simeq \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M) \times \mathfrak{X}^*(M)$. The restriction to this leaf is called *zero-momentum symplectic reduction*.

Turning to the semi-direct product reduction, we now have Poisson embeddings of $T^*\mathrm{Dens}(M)$ in $T^*\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ and of $T^*\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$ in $\mathfrak{s}^*$. The combined embedding of $T^*\mathrm{Dens}(M)$ as a symplectic leaf in $\mathfrak{s}^*$ is given by the map

$$(\rho, \theta) \mapsto (\rho\,\mathrm{d}\theta, \rho). \tag{29}$$

This gives a Hamiltonian action of S (or $\mathfrak{s}$) on the zero-momentum symplectic leaf $T^*\mathrm{Dens}(M)$ sitting inside $T^*\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$, which in turn sits inside $\mathfrak{s}^*$.

Since the group S has a natural symplectic action on its dual Lie algebra $\mathfrak{s}^*$ and since $\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M) \simeq \mathrm{Dens}(M)$ is an orbit in $H^s(M)$, we have, by restriction, a natural action of S on $T^*\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$. Furthermore, since the momentum map associated with S acting on $\mathfrak{s}^*$ is the identity, the Poisson embedding map (27) is the momentum map for S acting on $T^*\mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$. Thus, the momentum map of the group S acting on $T^*\mathrm{Dens}(M)$ is given by (29).

This considerations are summarized in the following theorem:

Theorem 4.13. *The inverse of the Madelung transform viewed as a momentum map (Section 4.4) can be regarded as the semi-direct product reduction and the Poisson embedding $T^*\mathrm{Dens}(M) \to \mathfrak{s}^*$ described above.*

This result explains Fusca's [8] observation that the inverse Madelung transform can be interpreted as a momentum map.

Acknowledgements. Boris Khesin is grateful to the IHES in Bures-sur-Yvette and the Weizmann Institute in Rehovot for their support and kind hospitality. Boris Khesin was also partially supported by an NSERC research grant. Part of this work was done while Gerard Misiołek held the Ulam Chair visiting Professorship in University of Colorado at Boulder. Klas Modin was supported by EU Horizon 2020 Grant No. 691070, by the Swedish Foundation for International Cooperation in Research and Higher Eduction (STINT) Grant No. PT2014-5823, and by the Swedish Research Council (VR) Grant No. 2017-05040.

Appendix A: The Functional-Analytic Setting

The infinite-dimensional geometric constructions in this paper are rigorously carried out in any reasonable function space setting in which the topology is at least as strong as C^1, satisfies the functorial axioms of Palais [19] and admits a Hodge decomposition. The choice of the Sobolev spaces is very convenient for the purposes of this paper because many of the technical details which were used (explicitly or implicitly) in the proofs can be readily traced in the literature. We briefly review the main points below.

As introduced in the main body of the paper the notation $\mathrm{Diff}^s(M)$ stands for the completion of the group of smooth C^∞ diffeomorphisms of an n-dimensional compact Riemannian manifold M with respect to the H^s topology where $s > n/2 + 1$. This puts the Sobolev lemma at our disposal and thus equipped $\mathrm{Diff}^s(M)$ becomes a smooth Hilbert manifold (when $s < \infty$) whose tangent space at the identity $T_e\mathrm{Diff}^s(M)$ consists of all H^s vector fields on M, see for example [6], Section 2.

Using the implicit function theorem the subgroup $\mathrm{Diff}^s_\mu(M) = \{\eta \in \mathrm{Diff}^s(M) : \eta^*\mu = \mu\}$ consisting of those diffeomorphisms that preserve the Riemannian volume form μ can then be shown to inherit the structure of a smooth Hilbert submanifold with $T_e\mathrm{Diff}^s_\mu(M) = \{v \in T_e\mathrm{Diff}^s : \operatorname{div} v = 0\}$, cf. for example [6], Sections 4 and 8.

Standard results on compositions and products of Sobolev functions ensure that both Diff^s and Diff^s_μ are topological groups with right translations $\xi \to \xi \circ \eta$

(resp., left translations $\xi \to \eta \circ \xi$ and inversions $\xi \to \xi^{-1}$) being smooth (resp., continuous) as maps in the H^s topology, cf. [19], Chapters 4 and 9. Furthermore, the natural projection

$$\pi : \mathrm{Diff}^{s+1}(M) \to \mathrm{Diff}^{s+1}(M)/\mathrm{Diff}^{s+1}_{\mu}(M) \simeq \mathrm{Dens}^s(M)$$

given by $\eta \to \pi(\eta) = \eta^*\mu$ extends to a smooth submersion between $\mathrm{Diff}^{s+1}(M)$ and the space of right cosets, which can be identified with the space of probability densities on M of Sobolev class H^s (cf. Section 2 above). The metrics and symplectic structures discussed in this paper are based on the L^2 pairing and are "weak" with respect to the H^s topology. More technical details, as well as proofs of all these facts, can be found in [6, 19] and their bibliographies.

