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A Poisson–Lie Framework for Rational Reductions of the KP Hierarchy

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Abstract. We give a simple proof of I. Krichever's theorem on rational reductions of the Kadomtsev–Petviashvili hierarchy by using the Poisson–Lie structure on the group of pseudo-differential symbols.

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In this Letter we give a simple proof of the Krichever theorem on rational Kadomtsev–Petviashvili equations. In [Kr], Krichever showed that the standard KP hierarchy admits new reductions to the subspaces of *rational* pseudo-differential symbols, i.e., those pseudo-differential symbols that are ratios of two pure differential operators.

We show that the existence of such reductions is immediate within the Poisson–Lie framework for pseudo-differential symbols developed in [KZ, EKRRR]. We also discuss an analogue of the rational reductions in the Lie group of q-difference operators, as well as the corresponding Kac–Moody counterpart (see [PSTS, KM]).

1. Krichever's Theorem

Consider the space of rational pseudo-differential symbols on the circle, i.e., the space of all pseudo-differential symbols of the form $L_1^{-1}L_2$, where L_1 and L_2 are mutually prime monic (i.e., with leading coefficient 1) differential operators on the circle of degree m and $n+m$ respectively.

Let $\mathcal{K}_{m,n}$ be the space of pseudo-differential symbols of the form $M = D + v_0(z)D^0 + v_1(z)D^{-1} + \cdots$, such that the nth power of M is rational: $M^n = L_1^{-1}L_2$.

Now we consider the KP hierarchy

$$\frac{\partial M}{\partial t_k} = [M_+^k, M], \qquad \text{where } k = 2, 3, \ldots, \tag{1}$$

for such $M \in \mathcal{K}_{m,n}$. Here the index + means taking the pure differential part of the corresponding pseudo-differential symbol.

THEOREM 1 ([Kr]). *The KP hierarchy is well-defined on the space $\mathcal{K}_{m,n}$, i.e., this space is invariant for the equations above for any k.*

As an example of this formalism, it is shown in [Kr] that the nonlinear Schrödinger equation can be viewed as a rational KP reduction to the submanifold $\mathcal{K}_{1,2}$.

2. The Lie Group of Pseudodifferential Symbols

In this and the next sections we describe the group of pseudodifferential symbols of real degrees following [KZ] (see also [EKRRR]). It turns out to be a Poisson–Lie group, equipped with a Poisson structure given by the quadratic Gelfand–Dickey bracket.

Consider the set of all monic pseudo-differential symbols of arbitrary degree, i.e., formal Laurent series of the following type:

$$\mathbf{G} = \bigcup_{\lambda \in \mathbb{R}} \mathbf{G}_\lambda, \; \mathbf{G}_\lambda = \left\{ L \mid L = D^\lambda + \sum_{k=-\infty}^{-1} u_k(z) D^{\lambda+k} \right\},$$

where $u_k \in C^\infty(S^1)$. Given λ, the expression for L is to be understood as a conveniently written form for a semi-infinite sequence of functions $\{u_k\}$ on the circle. The number λ is called the degree (or order) of symbol L.

This infinite-dimensional manifold $\mathbf{G}$ can be equipped with a group structure. The product of two such symbols is defined according to the Leibniz rule

$$D \circ f(z) = f(z) D + f'(z)$$

(which explains the meaning of the symbol $D = \mathrm{d}/\mathrm{d}z$). Namely, for an arbitrary (real) power of D one has:

$$D^\lambda \circ f(z) = f(z) D^\lambda + \sum_{\ell \geqslant 1} \binom{\lambda}{\ell} f^{(\ell)}(z) D^{\lambda-\ell},$$

where

$$\binom{\lambda}{\ell} = \frac{\lambda(\lambda-1)\cdots(\lambda-\ell+1)}{\ell!}.$$

It is easy to see that each coefficient of the product of two symbols is a differential polynomial in coefficients of the factors. We call $\mathbf{G}$ the group of classical pseudo-differential symbols, and it will be the phase space for the integrable systems below.

3. The Poisson Structure on the Group

The *quadratic Gelfand–Dickey Poisson structure* on the group $\mathbf{G} = \bigcup_{\lambda\in\mathbb{R}} \mathbf{G}_\lambda$ is defined as follows:

(a) The degree function λ is its Casimir function, i.e., the hyperplanes $\mathbf{G}_\lambda$ are Poisson submanifolds for this Poisson structure. In other words, the Poisson bracket of two functions at a given point is determined by the function restrictions to the subset $\mathbf{G}_\lambda$ of symbols of fixed degree $\lambda = \text{const}$.

