Available online at www.sciencedirect.com
SCIENCE DIRECT

Advances in Mathematics 176 (2003) 116–144

ADVANCES IN Mathematics

http://www.elsevier.com/locate/aim

Euler equations on homogeneous spaces and Virasoro orbits

Boris Khesin[a,*] and Gerard Misiołek[b]

[a] Department of Mathematics, University of Toronto, Toronto, Ont., M5S 3G3 Canada
[b] Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA

Received 20 July 2001; accepted 22 March 2002
Communicated by Pavel Etingof

Abstract

We show that the following three systems related to various hydrodynamical approximations: the Korteweg–de Vries equation, the Camassa–Holm equation, and the Hunter–Saxton equation, have the same symmetry group and similar bihamiltonian structures. It turns out that their configuration space is the Virasoro group and all three dynamical systems can be regarded as equations of the geodesic flow associated to different right-invariant metrics on this group or on appropriate homogeneous spaces. In particular, we describe how Arnold's approach to the Euler equations as geodesic flows of one-sided invariant metrics extends from Lie groups to homogeneous spaces.

We also show that the above three cases describe all generic bihamiltonian systems which are related to the Virasoro group and can be integrated by the translation argument principle: they correspond precisely to the three different types of generic Virasoro orbits. Finally, we discuss interrelation between the above metrics and Kahler structures on Virasoro orbits as well as open questions regarding integrable systems corresponding to a finer classification of the orbits.

MSC: 37K65; 53D25

Keywords: Euler equations; Geodesic flows; Bi-Hamiltonian structures; Virasoro orbits

*Corresponding author.
E-mail addresses: khesin@math.toronto.edu (B. Khesin), misiolek.1@nd.edu (G. Misiołek).

0001-8708/03/$ - see front matter
doi:10.1016/S0001-8708(02)00063-4

1. Introduction

One of the main mechanisms of integrability of evolution equations is the presence of two compatible Hamiltonian structures. In this paper, we compare Hamiltonian properties of three extensively studied nonlinear equations of mathematical physics, related to various hydrodynamical approximations: the Korteweg–de Vries equation

$$u_t = -3uu_x + cu_{xxx}, \tag{1.1}$$

the Camassa–Holm equation

$$u_t - u_{txx} = -3uu_x + 2u_xu_{xx} + uu_{xxx} + cu_{xxx} \tag{1.2}$$

derived as a shallow water equation in [CH] (see also the paper [FF]), and the Hunter–Saxon equation [HS]

$$u_{txx} = -2u_xu_{xx} - uu_{xxx}, \tag{1.3}$$

describing weakly nonlinear unidirectional waves. All three equations are known to be bihamiltonian and to possess infinitely many conserved quantities, as well as remarkable soliton or soliton-like solutions. A motivation for our paper was the paper [BSS], which described scattering theory for all three equations in a unified way.

As we show in this paper, the main reason why such a common treatment is possible is that all these equations have the same symmetry group. It turns out to be the Virasoro group, a one-dimensional extension of the group of smooth transformations of the circle. More precisely, the Virasoro group serves as the configuration space, and all three equations can be regarded as equations of the geodesic flow related to different right-invariant metrics on this group (in the case of KdV and CH) or on an associated homogeneous space (in the case of HS). (Here, we will be mostly concerned with the periodic case, though many statements can be extended to the case of rapidly decaying potentials on the real line.)

The main goal of this paper is to give a description of the three equations as *bihamiltonian* systems on the dual to the Virasoro algebra, and to relate them to the geometry of the Virasoro coadjoint orbits. One of the corresponding Hamiltonian (or Poisson) structures is provided by the linear Lie–Poisson bracket, and it is the same for all three equations. The other Poisson structure is constant and can be viewed as a "linear structure frozen at a point". The corresponding "freezing points" are different for each equation. We will also see that, in a sense, Eqs. (1.1)–(1.3) exhaust all generic possibilities, and among them the Camassa–Holm equation (1.2) is the "most general" equation that can be obtained by the "freezing argument" method on the dual Virasoro space.

We tried to make the paper self-contained, including in it necessary background on the Euler equations and the classification of Virasoro orbits. For additional information we refer the interested reader to the expositions in [Seg] or [AK], as well as to the original papers listed in the bibliography.

In [Arn] Arnold suggested a general framework for the Euler equations on an arbitrary (possibly infinite-dimensional) group, which we recall below. In this framework the Euler equation describes a geodesic flow with respect to a suitable one-sided invariant Riemannian metric on the given group.

In Section 2 we show how Arnold's approach to the Euler equation works for the Virasoro group and provides a natural geometric setting for the Korteweg–de Vries and Camassa–Holm equations. In Section 3 we give a Hamiltonian reformulation of the Euler equation. In Section 4 we extend this approach to include geodesic flows on homogeneous spaces and then use it to describe the Hunter–Saxton equation and its relatives.[1] This extension might be thought of as a version of the Hamiltonian formalism for homogeneous spaces developed in [GS,Th], which is applied to the case of a degenerate metric and in our infinite-dimensional situation.

In Sections 5 and 6 we develop the bihamiltonian formalism for the Euler systems. We show why the three equations above represent three main classes (and exhaust the rotation-invariant) bihamiltonian systems on the Virasoro algebra that can be integrated by means of the "freezing" (called also, the translation of) argument method. It turns out that the above three equations correspond to three different types of Virasoro coadjoint orbits of low codimensions.

In our classification of Poisson pairs and the corresponding equations we rely heavily on the classification of the Virasoro orbits, and we recall it in the appendix. An interesting open question is to extend the classification of the equations to orbits of higher codimension, as well as to show how the discrete invariant of Virasoro orbits manifests itself in the related bihamiltonian systems.

2. The Euler properties of the KdV, CH, and HS equations

The main objects in our consideration will be the group $Diff(S^1)$ of all diffeomorphisms of a circle, its Lie algebra $vect(S^1)$ of vector fields, and their central extensions. The following nontrivial one-dimensional extension of the algebra of vector fields has a special name.

Definition 2.1. The *Virasoro algebra* is an extension of the Lie algebra $vect(S^1)$ of vector fields on the circle:

$$vir = vect(S^1) \oplus \mathbb{R}$$

with the commutator between pairs (consisting of a vector field and a real number) given by

$$[(v(x)\partial_x, b), (w(x)\partial_x, c)] = ((-vw_x + v_x w)(x)\partial_x, C(v\partial_x, w\partial_x)),$$

[1] In particular, the Harry Dym equation [HZh,Kru] can be found as one of the equations in the bihamiltonian hierarchy associated with the HS system. This equation was also considered in [BSS]. In this way, the Harry Dym equation also becomes associated to the geodesic interpretation.

where

$$C(v\partial_x, w\partial_x) = \int_{S^1} v w_{xxx}\, dx$$

is the *Gelfand–Fuchs cocycle*. (Here x is a coordinate on the circle, the subscript x stands for the derivative in x, and ∂_x denotes the vector field $\frac{\partial}{\partial x}$ on S^1.)

Define the following two-parameter family of quadratic forms, "$H^1_{\alpha,\beta}$-energies", on the Lie algebra *vir*:

$$\langle (v(x)\partial_x, b), (w(x)\partial_x, c) \rangle_{H^1_{\alpha,\beta}} = \int_{S^1} (\alpha v w + \beta v_x w_x)\, dx + bc. \tag{2.4}$$

The case $\alpha = 1$, $\beta = 0$ corresponds to the L^2 inner product, while $\alpha = \beta = 1$ corresponds to H^1. Given $\alpha \neq 0$ and any β, extend the $H^1_{\alpha,\beta}$-energy to a right-invariant metric on the Virasoro group *Vir*. This group corresponds to the Virasoro algebra and is defined as follows.

Definition 2.2. The *Virasoro group Vir* is the product

$$Vir = Diff(S^1) \times \mathbb{R},$$

where the group multiplication between the pairs is given by

$$(\psi(x), a) \circ (\phi(x), b) = ((\psi \circ \phi)(x), a + b + B(\psi, \phi))$$

and

$$B(\psi, \phi) := \int_{S^1} \log((\psi \circ \phi)_x)\, d \log(\phi_x)$$

is the *Bott cocycle*. (Here the diffeomorphisms of S^1 are described by functions, e.g., $x \mapsto \phi(x)$.)

Having equipped the Virasoro group with those right-invariant metrics, one can consider the geodesic flows they generate.

Theorem 2.3. (1) [OK] *The KdV equation is the Euler equation, describing the geodesic flow on the Virasoro group with respect to the right-invariant L^2-metric.*

(2) [Mi] *The CH equation is the Euler equation for the geodesic flow on the same group with respect to the right-invariant Sobolev H^1-metric.*

It turns out that one can give a similar description of the Hunter–Saxton equation as a geodesic flow on a *homogeneous space* related to the Virasoro algebra. Consider the $\dot{H}^1$-quadratic form (which is the $H^1_{\alpha,\beta}$-form with $\alpha = 0, \beta = 1$) on the

Virasoro algebra:

$$\langle (v(x)\partial_x, b), (w(x)\partial_x, c) \rangle_{\dot{H}^1} = \int_{S^1} v_x w_x \, dx + bc. \tag{2.5}$$

Although this form is degenerate, as is the corresponding right-invariant metric on the Virasoro group, one can define a nondegenerate metric by descending on an appropriate quotient space.