References

1. Arnold, V.I.: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)* **16**, 319–361, 1966
2. Arnold, V.I., Khesin, B.A.: *Topological Methods in Hydrodynamics*, vol. 125. Applied Mathematical Sciences, Springer, New York 1998
3. Bauer, M., Bruveris, M., Michor, P.W.: Uniqueness of the Fisher–Rao metric on the space of smooth densities. *Bull. Lond. Math. Soc*. **48**, 499–506, 2016
4. Bush, J.W.: Quantum mechanics writ large. *Proc. Natl. Acad. Sci. USA* **107**, 17455–17456, 2010
5. Couder, Y., Protiere, S., Fort, E., Boudaoud, A.: Dynamical phenomena: walking and orbiting droplets. *Nature* **437**, 208, 2005
6. Ebin, D.G., Marsden, J.E.: Groups of diffeomorphisms and the notion of an incompressible fluid. *Ann. Math*. **92**, 102–163, 1970
7. Foskett, M.S., Holm, D.D., Tronci, C.: Geometry of nonadiabatic quantum hydrodynamics. arXiv:1807.01031 2018
8. Fusca, D.: The Madelung transform as a momentum map. *J. Geom. Mech*. **9**, 157–165, 2017
9. Hasimoto, H.: A soliton on a vortex filament. *J. Fluid Mech*. **51**, 477–485, 1972
10. Khesin, B., Lenells, J., Misiołek, G., Preston, S.C.: Geometry of diffeomorphism groups, complete integrability and geometric statistics. *Geom. Funct. Anal*. **23**, 334–366, 2013
11. Khesin, B., Misiolek, G., Modin, K.: Geometric hydrodynamics via Madelung transform. *Proc. Natl. Acad. Sci. USA* **115**, 6165–6170, 2018
12. Kibble, T.W.B.: Geometrization of quantum mechanics. *Commun. Math. Phys*. **65**, 189–201, 1979
13. Lenells, J.: Spheres, Kähler geometry and the Hunter-Saxton system. *Proc. R. Soc. A Math. Phys. Eng. Sci*. **469**, 20120726, 2013
14. Madelung, E.: Quantentheorie in hydrodynamischer form. *Z. Phys*. **40**, 322–326, 1927
15. Molitor, M.: On the relation between geometrical quantum mechanics and information geometry. *J. Geom. Mech*. **7**, 169–202, 2015
16. Moser, J.: On the volume elements on a manifold. *Trans. Am. Math. Soc*. **120**, 286–294, 1965
17. Nelson, E.: Review of stochastic mechanics. *J. Phys. Conf. Ser*. **361**, 012011, 2012
18. Otto, F.: The geometry of dissipative evolution equations: the porous medium equation. *Commun. Partial Differ. Equ*. **26**, 101–174, 2001
19. Palais, R.S.: *Foundations of Global Non-linear Analysis*. Benjamin, New York 1968
20. Shashikanth, B.N.: Vortex dynamics in $\mathbb{R}^4$. *J. Math. Phys*. **53**, 013103, 2012

21. von Renesse, M.-K.: An optimal transport view of Schrödinger's equation. *Can. Math. Bull*. **55**, 858–869, 2012
22. Wu, H., Wunsch, M.: Global existence for the generalized two-component Hunter–Saxton system. *J. Math. Fluid Mech*. **14**, 455–469, 2011

Boris Khesin
Department of Mathematics,
University of Toronto,
Toronto
ON
M5S 2E4 Canada.
e-mail: khesin@math.toronto.edu

and

Gerard Misiołek
Department of Mathematics,
University of Notre Dame,
Notre Dame
IN
46556 USA.
e-mail: gmisiole@nd.edu

and

Klas Modin
Department of Mathematical Sciences,
Chalmers University of Technology and University of Gothenburg,
412 96 Gothenburg
Sweden.
e-mail: klas.modin@chalmers.se

(*Received July 20, 2018 / Accepted May 4, 2019*)
Published online May 16, 2019