(b) The subset $\lambda = \text{const}$ is an affine space, so we can identify the tangent space to this subset

$$\mathbf{G}_\lambda = \left\{ L \mid L = \left(1 + \sum_{k=-\infty}^{-1} u_k(z) D^k \right) \circ D^\lambda \right\}$$

with the set of operators of the form $\delta L = (\sum_{k=-\infty}^{-1} \delta u_k D^k) \circ D^\lambda$.

We can also identify the cotangent space with the space of operators of the form $X = D^{-\lambda} \circ DO$, where DO is a pure differential (i.e., polynomial in D) operator using the following pairing:

$$F_X(\delta L) := \langle X, \delta L \rangle = \mathrm{Tr}(\delta L \circ X).$$

Here the product $\delta L \circ X$ is a symbol $\sum p_k(z) D^k$ of an integer degree, and its trace Tr is defined as the integral of $p_{-1}(z)dz$ over the circle.

(c) Now it is sufficient to define the bracket on linear functionals, and

$$\{F_X, F_Y\}\big|_L := F_Y(V_{F_X}(L)),$$

where V_{F_X} is the following Hamiltonian mapping $F_X \longmapsto V_{F_X}(L)$ (from the cotangent space $\{X\}$ to the tangent space $\{\delta L\}$):

$$V_{F_X}(L) = (LX)_+ \, L - L(XL)_+ \; .$$

THEOREM 2 ([KZ]). *The group* $\mathbf{G}$ *is a Poisson–Lie group, whose Poisson structure is the quadratic Gelfand–Dickey structure.*

Recall that by definition of a Poisson–Lie group, the group product $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ is a Poisson morphism, i.e., it takes the natural Poisson structure on the product $\mathbf{G} \times \mathbf{G}$ into the Poisson structure on $\mathbf{G}$ itself. Also, the map $\mathbf{G} \to \mathbf{G}$ of taking the group inverse is an anti-Poisson morphism, as it changes the sign of the Poisson bracket.

4. Integrable Hierarchies on the Group

Consider the following family of Hamiltonian functions $\{H_k\}$ on $\mathbf{G}$ parametrized by an integer parameter $k = 1, 2, \ldots$: the value of H_k at the pseudo-differential symbol

$L \in \mathbf{G}$ of degree λ is

$$H_k(L) := \frac{\lambda}{k}\,\mathrm{Tr}(L^{k/\lambda}).$$

Here any real power of L is a uniquely defined element in $\mathbf{G}$, since the group $\mathbf{G}$ is quasi-unipotent and its exponential map is one-to-one [KZ].

It is well known (and easy to see from the formulas above) that the corresponding Hamiltonian equations with respect to the quadratic Gelfand–Dickey Poisson structure form the following system:

$$\frac{\partial L}{\partial t_k} = [(L^{k/\lambda})_+, L], \quad k = 1, 2, \ldots \tag{2}$$

This is an infinite system of commuting flows on the coefficients of L, $\deg L = \lambda$. (Here '+' means taking the differential part of a symbol of integral degree.) For $\lambda = 1$ it is the standard Kadomtsev–Petviashvili hierarchy of commuting flows. We call the above general system on $\mathbf{G}$ the *universal KP hierarchy*.

Note, that a slight difference with the classical case is that these equations are considered on $\mathbf{G}_\lambda$ not only for integral λ, but for any real λ, or, in other words, on the whole group $\mathbf{G}$. However, below we are going to use only rational powers of pseudodifferential operators, which constitute a rather classical framework.

5. Poisson Submanifolds

The classical definition of the Gelfand–Dickey (or Adler–Gelfand–Dickey) structure is given in the case when λ is a fixed positive integer n and L is a differential operator, cf. [Ad, Di].

One can see that the set $\mathcal{L}_n = \{L | L_+ = L\}$ of pure differential operators is a Poisson submanifold in the Poisson 'hyperplane' $\mathbf{G}_n$ of all monic pseudo-differential symbols of the same degree n. Indeed, for any operator $L = D^n + u_{-1}(z)D^{n-1} + \cdots + u_{-n}(z)$ and an arbitrary symbol $X = DO \circ D^{-n}$ the corresponding Hamiltonian vector $V_{F_X}(L) = (LX)_+\, L - L(XL)_+$ is a differential operator of the order $n-1$ and, hence, all Hamiltonian fields keep the submanifold $\mathcal{L}_n$ of such monic differential operators of order n invariant.

The restriction of the universal hierarchy (2) to the Poisson submanifolds $\mathcal{L}_n$ gives the so-called n-KdV hierarchy

$$\frac{\partial L}{\partial t_k} = [(L^{k/n})_+, L], \quad k = 1, 2, \ldots \tag{3}$$

Thus the universal KP hierarchy interpolates between the standard KP one (for $\lambda = 1$) and the n-KdV flows (for $\lambda = n$), see [EKRRR, KZ].