Theorem 2.4. *The HS equation is the equation describing the geodesic flow on the homogeneous space $Vir/Rot(S^1)$ of the Virasoro group modulo rotations with respect to the right-invariant homogeneous $\dot{H}^1$ metric.*

Note that one can also obtain the HS equation by considering the smaller homogeneous space $Diff(S^1)/Rot(S^1)$ of all diffeomorphisms of the circle modulo rotations, see the end of Section 4.

These three equations essentially exhaust the list of integrable systems associated with the Virasoro algebra and integrated by the freezing argument method, as we discuss below. Note that their degenerations include, e.g., the inviscid Burgers equation (it corresponds to the L^2-metric on the "centerless Virasoro" group, $Diff(S^1)$).

Remark 2.5. Before proving the theorems, we recall the general set-up for the Euler equation on an arbitrary Lie group, suggested by Arnold [Arn]. Consider a (possibly infinite-dimensional) Lie group G, which can be thought of as the configuration space of some physical system. (Examples from [Arn]: $SO(3)$ for a rigid body or the group $SDiff(M)$ of volume-preserving diffeomorphisms for an ideal fluid filling a domain M.) The tangent space at the identity of the Lie group G is the corresponding Lie algebra $\mathfrak{g}$. Fix some (positive definite) quadratic form, the "energy", on $\mathfrak{g}$. We consider right translations of this quadratic form to the tangent space at any point of the group (the "translational symmetry" of the energy). This way the energy defines a right-invariant Riemannian metric on the group G. The geodesic flow on G with respect to this energy metric represents the extremals of the least action principle, i.e. the actual motions of our physical system.[2]

To describe a geodesic on the *Lie group* with an initial velocity $v(0)$, we transport its velocity vector at any moment t to the identity of the group (by using the right translation). In this way we obtain the evolution law for $v(t)$, given by a (nonlinear) dynamical system $dv/dt = F(v)$ on the *Lie algebra* $\mathfrak{g}$.

Definition 2.6. The system on the Lie algebra $\mathfrak{g}$, describing the evolution of the velocity vector along a geodesic in a right-invariant metric on the Lie group G, is called the *Euler equation* corresponding to this metric on G.

[2] For a rigid body one has to consider left translations, but in our exposition we stick to the right-invariant case in view of its applications to the groups of diffeomorphisms.

In particular, the above scheme works for the Virasoro group (see Theorem 2.3) and allows one to describe the Korteweg–de Vries and Camassa–Holm equations as geodesic equations on that group. It also can be extended to include geodesic flows on homogeneous spaces and to describe the Hunter–Saxton equation, as we discuss below.

3. Hamiltonian framework for the Euler equations

We start with preliminaries on Lie algebras and Poisson structures.

Definition 3.1. The dual space $\mathfrak{g}^*$ to any Lie algebra $\mathfrak{g}$ carries a natural *Lie–Poisson structure*:

$$\{f,g\}_{\mathrm{LP}}(m) \coloneqq \langle [df,dg],m \rangle$$

for any $m \in \mathfrak{g}^*$ and any two smooth functions f,g on $\mathfrak{g}^*$. (Here the differentials are taken at the point m, and $\langle \cdot,\cdot \rangle$ is a natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.)

In other words, the Lie–Poisson bracket of two linear functions on $\mathfrak{g}^*$ is equal to their commutator as elements of the Lie algebra $\mathfrak{g}$ itself.

Proposition 3.2. *The Hamiltonian vector field on $\mathfrak{g}^*$ corresponding to a Hamiltonian function f and computed with respect to the Lie–Poisson structure has the following form*:

$$\frac{dm}{dt} = ad^*_{df}m. \tag{3.6}$$

Proof. Let $dm/dt = X_f$ be the corresponding Hamiltonian field. Then for any function $g \in C^\infty(\mathfrak{g}^*)$ one has the identities

$$i_{X_f}dg|_m = L_{X_f}g|_m = \{f,g\}_{\mathrm{LP}}(m) = \langle [df,dg],m \rangle = \langle dg, ad^*_{df}m \rangle.$$

This implies that $X_f = ad^*_{df}m$. $\square$

Remark 3.3. The differential-geometric description of the Euler equation as a geodesic flow on a Lie group has a Hamiltonian reformulation.

Fix the notation $E(v) = \frac{1}{2}\langle v, Av \rangle$ for the energy quadratic form on $\mathfrak{g}$ which we used to define the Riemannian metric. Identify the Lie algebra and its dual with the help of this quadratic form. This identification $A : \mathfrak{g} \to \mathfrak{g}^*$ (called the *inertia operator*) allows one to rewrite the Euler equation on the dual space $\mathfrak{g}^*$, see Fig. 1.

It turns out that the Euler equation on $\mathfrak{g}^*$ is Hamiltonian with respect to the Lie–Poisson structure [Arn]. Moreover, the corresponding Hamiltonian function is minus the energy quadratic form lifted from the Lie algebra to its dual space by the

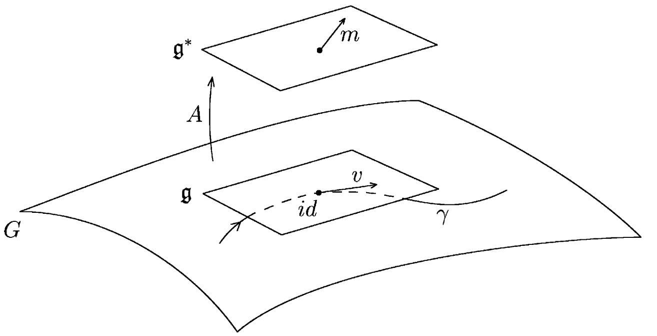

Fig. 1. The vector v in the Lie algebra $\mathfrak{g}$ traces the evolution of the velocity vector of a geodesic γ on the group. The inertia operator A sends v to a vector m in the dual space $\mathfrak{g}^*$.

same identification: $-H(m) = -\frac{1}{2}\langle A^{-1}m, m\rangle$, where $m = Av$. Here, we are going to take it as the *definition* of the Euler equation (we use the Proposition above and the observation $dH(m) = A^{-1}m$).

Definition 3.4. The *Euler equation on* $\mathfrak{g}^*$ corresponding to the Hamiltonian $-H(m) = -\frac{1}{2}\langle A^{-1}m, m\rangle$ is given by the following explicit formula:

$$\frac{dm}{dt} = -\mathrm{ad}^*_{A^{-1}m} m$$

as an evolution of a point $m \in \mathfrak{g}^*$.

Remark 3.5. The underlying reason for the Hamiltonian reformulation is the fact that any geodesic problem in *Riemannian* geometry can be described in terms of *symplectic* geometry. Geodesics on M are extremals of a quadratic Lagrangian (metric) on TM. They can also be described by the Hamiltonian flow on T^*M for the quadratic Hamiltonian function obtained from the Lagrangian via the Legendre transform.

If the manifold is a group G with a right-invariant metric then there exists the group action on the tangent bundle TG, as well as on the cotangent bundle T^*G. By taking the quotient with respect to the group action, we obtain from the (symplectic) cotangent bundle T^*G the Lie–Poisson structure on the cotangent space $T^*G|_e = \mathfrak{g}^*$, i.e., on the dual to the Lie algebra. The Hamiltonian function on T^*G is dual to the Riemannian metric (viewed as a form on TG), and its restriction to $\mathfrak{g}^*$ is the quadratic form $H(m) = \frac{1}{2}\langle A^{-1}m, m\rangle$, $m \in \mathfrak{g}^*$.

The geodesics of a left-invariant metric on G correspond to the Hamiltonian function $H(m)$, while those of a right-invariant metric correspond to $-H(m)$.

Now we are ready to prove Theorem 2.3 on the Eulerian nature of the KdV and CH equations in the following slightly more general setting.

Theorem 3.6 ($= 2.3'$). *The Euler equation describing the geodesic flow on the Virasoro group with respect to the right-invariant $H^1_{\alpha,\beta}$-metric with $\alpha \neq 0$ has the form*

$$\alpha(v_t + 3vv_x) - \beta(v_{xxt} + 2v_x v_{xx} + vv_{xxx}) - bv_{xxx} = 0, \tag{3.7}$$

$$b_t = 0.$$

Remark 3.7. By choosing $\alpha = 1, \beta = 0$ one obtains the KdV equation, related to the L^2-metric on the Virasoro algebra [OK]. Similarly, for $\alpha = \beta = 1$ one recovers a general form of the CH equation [Mi]. Note that by shifting $v \mapsto v + \text{const.}$ we get another form of the CH equation, in which the term v_{xxx} is replaced by v_x. Finally, if $\alpha = 0, \beta = 1$ then Eq. (3.7) becomes the HS (Hunter–Saxton) equation, which we discuss in the next section.

The case $b = 0$ corresponds to considering the nonextended Lie algebra $vect(S^1)$ of vector fields rather than the Virasoro algebra vir. Depending on the values of α and β one obtains either the inviscid Burgers (also called, Hopf) equation $v_t + 3vv_x = 0$ or the nonextended CH equation.

Proof of Theorem 3.6. Recall that the Virasoro coadjoint action can be computed as follows. Let $\{(u(dx)^2, a) \mid u \in C^\infty(S),\ a \in \mathbb{R}\}$ be the dual space to the Virasoro algebra with the natural pairing given by

$$\langle (u(dx)^2, a), (w\partial_x, c) \rangle = \int_{S^1} uw\, dx + ac.$$

(Here we denote by $u(x)(dx)^2$ (or, shorter, by $u(dx)^2$) a quadratic differential on the circle.) The coadjoint operator is defined by the identity

$$\langle \mathrm{ad}^*_{(v\partial_x, b)}(u(dx)^2, a), (w\partial_x, c) \rangle = \langle (u(dx)^2, a), [(v\partial_x, b), (w\partial_x, c)] \rangle.$$

Using the definition of the Virasoro commutator and integrating by parts we obtain that the right-hand side is equal to

$$\int_{S^1} w(2uv_x + u_x v - av_{xxx})\, dx.$$

Thus the coadjoint operator is

$$\mathrm{ad}^*_{(v\partial_x, b)}(u(dx)^2, a) = ((2uv_x + u_x v - av_{xxx})(dx)^2, 0). \tag{3.8}$$

The $H^1_{\alpha,\beta}$-energy (2.4) on the Virasoro algebra

$$\langle (v\partial_x, b), (w\partial_x, c) \rangle = \int_{S^1} (\alpha vw + \beta v_x w_x)\, dx + bc = \int_{S^1} v\Lambda w\, dx + bc.$$

corresponds to the general inertia operator $A : vir \to vir^*$, given by

$$(v\partial_x, b) \mapsto ((\Lambda v)(dx)^2, b),$$

where $\Lambda := \alpha - \beta\partial_x^2$ is a second-order differential operator. This operator is nondegenerate on vir for $\alpha \neq 0$, while for $\alpha = 0$ it has a nontrivial kernel consisting of constant vector fields on S^1.

Now the Euler equation

$$\frac{d}{dt}(u(dx)^2, a) = -\mathrm{ad}^*_{A^{-1}(u(dx)^2, a)}(u(dx)^2, a)$$

on vir^* (see Definition 3.4) assumes the form

$$\frac{d}{dt}(u(dx)^2, a) = -((2u\Lambda^{-1}u_x + u_x\Lambda^{-1}u - a\Lambda^{-1}u_{xxx})(dx)^2, 0).$$

(Here we substituted $(v\partial_x, b) = A^{-1}(u(dx)^2, a) = ((\Lambda^{-1}u)(dx)^2, a)$ into the expression for ad^*.)

In terms of $v = \Lambda^{-1}u$ (the first component of) this equation becomes

$$\frac{d}{dt}(\Lambda v) = -2(\Lambda v)v_x - (\Lambda v_x)v + bv_{xxx},$$

which is equivalent to Eq. (3.7), since $\Lambda = \alpha - \beta\partial_x^2$. For the second component we find that b does not change in time: $b_t = 0$. □

Remark 3.8. In the proof we assumed that the inertia operator is invertible. In the next section we discuss the precise relation of the geodesic and Hamiltonian approaches in the case of a degenerate metric and show what reductions are necessary for the corresponding Euler equation to make sense.

4. The Euler equations on homogeneous spaces

Let G be a Lie group and K its subgroup. Consider the space G/K of *right* cosets $\{Kg \mid g \in G\}$. Then the group G acts on them on the right. Here, we are going to develop the formalism for the Euler equation, describing the geodesic flow on G/K with respect to a right-invariant metric.

One immediately encounters the following difficulty: not every *right-invariant* metric on the group G, degenerate along K at the identity, descends to a metric on the space of right cosets G/K (see Example 4.3a). To formulate the condition, which the degenerate metric should satisfy, let us consider the corresponding problem at the level of Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra, and $A : \mathfrak{g} \to \mathfrak{g}^*$ a degenerate inertia operator. Suppose that the kernel of A is a Lie subalgebra $\mathfrak{k}$. (In other words, the corresponding energy form $E(v) = \frac{1}{2}\langle v, Av\rangle$ vanishes for all $v \in \mathfrak{k}$.) Consider the right-invariant degenerate metric E_G on the group G obtained by translating the quadratic form E from identity to any point of the group.

Theorem 4.1. *The right-invariant form E_G on a group G descends to a form on the space G/K of right cosets if and only if the quadratic form E on the Lie algebra $\mathfrak{g}$ vanishes on the Lie subalgebra $\mathfrak{k}$ and is Ad-invariant with respect to the action of this subalgebra.*

Remark 4.2. The condition of *Ad*-invariance for E reads as follows:

$$\langle ad_w v, Au \rangle = -\langle v, A(ad_w u) \rangle$$

for all $w \in \mathfrak{k}$ and any $u, v \in \mathfrak{g}$. The above is an infinitesimal version of the invariance of E with respect to the subgroup action:

$$\langle (Ad_k v), A(Ad_k u) \rangle = \langle v, Au \rangle$$

for all $k \in K$.

Example 4.3. (a) (*Rotations of a rod*). The configuration space of a rod in $\mathbb{R}^3$ fixed at its center of mass is S^2. It can be obtained from the configuration space of a rigid body by moding out rotations about one of its axes: $S^2 = S^1 \diagdown SO(3)$.

Suppose that $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a degenerate inertia operator with one vanishing eigenvalue. The corresponding eigenvector generates the one-dimensional rotation subgroup S^1 in $SO(3)$. It is not difficult to see that the bi-invariance condition imposes the following restriction on A: its nonvanishing eigenvalues must be equal. (Indeed, the S^1-action sends one of these two eigenvectors to the other.) Then the corresponding degenerate metric on $SO(3)$ descends to (a multiple of) the standard metric on the sphere, which is the space of cosets: $S^2 = S^1 \diagdown SO(3)$. Geodesics with respect to the standard metric on S^2 are the great circles. These geodesics describe all free motions of the rod.[3] (We can see that the only parameter of the rod is its length. In terms of the inertia operator, this corresponds to the choice of the nonzero eigenvalue.)

Note that the inertia operator $A = diag(\lambda, \mu, 0)$ with $\lambda \neq \mu$ does not correspond to any physical object. The corresponding degenerate metric on $SO(3)$ is not S^1-invariant, and hence it does not descend to S^2.

(b) (The Hunter–Saxton equation). Consider the group of diffeomorphisms $Diff(S^1)$ and its quotient $Diff(S^1)/Rot(S^1)$ by the subgroup of rotations $Rot(S^1)$. Consider the degenerate quadratic form on the corresponding Lie algebra $vect(S^1)$

[3] For the rod, as well as for a rigid body mentioned above, we consider the left-invariant metrics, and hence, left cosets $S^1 \diagdown SO(3)$. For the group of diffeomorphisms, we study the right-invariant metrics and the space of right cosets $Diff(S^1)/Rot(S^1)$.

given by the homogeneous $\dot{H}^1$-energy

$$E_0(v\partial_x) = \frac{1}{2}\int_{S^1} v_x^2 \, dx.$$

(Similarly, for the Virasoro algebra, consider the energy $E_0(v\partial_x, b) = \frac{1}{2}(\int_{S^1} v_x^2 \, dx + b^2)$.)

This energy vanishes on constant vector fields. Those fields generate the subgroup $Rot(S^1)$ of rotations of the circle S^1. One can see that the form $E_0(v\partial_x)$ is bi-invariant with respect to the circle action, since the energy is invariant with respect to translations $x \mapsto x + const$. (The same holds for the Gelfand–Fuchs cocycle, and the energy $E_0(v\partial_x, b)$ on the extended algebra.) Hence the corresponding right-invariant metric on $Diff(S^1)$ descends to the quotient $Diff(S^1)/Rot(S^1)$. We will see that the geodesics with respect to this metric are described by the HS equation.

Remark 4.4. One hopes that a certain modification of this approach can be applied to generalized flows in [Bre], where the fluid particles in 3D can move freely and independently along one coordinate. The corresponding subgroup K here might be that of fiberwise diffeomorphisms along a coordinate.

Proof of Theorem 4.1. First of all we note that the quadratic form E on $\mathfrak{g}$ induced from a nondegenerate form on the quotient $\mathfrak{g}/\mathfrak{k}$ is degenerate exactly along $\mathfrak{k}$ (i.e., $\mathfrak{k}$ is its null subspace).

Let $\mathfrak{k}$ be a Lie subalgebra and consider the quotient of the corresponding groups G/K. Note that the restriction of the energy form to the subgroup K is zero. Indeed, the latter is nothing but the right translation from the identity of the energy form on the subalgebra $\mathfrak{k}$.