6. More Poisson Submanifolds: Krichever's Theorem

Now one can reformulate the theorem of Krichever as follows:

THEOREM 3. *The submanifolds of rational pseudo-differential symbols are Poisson submanifolds in* **G**. *In particular, any Hamiltonian system on the group (with respect to the group Poisson–Lie structure) leaves those submanifolds invariant.*

Proof. The properties of a Poisson–Lie group imply that the products of all group elements belonging to two given Poisson submanifolds form again a Poisson submanifold in such a group. Similarly, the inverses of points of a Poisson submanifold form a Poisson submanifold in a Poisson–Lie group.

Apply this general observation to the submanifolds $\mathcal{L}_n \subset \mathbf{G}$. As we discussed above, the submanifold $\mathcal{L}_n$ of purely differential operators of order n is Poisson for any n. Then the submanifold $\mathcal{L}_n^{-1}$ of the inverses of differential operators of order n is also Poisson, as well as the products $\mathcal{L}_n^{-1}\mathcal{L}_m$ for any n and m. □

COROLLARY 4 (= Theorem 1). *The KP hierarchy (1) is well-defined on the sets* $\mathcal{K}_{n,m}$ *of* nth *roots of rational pseudo-differential symbols.*

Indeed, the hierarchy (3) is well-defined on the set of rational pseudo-differential symbols. The KP hierarchy (1) on $M \in \mathcal{K}_{n,m}$ is obtained from (3) by taking the nth root of the symbols: $M = L^{1/n}$, and the latter operation is uniquely defined for every group element. □

Remark 1. An argument similar to the one employed in Theorem 3 above was used in the proof of Proposition 3 in [EOR] in the study of analogues of Benney's equations and the Poisson structures associated with them. We would also like to mention the related work on symmetries of various KP-type hierarchies (see [AGNP] and references therein). In particular, in the latter paper the authors show how to recover the Davey–Stewartson system from constrained KP hierarchies. It would be very interesting to find its Lie–Poisson meaning, as the Davey–Stewartson and nonlinear Schrödinger equations are related in a way similar to the KP and KdV systems.

7. *Q*-Analogues and Discussion

Many facts in the group and Poisson approaches have a q-difference counterpart, following [PSTS].

First, note that instead of pseudo-differential symbols on the circle one can consider those symbols on $\mathbb{C}^*$, i.e., their coefficients can be assumed to be Laurent polynomials in z. The latter model is more suitable for introducing q-analogues of the symbols.

Remark 2. Denote by D_q the shift operator on functions: $D_q f(z) = f(qz)$. Then its commutation relation with the operator of multiplication by a function is the following: $D_q \circ f = (D_q f) D_q$, very similar to the Leibniz rule above. Now one can construct the group of q-pseudo-difference symbols:

$$\mathbf{G}_q = \bigcup_{\lambda \in \mathbb{C}} \mathbf{G}_{q,\lambda}, \ \mathbf{G}_{q,\lambda} = \left\{ L \mid L = D_q^{\lambda} + \sum_{k=-\infty}^{-1} u_k(z) D_q^{\lambda+k} \right\},$$

see [PSTS] (or [KLR], where a different basis for the symbols was used). It turns out that this group carries a Poisson structure (though not a Poisson–Lie one). Assume q to be generic. The corresponding Hamiltonian equations have the same form (2), and can be restricted to (the Poisson submanifold of) 'pure q-difference operators,' i.e., to those, containing only non-negative powers of D_q, see [PSTS].

The q-analogue of rational pseudo-differential symbols is given by the following Proposition 3.14 of [PSTS]: the submanifolds of $\mathbf{G}_q$ consisting of the operators

$$\left\{ L = D_q^{\lambda} + u_{-1}(z) D_q^{\lambda-1} + \cdots u_{-n}(z) D_q^{\lambda-n} \right\}$$

are Poisson (and hence the equations can be restricted to them).

Remark 3. The Poisson submanifolds $\mathcal{L}_n \subset \mathbf{G}$ (with the quadratic Gelfand–Dickey bracket on them) of pure differential operators arise as a result of Hamiltonian reduction from the dual spaces to affine $\widehat{\mathfrak{gl}}_n$-algebras in the classical Drinfeld–Sokolov construction. (The corresponding Poisson algebras of functions on $\mathcal{L}_n$ are also called the classical W_n-algebras.) It would be interesting to find an affine analogue for the Poisson submanifolds $\mathcal{L}_n^{-1}\mathcal{L}_m \subset \mathbf{G}$ of rational pseudo-differential symbols (as well as their q-analogues) in the spirit of the universal Drinfeld–Sokolov reduction from the affine algebra $\widehat{\mathfrak{gl}}_{\lambda}$ (for any real or complex λ), cf. [KM, PSTS2].

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