We would like to compare the right-invariant metric E_G on G at two different points $k_1 g$ and $k_2 g$ of the same coset Kg. Then the element $\bar{k} = k_2 k_1^{-1} \in K$ sends k_1 to k_2 by means of the left translation. This translation also identifies the tangent spaces to G along the same coset Kg, see Fig. 2. The energy form E_G is invariant under this identification, since it is bi-invariant with respect to the action of elements of K. Finally, note that the energy E_G is degenerate along cosets. (Indeed, it vanishes on K, the "identity coset", and it is invariant with respect to right translations, which shuffle the cosets.)

Therefore, the corresponding energy form descends to the coset space. □

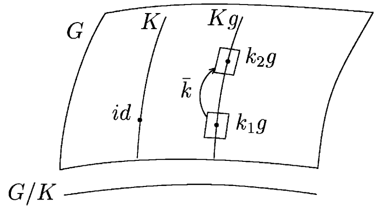

Fig. 2. Defining a right-invariant form on the space of right cosets G/K.

Remark 4.5. From the Hamiltonian point of view, the geodesic picture on a homogeneous space corresponds to a Hamiltonian reduction of the nondegenerate case with respect to the subgroup K action.

More precisely, consider the cotangent bundle $T^*(G/K)$ of the metric space G/K. Similarly to the nondegenerate case, look at the fiber $(\mathfrak{g}/\mathfrak{k})^*$ over the "identity coset" K. This space $(\mathfrak{g}/\mathfrak{k})^*$ can be naturally identified with the image $L = im\, A \subset \mathfrak{g}^*$ of the degenerate inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$, for which $\mathfrak{k} = ker A$, (or, equivalently, with the subspace $L = \mathfrak{k}^{\perp} \subset \mathfrak{g}^*$, the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$).

The subgroup K, being a stabilizer of the "identity coset"and hence of $L = (\mathfrak{g}/\mathfrak{k})^*$, acts on L respecting the Poisson structure. Therefore, there is a natural Poisson structure on the quotient $L/K \coloneqq im\, A/ad^*K$.

Furthermore, one can define the corresponding Hamiltonian function on L by the same formula as above:

$$H_L(m) = \tfrac{1}{2}\langle m, A^{-1}m \rangle. \tag{4.9}$$

This function is ad^*K-invariant and hence is well-defined on the quotient L/K.

Suppose A generates a right-invariant and ad$K$-invariant metric on the group G (i.e., satisfies the conditions of Theorem 4.1), so that it makes sense to consider geodesics of the corresponding metric on G/K.

Then the above consideration provides the following limiting (degenerate) case of Arnold's theorem (cf. Definition 3.4).

Theorem 4.6. *The Euler equation, which corresponds to the inertia operator A and describes the geodesic flow on the space G/K of right cosets, has the following Hamiltonian form on $L/K = im\, A/ad^*K$: it is the quotient with respect to the K-action of the restriction to $L \subset \mathfrak{g}^*$ of the following Hamiltonian equation on $\mathfrak{g}^*$:*

$$\frac{dm}{dt} = -ad^*_{A^{-1}m} m$$

for $m \in L = im\, A$.

Now we are ready to complete the argument showing the Eulerian nature of the Hunter–Saxton equation.

Theorem 4.7 $(= 2.4')$. *The HS equation*

$$v_{txx} = -2v_x v_{xx} - v v_{xxx}$$

is a well-defined equation on the equivalence classes of periodic functions

$$\{v(x) \sim v(x+p) + q \text{ for any } p, q\}.$$

It describes the geodesic flow on the homogeneous space $Vir/Rot(S^1)$ of the Virasoro group modulo rotations with respect to the right-invariant homogeneous $\dot{H}^1$ metric.

Proof. The homogeneous $\dot{H}^1$-metric on the Virasoro algebra is related to the degenerate inertia operator $A : vir \to vir^*$ sending

$$(v\partial_x, b) \mapsto (-(\partial_x^2 v)(dx)^2, b).$$

Its image $L \subset vir^*$ consists of pairs $(u(dx)^2, a)$, where functions u have zero mean:

$$L = \left\{ (u(dx)^2, a) \Big| \int_{S^1} u(x)\, dx = 0 \right\}.$$

The action of the subgroup $K = Rot(S^1)$ identifies those functions u that differ by a rotation: $u(x) \sim u(x+p)$. Thus, we come to a Hamiltonian equation on L/K.

For explicit calculations, recall that the present case corresponds to setting $\alpha = 0$, $\beta = -1$ in the proof of Theorem 3.6. Choosing these values in Eq. (3.7), one arrives at the HS equation

$$v_{txx} = -2v_x v_{xx} - v v_{xxx} - b v_{xxx}$$

on *vir*. In order to obtain equation describing the evolution of v rather than that of v_{xx} we observe that

$$2v_x v_{xx} + v v_{xxx} + b v_{xxx} = \left(v v_{xx} + \tfrac{1}{2}(v_x)^2 + b v_{xx}\right)_x$$

and hence, integrating both sides in x, we obtain

$$v_{tx} = -v v_{xx} - \tfrac{1}{2}(v_x)^2 - b v_{xx} + r,$$

where r is an arbitrary constant. This constant is uniquely determined by the condition that the right-hand side is a complete derivative, i.e., by $\int_{S^1} ((v_x)^2/2 + r_0)\, dx = 0$. Then

$$v_t = -v v_x + \partial_x^{-1}((v_x)^2/2 + r_0) - b v_x + q,$$

where q is an arbitrary constant.

Whence v_t (and hence the evolution of v) is defined only up to addition of a parameter q and a multiple of v_x. (The latter is the velocity of the rotation subgroup: $v_x = \frac{d}{dp}|_{p=0} v(x+p)$.) This manifests the fact that the evolution of v is defined on the equivalence classes $\{v(x+p)+q\}$. The inertia operator A sends this equation on classes to the Hamiltonian equation on the quotient L/K.

Note that the equivalence classes above absorb the term bv_x (respectively, bv_{xxx} for v_{txx}: consider a shift $v \mapsto v + \text{const.}$) and one obtains the HS equation in its standard form (1.3). □

One should notice that setting $b = 0$ corresponds to the Euler equation on the quotient $Diff(S^1)/Rot(S^1)$. Thus in the homogeneous case the consideration of the central extension does not give anything new, since the Euler equations for $Diff(S^1)/Rot(S^1)$ and $Vir/Rot(S^1)$ are equivalent.

5. Bihamiltonian structures for the equations

To formulate our next result, we need to recall some generalities on bihamiltonian systems.

Definition 5.1. Assume that a manifold M is equipped with two Poisson structures $\{.,.\}_0$ and $\{.,.\}_1$. They are said to *be compatible* (or, form a *Poisson pair*) if all of their linear combinations $\{.,.\}_0 + \lambda\{.,.\}_1$ are also Poisson structures.

A dynamical system $dm/dt = F(m)$ on M is called *bi-Hamiltonian* if the vector field F is Hamiltonian with respect to both structures $\{.,.\}_0$ and $\{.,.\}_1$.

Consider the dual space $\mathfrak{g}^*$ to a Lie algebra $\mathfrak{g}$. As we discussed above, it is equipped with the *Lie–Poisson structure*:

$$\{f,g\}_{\mathrm{LP}}(m) \coloneqq \langle [df,dg], m \rangle,$$

where $m \in \mathfrak{g}^*$ and f,g are two arbitrary functions on $\mathfrak{g}^*$.

Now fix a point m_0 in $\mathfrak{g}^*$. One can associate to this point another Poisson bracket on $\mathfrak{g}^*$ as follows, see [Ma].

Definition 5.2. The *constant Poisson bracket* associated to a point $m_0 \in \mathfrak{g}^*$ is the bracket $\{.,.\}_0$ on the dual space $\mathfrak{g}^*$ defined by

$$\{f,g\}_0(m) \coloneqq \langle [df,dg], m_0 \rangle$$

for any two smooth functions f,g on the dual space, and any $m \in \mathfrak{g}^*$. The differentials df, dg of the functions f,g are taken, as above, at the point m and are regarded as elements of the Lie algebra itself.

The constant bracket depends on the choice of the "freezing" point m_0, while the Lie–Poisson bracket is defined by the Lie algebra structure only. Note that the brackets $\{.,.\}_{\mathrm{LP}}$ and $\{.,.\}_0$ coincide at the point m_0 itself, and, moreover, the bivector defining the constant bracket $\{.,.\}_0$ is the same at all points m.

Proposition 5.3. *The brackets $\{.,.\}_{\mathrm{LP}}$ and $\{.,.\}_0$ are compatible for every "freezing" point m_0.*

Proof. Indeed, any linear combination $\{.,.\}_\lambda \coloneqq \{.,.\}_{\mathrm{LP}} + \lambda\{.,.\}_0$ is again a Poisson bracket, since it is just the linear Lie–Poisson structure $\{.,.\}_{\mathrm{LP}}$ translated from the origin to the point $-\lambda m_0$. $\square$

Remark 5.4. Explicitly, the Hamiltonian equation on $\mathfrak{g}^*$ with the Hamiltonian function f and computed with respect to the constant Poisson structure frozen at a point $m_0 \in \mathfrak{g}^*$ has the following form:

$$\frac{dm}{dt} = ad^*_{df} m_0, \tag{5.10}$$

as a modification of Proposition 3.2 shows.

Now we can formulate another main result.

Theorem 5.5. *The Euler equation* (3.7) *for the* $H^1_{\alpha,\beta}$*-metric* (*with* $\alpha \neq 0$) *on the Virasoro group is bihamiltonian on the dual* vir^* *of the Virasoro algebra. The corresponding "freezing" point in* vir^* *is* $(\frac{\alpha}{2}(dx)^2, \beta)$.

In the appendix, we show that the dual space $vir^* = \{(u(x)(dx)^2, a)\}$ can be thought of as the space of Hill's operators $\{-a\partial_x^2 + u(x)\}$. In these terms the above theorem can be stated as follows: The $H^1_{\alpha,\beta}$-metric on vir given by the inertia operator $\Lambda = \alpha - \beta\partial_x^2$ is bihamiltonian on vir^* with "freezing" at the point $\alpha/2 - \beta\partial_x^2$.

Remark 5.6. The KdV and CH equations are bihamiltonian on the Virasoro dual. The corresponding "freezing" points $(u_0(dx)^2, a_0)$ in vir^* are $(u_0 = 1/2, a_0 = 1)$ for the CH equation and $(u_0 = 1/2, a_0 = 0)$ for the KdV equation, see Fig. 3, as they are related to the H^1- and L^2-energies, respectively.

To describe bihamiltonian nature of the HS equation on the reduced space L/K, discussed in Theorems 4.6 and 4.7, one should consider the following analog of the constant Poisson structure. Take the Lie–Poisson structure on vir^* "frozen" at the point $(u_0 = 0, a_0 = 1)$ and then push it forward to the corresponding quotient space for the HS equation. An alternative way to show integrability (rather than the bihamiltonian property) of this equation is to use integrability of CH and an infinite-dimensional version of the formalism developed in [GS,Th].

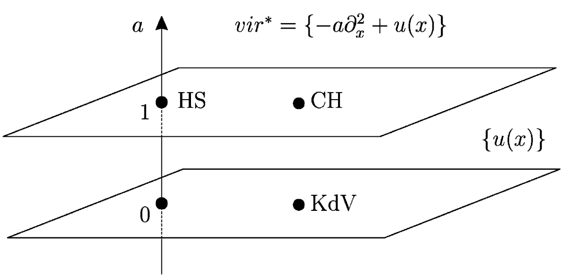

Fig. 3. Locations of the "freezing" points for the KdV, CH, and HS equations in the Virasoro dual vir^*.

Question 5.7. Which metrics on the Virasoro group (or which quadratic forms on the Virasoro algebra) correspond to the bihamiltonian system on *vir** with "freezing" at a point $(u_0(x)(dx)^2, a_0)$ for nonconstant $u_0(x)$?[4] For which $u_0(x)$ are these metrics positive definite (i.e., Riemannian rather than pseudo-Riemannian)?

Proof of Theorem 5.5. Let $F(u,a)$ be a function on *vir** and let $(v\partial_x, b) = (\delta F/\delta u, \delta F/\delta a)$ be a (variational) derivative of F at $(u(dx)^2, a)$. Then the Hamiltonian equation with Hamiltonian function F computed with respect to the constant Poisson structure "frozen" at $(u_0(dx)^2, a_0)$ has the form

$$\frac{d}{dt}(u(dx)^2, a) = \mathrm{ad}^*_{(v\partial_x, b)}(u_0(dx)^2, a_0) = ((2u_0 v_x - a_0 v_{xxx})(dx)^2, 0).$$

(Here we used Remark 5.4, the explicit form (3.8) of *ad** for the Virasoro algebra, and the fact that $u_0 = \mathrm{const}$.)

For the first component of this equation one has

$$\frac{du}{dt} = (2u_0 - a_0\partial_x^2)\partial_x\left(\frac{\delta F}{\delta u}\right).$$

Setting $u_0 = \alpha/2$ and $a_0 = \beta$ we obtain $2u_0 - a_0\partial_x^2 = \alpha - \beta\partial_x^2 = \Lambda$, and this simplifies the equation to

$$\frac{du}{dt} = \Lambda\left(\partial_x\left(\frac{\delta F}{\delta u}\right)\right). \tag{5.11}$$

To prove the theorem one needs to show that for any $\alpha \neq 0$ and any β the Euler equation (3.7) can also be expressed in form (5.11) for an appropriate Hamiltonian function F.

Next, consider the Hamiltonian function F of the form

$$F(u,a) = -\int\left(\frac{\alpha}{3}(\Lambda^{-1}u)^3 + \frac{1}{4}(\Lambda^{-1}u)^2 u + \frac{a}{2}(\Lambda^{-1}u_x)^2\right)dx.$$

(The operator $\Lambda \coloneqq \alpha - \beta\partial_x^2$ is invertible for $\alpha \neq 0$.) By definition, the variational derivative $(\delta F/\delta u, \delta F/\delta a) \in vir$ of the functional F is determined by the following identity satisfied for any $(\xi(dx)^2, c) \in vir^*$:

$$\left\langle (\xi(dx)^2, c), \left(\frac{\delta F}{\delta u}, \frac{\delta F}{\delta a}\right)\right\rangle = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F(u + \varepsilon\xi, a + \varepsilon c).$$

[4] After this paper was submitted, Zakharevich found a formula for the corresponding quadratic form on the Lie algebra in terms of Bloch solutions of the operator $-\partial_x^2 + u(x)$ [Za].

Since we need only the partial variational derivative $\delta F/\delta u$, we compute:

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon\xi, a)\\
&\quad = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int\left(\frac{\alpha}{3}(\Lambda^{-1}(u+\varepsilon\xi))^3+\frac{1}{4}(\Lambda^{-1}(u+\varepsilon\xi))^2(u+\varepsilon\xi)+\frac{a}{2}(\Lambda^{-1}(u+\varepsilon\xi)_x)^2\right)dx\\
&\quad = -\int \xi\cdot\left(\alpha\Lambda^{-1}(\Lambda^{-1}u)^2+\frac{1}{4}(\Lambda^{-1}u)^2+\frac{1}{2}\Lambda^{-1}((\Lambda^{-1}u)u)-a\Lambda^{-2}u_{xx}\right)dx.
\end{aligned}$$

Thus, we have found that

$$\frac{\delta F}{\delta u}=-\left(\alpha\Lambda^{-1}(\Lambda^{-1}u)^2+\frac{1}{4}(\Lambda^{-1}u)^2+\frac{1}{2}\Lambda^{-1}((\Lambda^{-1}u)u)-a\Lambda^{-2}u_{xx}\right)\partial_x.$$

Now, we substitute the variational derivative $\delta F/\delta u$ into Eq. (5.11) and then rewrite the obtained equation on the algebra *vir*, rather than on its dual *vir**. The latter corresponds to rewriting the equation in terms of $(v\partial_x, b)=\Lambda^{-1}(u(dx)^2, a)$, i.e., in terms of an unknown function $v=\Lambda^{-1}u$ and setting $b=a$.

Finally, applying Λ to the equation we obtain

$$\Lambda\left(\frac{dv}{dt}\right)=-\left(2\alpha vv_x+\frac{1}{2}\Lambda(vv_x)+\frac{1}{2}v\Lambda v_x+\frac{1}{2}v_x\Lambda v-bv_{xxx}\right).$$

Recalling that $\Lambda=\alpha-\beta\partial_x^2$ and collecting the terms we recover Eq. (3.7). □

To explain in what sense the above equations are generic bihamiltonian systems on *vir** (obtained by the freezing argument method), we need to consider symplectic leaves of the above Poisson structures.

6. Hierarchies of Hamiltonians from compatible structures

Recall that the symplectic leaves, i.e., maximal nondegenerate submanifolds, of the Lie–Poisson structure are the coadjoint orbits of the group action on $\mathfrak{g}^*$ (see, e.g., [Kir] or this also follows from Proposition 3.2). Therefore the functions constant on symplectic leaves (called *Casimir functions*) of the Lie–Poisson bracket are those functions on the dual space $\mathfrak{g}^*$ that are invariant under the coadjoint action. The tangent plane to the group coadjoint orbit at the point m_0, as well as all the planes in $\mathfrak{g}^*$ parallel to this tangent plane are the symplectic leaves of the constant bracket frozen at the point m_0 .

Definition 6.1. The codimension of the coadjoint orbit passing through m_0 will be called the *codimension of the Poisson pair* $\{.,.\}_0$ and $\{.,.\}_{\mathrm{LP}}$.

It turns out that there are no Poisson pairs of codimension 0 or 1 in the (smooth) Virasoro dual vir^*, and that the Poisson pairs of codimension 2 can all be classified.

Theorem 6.2. *All Poisson pairs* $\{.,.\}_0$ *and* $\{.,.\}_{LP}$ *on* vir^* *of codimension* 2 *belong to one of three classes according to the orbit type of the "freezing" point* $(u_0(dx)^2, a_0)$. *These classes can be represented by the points* (a) $((dx)^2/2, 1)$, (b) $((dx)^2/2, 0)$, *and* (c) $(0, 1)$.

Proof. First we observe that for any Lie algebra $\mathfrak{g}$ the list of coadjoint orbits in $\mathfrak{g}^*$ provides the list of normal forms for the constant and Lie–Poisson pairs as well. Indeed, let a point $\bar{m}_0 \in \mathfrak{g}^*$ be a normal form for all points m_0 belonging to the same coadjoint orbit as $\bar{m}_0$. This means that m_0 can be mapped by the group coadjoint action to its normal form: $\bar{m}_0 = Ad_g^* m_0$. The group action $Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear operator on $\mathfrak{g}^*$, which preserves the Lie–Poisson bracket on $\mathfrak{g}^*$. It also maps the constant bracket frozen at m_0 to that frozen at $\bar{m}_0$. Thus the group action Ad_g^* sends one Poisson pair to the other.

The proof of Theorem 6.2 is based on the Virasoro orbit classification, which we recall in the appendix. Notice that a "cocentral value" a_0 is invariant on the orbits in vir^*. This allows us to fix a_0 and consider the orbits in the hyperplane $\{(u(x)(dx)^2, a) \mid a = a_0\}$.

As shown in Corollary A.8 of the appendix, for $a_0 \neq 0$ there are exactly two types of Virasoro orbits of codimension 1 in this hyperplane that correspond to either a generic or a Jordan 2×2 block holonomy for Hill's operator $-a_0\partial_x^2 + u(x)$. If we discard the discrete invariant of these orbits (see the appendix), representatives for their normal forms can be chosen as stated in (a) and (b) parts of the theorem.

If $a_0 = 0$ there is just one orbit type of codimension 1 in the corresponding hyperplane, and represented by (c), see Remark A.2. Note that in the whole dual space those orbits have codimension 2, taking extra dimension a into account. □

Given a Poisson pair one can generate a bihamiltonian dynamical system, by producing a sequence of Hamiltonians in involution, according to the following *Lenard scheme*, see [Mag]. Let $\{.,.\}_\lambda := \{.,.\}_0 + \lambda\{.,.\}_1$ be the Poisson bracket on a manifold M for any λ. Denote by h_λ its Casimir function on M parameterized by λ. This means that $\{h_\lambda, f\}_\lambda = 0$ for any function f. Expand h_λ in a power series: $h_\lambda = h_0 + h_1\lambda + \cdots$, where each coefficient h_j is a function on M. The following theorem is well-known.

Theorem 6.3. *The functions* h_j, $j = 1, 2, \ldots$ *are Hamiltonians of a hierarchy of bihamiltonian systems. In other words, each function* h_j *generates the Hamiltonian field* X_j *with respect to the Poisson bracket* $\{.,.\}_1$ (*i.e.*, X_j *satisfies* $L_{X_j} f = \{h_j, f\}_1$ *for any* f), *which is also Hamiltonian for the other bracket* $\{.,.\}_0$ *with Hamiltonian function* $-h_{j+1}$ (*i.e.*, $L_{X_j} f = -\{h_{j+1}, f\}_0$ *for any* f). *Other functions* h_i, $i \neq j$ *are first integrals of the corresponding dynamical systems* X_j.

In other words, the functions $h_j,\ j = 0, 1, \ldots$ are in involution with respect to each of the two Poisson brackets $\{.,.\}_0$ and $\{.,.\}_1$.

Proof. Substituting the power series for h_λ into the Casimir condition we obtain:

$$0 = \{h_\lambda, f\}_\lambda = \{h_0 + h_1\lambda + \cdots, f\}_0 + \lambda\{h_0 + h_1\lambda + \cdots, f\}_1.$$

Collecting the terms at $\lambda^0, \lambda^1, \lambda^2, \ldots$ we obtain a sequence of identities:

$$\{h_0, f\}_0 = 0, \quad \{h_1, f\}_0 + \{h_0, f\}_1 = 0, \quad \{h_2, f\}_0 + \{h_1, f\}_1 = 0, \ldots$$

for any function f. The first identity expresses the fact that h_0 is a Casimir function for the bracket $\{.,.\}_0$. The next one says that the Hamiltonian field for h_1 with respect to $\{.,.\}_0$ coincides with the Hamiltonian field for $-h_0$ and the bracket $\{.,.\}_1$, and so on.

To see that every function h_i is a first integral for the equation generated by h_j with respect to each bracket, we check that $\{h_i, h_j\}_k = 0, k = 0, 1$. Indeed, e.g., if $i < j$

$$\{h_i, h_j\}_1 = -\{h_{i+1}, h_j\}_0 = \{h_{i+1}, h_{j-1}\}_1 = \cdots = 0,$$

since we finally obtain the bracket (either $\{.,.\}_0$ or $\{.,.\}_1$) of one of the functions h_l with itself. $\square$

Thus the choice of Casimir functions h_λ determines the corresponding (hierarchy of) dynamical systems. By combining Theorems 6.2 and 6.3 we get the following:

Corollary 6.4. *The three types of Poisson pairs of codimension* 2 *on the Virasoro algebra correspond to the three integrable systems*: *CH*, *KdV*, *and HS. These three systems represent all generic Hamiltonian systems on vir** (*modulo the ambiguity in the choice of Casimir*), *which can be integrated by the freezing argument method.*

Remark 6.5. The corresponding "freezing" points in *vir** represent all three types of the Virasoro coadjoint orbits of codimension 2.

If the "freezing" point $(u_0(dx)^2, a_0)$ is generic, one obtains an equation "equivalent" to the CH equation. In this sense, the CH equation is the most general equation, which is encountered by applying the freezing argument method of integration; this is case (a) in Theorem 6.2.

Two other equations can be recovered by confining the "freezing" point to special hypersurfaces in *vir**. (In turn, these hypersurfaces are foliated into coadjoint orbits. Those orbits are of codimension 1 in the hypersurfaces, and hence of total codimension 2 in *vir**. The classification of the orbits will be discussed in detail in Appendix.) A generic point on the hyperplane $a_0 = 0$ produces the KdV equation; see case (b) in Theorem 6.2. Case (c) in the same theorem corresponds to the HS equation if we consider the cone-like Virasoro orbits in the $a_0 \neq 0$-case (see the appendix and Fig. 4). The latter (e.g., "freezing" at the point $(u_0(dx)^2, a_0) = (0, 1)$)

corresponds to the Euler equation with a degenerate metric on the group, which we discussed in Section 4.

One could also consider a more subtle Virasoro orbit classification, where one distinguishes between the two types of generic orbits in *vir**: hyperbolic and elliptic ones, according to the eigenvalues of the monodromy, as well as between the orbits which differ by the discrete invariant (see Corollary A.8). All elliptic orbits with arbitrary values of the discrete invariant can be represented by Hill's operators with *constant coefficients*. However, just one of the hyperbolic and one of the Jordan block classes has such representatives, while others do not.

Note that the bihamiltonian equations corresponding to elliptic orbits with different discrete invariants are almost the same: these are the CH equations with different coefficients. It would be very interesting to see whether an analogous similarity holds for the hyperbolic and Jordan block orbits with different discrete invariants.

Remark 6.6. When the symplectic leaves of the bracket $\{.,.\}_\lambda$ are of codimension 1, then the choice of a Casimir function is essentially unique for every λ. (Any two Casimir functions for every fixed λ are functionally dependent.) Therefore, the choice of the Poisson pair itself defines the bihamiltonian system (modulo the mentioned functional dependence of the initial Hamiltonian), provided that the symplectic leaves are hypersurfaces.

This is indeed the case for the Virasoro coadjoint orbits discussed above, which have codimension 1 for a fixed cocentral value a. It turns out that a natural Casimir function $h_\lambda(u(x)(dx)^2, a)$ corresponding to the KdV Poisson pair on the dual space to the Virasoro algebra is the trace of the monodromy operator associated to Hill's operator $-a\partial_x^2 + u(x) - \lambda^2$. It generates the first integrals of the KdV equation, see Remark A.11 in the appendix. Similarly, one can expand Casimir functions for the two other integrable cases, the CH and HS equations.

Note that for orbits of higher codimension one can start with several Casimirs and consider several Lenard schemes to generate the sequences of Hamiltonians.

Remark 6.7. A generic Virasoro coadjoint orbit $Diff(S^1)/S^1$ can be equipped with a complex structure and a two-parameter family of compatible (pseudo) Kahler metrics [Kir2].

This family of Kahler metrics has a simple origin: a generic Virasoro orbit has codimension 2, i.e., it is locally included in a two-parameter family of orbits, each equipped with its own symplectic structure compatible with the complex structure. Alternatively, one could consider a two-parameter family of symplectic structures on the same orbit, given by the Hamiltonian operators $a\partial_x^3 + b\partial_x$.

It turns out that the restriction of the two-parameter family of $H^1_{\alpha,\beta}$-metrics on *vir** to a generic Virasoro orbit $Diff(S^1)/S^1$ coincides with the family of Kahler metrics on the orbits. Proof is achieved by comparison with formula (7) in [Kir2] for those homogeneous metrics at one point of the orbit.

This is yet another fact manifesting a special role of the $H^1_{\alpha,\beta}$-metrics in Virasoro geometry.

Acknowledgments

We are grateful for the hospitality to the Isaac Newton Institute (Cambridge, UK) and the Fields Institute (Toronto, Canada), where most of this work was done. We are indebted to P. Pushkar and D. Novikov for drawing the figures. Special thanks go to I. Zakharevich for many useful remarks, as well as for illuminating discussions on Theorem 4.7.

The work of B.K. was partially supported by an Alfred P. Sloan Research Fellowship and by an NSERC research grant. The work of G.M. was supported in part by NSF Grant DMS-9970857.

Appendix A. Classification of Virasoro orbits

In this section, we recall the classification result for Virasoro coadjoint orbits (see, e.g., [Kir,Seg] or the book [GR]). The dual spaces to the infinite-dimensional Lie algebras considered below are always understood as smooth duals, i.e. identified with appropriate spaces of smooth functions.

(A) *Classification of quadratic differentials*: We start with the nonextended group of diffeomorphisms of the circle. Let $Diff(S^1)$ be the group of all orientation-preserving diffeomorphisms of S^1 and let $vect(S^1)$ be its Lie algebra.

Proposition A.1 (Kirillov [Kir]). *The dual space $vect(S^1)^*$ is naturally identified with the space of quadratic differentials $\{u(x)(dx)^2\}$ on the circle. The pairing is given by the formula*:

$$\langle u(x)(dx)^2, v(x)\partial_x \rangle = \int_{S^1} u(x)v(x)\, dx$$

for any vector field $v(x)\partial_x \in vect(S^1)$. The coadjoint action coincides with the action of a diffeomorphism on the quadratic differential: for a diffeomorphism $\varphi \in Diff(S)$ the action is

$$Ad^*_\varphi : u(dx)^2 \mapsto u(\varphi)\cdot \varphi_x^2 (dx)^2 = u(\varphi)\cdot (d\varphi)^2.$$

Hence, for instance, if $u(x) > 0$ for all $x \in S^1$ then the square root $\sqrt{u(x)(dx)^2}$ transforms under a diffeomorphism as a differential 1-form. In particular, $\Phi(u(x)(dx)^2) = \int_{S^1} \sqrt{u(x)}\, dx$ is a Casimir function (i.e., an invariant of the coadjoint action). One can see that there is only one Casimir function in this case, since the corresponding orbit has codimension 1 in the dual space $vect(S^1)^*$. Indeed, a diffeomorphism action sends the quadratic differential $u(x)(dx)^2$ to the constant quadratic differential $C(dx)^2$, where the constant C is the average value of the 1-form $\sqrt{u(x)}\,dx$ on the circle:

$$C = \frac{1}{2\pi}\int_{S^1}\sqrt{u(x)}\,dx.$$

On the other hand, if a differential $u(x)(dx)^2$ changes sign on the circle, then the integral $\int_a^b \sqrt{|u(x)|}\,dx$, evaluated between any two consecutive zeros a and b of the function $u(x)$, is invariant. In particular, the coadjoint orbit of such a differential $u(x)(dx)^2$ has necessarily codimension higher than 1.

Remark A.2. In our study of the KdV equation we pick the "freezing" point in the dual space $vect(S^1)^*$ to be $u_0 = 1/2$. (Actually, we consider the dual space to the Virasoro algebra, but we choose the cocentral term equal to zero, so that the "freezing" point $(u_0(dx)^2, a_0) = ((dx)^2/2, 0)$ belongs to the dual space to the Lie algebra of vector fields.) Other values of C give equivalent equations, differing from the KdV by scaling only.

(B) *Virasoro dual and Hill's operators*: Let vir be the Virasoro algebra. We can think of its dual space as the space of pairs $vir^* = \{(u(x)(dx)^2, a)\}$ consisting of a quadratic differential and a real number (cocentral term). It is more convenient, however, to regard such pairs as Hill's operators, i.e. differential operators $-a\partial_x^2 + u(x)$, as we will see below.

Proposition A.3. *The Virasoro coadjoint group action is given by the formula*

$$Ad^*_{(\varphi,b)} : (u(dx)^2, a) \mapsto (u(\varphi)\cdot\varphi_x^2(dx)^2 - aS(\varphi)(dx)^2, a), \tag{A.1}$$

where

$$S(\varphi) = \frac{\varphi_x\varphi_{xxx} - \frac{3}{2}\varphi_{xx}^2}{\varphi_x^2}$$

is the Schwarzian derivative of φ.

This group action on Hill's operators:

$$Ad^*_{(\varphi,b)} : -a\partial_x^2 + u(x) \mapsto -a\partial_x^2 + u(\varphi)\cdot\varphi_x^2 - aS(\varphi)$$

has the following geometric interpretation (see, e.g., [Kir,Ovs,Seg]). Fix $a = -1$ and consider Hill's operators of the form $\partial_x^2 + u(x), x \in S^1$. Let f and g be two independent solutions of the corresponding differential equation

$$(\partial_x^2 + u(x))y = 0$$

for an unknown function y. Although the equation has periodic coefficients, the solutions need not necessarily be periodic, but instead are defined over $\mathbb{R}$. Consider the ratio $\eta := f/g : \mathbb{R} \to \mathbb{R}P^1$.

Proposition A.4. *The potential u is (one half of) the Schwarzian derivative of the ratio η:*

$$u = \frac{S(\eta)}{2}.$$

Proof. Note that the Wronskian $W(f,g) := fg_x - f_x g$ is constant, since it should satisfy the differential equation $W_x = 0$. Here we normalize W by setting $W = -1$. This additional condition allows one to find the potential u from the ratio η. Indeed, first one reconstructs the solutions f, g from the ratio η by differentiating:

$$\eta_x = \frac{f_x g - f g_x}{g^2} = \frac{-W}{g^2} = \frac{1}{g^2}.$$

Therefore,

$$g = \frac{1}{\sqrt{\eta_x}}, \qquad f = g \cdot \eta = \frac{\eta}{\sqrt{\eta_x}}.$$

Given two solutions f and g, one immediately finds the corresponding differential equation they satisfy by writing out the following 3×3-determinant:

$$\det \begin{bmatrix} y & f & g \\ y_x & f_x & g_x \\ y_{xx} & f_{xx} & g_{xx} \end{bmatrix} = 0.$$

Since f and g satisfy the equation $y_{xx} + u \cdot y = 0$, one obtains from the determinant above that

$$u = -\det \begin{bmatrix} f_x & g_x \\ f_{xx} & g_{xx} \end{bmatrix}.$$

The explicit formula for u expressed in terms of η turns out to be one half of the Schwarzian derivative of η. $\square$

Corollary A.5. *The Schwarzian derivative $S(\eta)$ is invariant with respect to a Möbius transformation $\eta \to (a\eta + b)/(c\eta + d)$, where $ad - bc = 1$.*

Proof. Indeed, for a given potential u the solutions f and g of the corresponding differential equation are not defined uniquely, but up to a transformation of the pair (f, g) by a matrix from $SL_2(\mathbb{R})$. Then the ratio η changes by a Möbius transformation. $\square$

Proposition A.6. *The Virasoro coadjoint action of a diffeomorphism φ on the potential $u(x)$ gives rise to a diffeomorphism change of coordinate in the ratio η:*

$$\varphi : \eta(x) \to \eta(\varphi(x)).$$

Proof. We look at the corresponding infinitesimal action on the solutions of the differential equation $(\partial_x^2 + u(x))y = 0$. For $\varphi(x) = x + \varepsilon v(x)$ close to the identity, consider the infinitesimal Virasoro action of φ on the potential $u(x)$:

$$u \mapsto u + \varepsilon \cdot \delta u \quad \text{where } \delta u = 2uv_x + u_x v - \tfrac{1}{2} v_{xxx}.$$

(cf. formula (3.8) for $a = \frac{1}{2}$). It is consistent with the following action on a solution y of the above differential equation:

$$y \mapsto y + \varepsilon \cdot \delta y \quad \text{where } \delta y = -\tfrac{1}{2} y v_x + y_x v.$$

The consistency means that $(\partial_x^2 + u + \varepsilon \cdot \delta u)(y + \varepsilon \cdot \delta y) = 0 + \mathcal{O}(\varepsilon^2)$.

Note that the action $\varepsilon \cdot \delta y = \varepsilon \cdot (-\frac{1}{2} y v_x + y_x v)$ is an infinitesimal version of the following action of the diffeomorphism $\varphi(x) = x + \varepsilon v(x)$ on y:

$$\varphi : y \mapsto y(\varphi)(\varphi_x)^{-1/2}.$$

Thus solutions to Hill's equation transform as forms of degree $-1/2$. Therefore, the ratio η of two solutions transforms as a function under a diffeomorphism action. $\square$

In short, to calculate the coadjoint action on the potential u one can first pass from this potential to the ratio of two solutions, then change the variable in the ratio, and finally take the Schwarzian derivative of the new ratio to reconstruct the new potential $Ad^*_{(\varphi,b)} u$.

All of the above considerations of Hill's operators were local in x. To describe the Virasoro orbits, we now recall that $u(x)$ is defined on a circle.

Theorem A.7 (Kirillov [Kir]; Segal [Seg]). *The coadjoint Virasoro orbits (for a given cocentral term $a \neq 0$) are enumerated by the conjugacy classes in $(\widetilde{SL}_2(\mathbb{R}) \setminus \{id\})/\mathbb{Z}_2$, the universal covering of $SL_2(\mathbb{R})$ without the identity and modulo the $\mathbb{Z}_2$-action.*

Proof. For a periodic potential u the solutions of $(\partial_x^2 + u)y = 0$ are quasiperiodic. In other words, the boundary values of the fundamental set of solutions $F := (f, g)$ on $[0, 2\pi]$ are related by a holonomy matrix $M \in SL_2(\mathbb{R})$: $F(2\pi) = F(0)M$. (Similarly, one can consider the "projective solution" $\Theta := (\eta, \eta_x)$, which consists of the solution ratio η and its derivative η_x with the corresponding holonomy $\mathscr{M}$ now in $PSL_2(\mathbb{R})$.) This holonomy matrix M changes to a conjugate matrix if $x_0 = 0$ is replaced by any point $x_0 \in S^1$ or if F is replaced by another system of solutions.

Now regard the ratio $\eta = f/g$ as a map $\eta : [0, 2\pi] \to \mathbb{R}P^1$ describing a motion ("rotation") along the circle $\mathbb{R}P^1 \approx S^1$. One can see that the condition $\eta_x \neq 0$ is equivalent to the condition $W \neq 0$ on the Wronskian. Choosing the negative sign of the Wronskian, $W < 0$, we can assume that the rotation always goes in the positive direction: $\eta_x = -W/g^2 > 0$.

By a diffeomorphism change of the parameter $x \mapsto \varphi(x)$, one can always turn the map $\eta : [0, 2\pi] \to \mathbb{R}P^1$ into a *uniform* rotation along $\mathbb{R}P^1$, while keeping the boundary values of $\eta(x)$ on the segment $[0, 2\pi]$ satisfying the holonomy relation $\Theta(2\pi) = \Theta(0)\mathscr{M}$. Furthermore, the number of rotations (the "winding number") for the map $\eta : [0, 2\pi] \to \mathbb{R}P^1$ does not change under a reparametrization by a diffeomorphism φ. Thus the orbits of the maps η (or, equivalently, of the potentials $\{u(x)\}$) are described by the conjugacy classes of matrices in the universal covering of $SL_2(\mathbb{R})$. The choice in the sign of the Wronskian reflects the $\mathbb{Z}_2$-action on this universal covering.

Finally, note that the identity matrix in the universal covering $\widetilde{SL}_2(\mathbb{R})$ (or in its projectivization) cannot be obtained as a holonomy matrix for the maps $\eta : [0, 2\pi] \to \mathbb{R}P^1$. Indeed, any map η starts at the identity and goes in the positive direction. Thus, no matter how slow the rotation, one always moves out from the identity. □

Corollary A.8. *The Virasoro orbits in the hyperplane* $\{-a\partial_x^2 + u(x) \mid a = a_0\} \subset vir^*$ *with fixed* $a \neq 0$ *are classified by the Jordan normal form of matrices in* $SL_2(\mathbb{R})$ *and a positive integer parameter, winding number. Matrices in the group* $SL_2(\mathbb{R})$ *split into three classes, whose normal forms are equivalent to the exponents of the following three classes:*

(i)
$$\begin{bmatrix} \mu & 0 \\ 0 & -\mu \end{bmatrix},$$

(ii)
$$\begin{bmatrix} 0 & \pm 1 \\ 0 & 0 \end{bmatrix} \quad \text{and}$$

(iii)
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

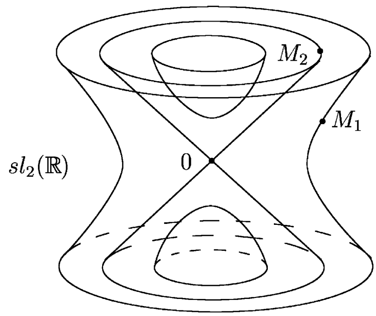

Fig. 4. The points M_1, M_2 and 0 in $sl_2(\mathbb{R})$ (which is a local picture of $SL_2(\mathbb{R})$) correspond to the Virasoro orbits of types (i), (ii) and (iii), respectively.

in the corresponding Lie algebra $sl_2(\mathbb{R})$, see Fig. 4. *The Virasoro orbit containing Hill's operator $\partial_x^2 + u(x)$ has the codimension in the hyperplane $\{a = a_0\} \subset vir^*$ that is equal to the codimension in $SL_2(\mathbb{R})$ of* (*the conjugacy class of*) *the holonomy matrix M corresponding to this Hill's operator. This codimension is* 1 *for classes* (i) *and* (ii), *and it is* 3 *for* (iii) *independ of the integer parameter.*

Remark A.9. Note that the exponents of real matrices with the normal form (i) split into rotation matrices (with $\mu \in i\mathbb{R}$) and hyperbolic rotations (with $\mu \in \mathbb{R}$), the elliptic and hyperbolic cases, cf. Remark A.11. In the paper, we regard these cases as belonging to the same general class (and similarly we do not distinguish between the cases ± 1 in type (ii)), since we are interested in the algebraic (rather than geometric) question of constructing the corresponding integrable equations.

One should also mention that the equality of codimension of the Virasoro coadjoint orbits in vir^* and (the conjugacy class of) the corresponding holonomy matrices in $SL_2(\mathbb{R})$ can be seen by checking the smooth dependence on a parameter in the above classification. (The versal deformations of the orbits can be given in terms of the Jordan–Arnold normal forms of the holonomy matrices depending on a parameter, cf. [OK2].) Alternatively, one can find the dimension of the corresponding stabilizers, see [Kir,Seg].

Regarded as homogeneous spaces, the orbits of type (i) are often denoted by $Diff(S^1)/S^1$, the notation $Diff(S^1)/\mathbb{R}^1$ stands for (ii) (and sometimes for the case $\mu \in i\mathbb{R}$ in (i)), and $Diff(S^1)/SL_2(\mathbb{R})$ corresponds to (iii).

Remark A.10. For applications to bihamiltonian systems we would like to describe all points in vir^* belonging to orbits of codimension at most 2. As we have shown above, in the smooth dual there are no orbits of codimension 0 or 1, as a is a Casimir function, and in each hyperplane the orbits are of codimension at least 1.[5]

[5] There exist Virasoro orbits of codimension 1 if in the dual space vir^* besides smooth elements we also admit singular ones, cf. [Wit]. In this paper, we consider the classification of the smooth dual elements only.

For $a_0 \neq 0$ the orbits are represented by the Hill's operators, whose holonomy matrices were classified above, while for $a_0 = 0$ they are quadratic differentials. Our choices of representatives for the orbits of codimension 1 for a fixed a_0 (i.e. of total codimension 2 in vir^*) will be as follows.

(a) For a generic point representing case (i) above we take Hill's operator $-\partial_x^2 + 1/2$ (i.e., $u_0 = (dx)^2/2, a_0 = 1$). It corresponds to the differential equation $(\partial_x^2 - 1/2)y = 0$ and has the holonomy matrix $diag(\exp(\pi\sqrt{2}), \exp(-\pi\sqrt{2}))$, the exponent of type (i). This freezing point corresponds to the CH equation.

(b) The matrix of type (ii) can be encountered in a generic 1-parameter family of matrices in $sl_2(\mathbb{R})$. Its exponent, a Jordan 2×2-block with the eigenvalue 1, can be represented as the holonomy matrix by the Hill's operator $-\partial_x^2$ (i.e., $u_0 = 0, a_0 = 1$). The latter point in vir^* corresponds (after an appropriate reduction) to the HS equation.

(c) The hyperplane $a_0 = 0$ in the Virasoro dual is the dual space $vect^*$ to the non-extended Lie algebra of vector fields. The orbits of codimension 1 in the space $vect^*$ are represented, e.g., by the quadratic differential $\frac{1}{2}(dx)^2$ (i.e., by the point $(u_0 = (dx)^2/2, a_0 = 0)$) as we discussed in Section A. Freezing the Poisson structure at the latter point leads to the KdV equation.

The above three cases are described in Corollary 6.4.

Remark A.11. Recall that the holonomy matrix M of Hill's operator $\partial_x^2 + u(x)$ changes to a conjugate one under the Virasoro action. This implies that $h(\partial_x^2 + u(x)) = \log(trace\, M)$ is a Casimir function on vir^*. One can use it to generate the KdV hierarchy via the Lenard scheme described in Theorem 6.3.

Recall that for the KdV equation the freezing point for the constant Poisson structure $\{.,.\}_0$ is $(a_0 = 0, u_0 = (dx)^2/2)$. Therefore, the Casimir function of the bracket $\{.,.\}_\lambda := \{.,.\}_{\mathrm{LP}} - \lambda^2\{.,.\}_0$ has the form $h_\lambda(\partial_x^2 + u(x)) := \log(trace\, M_\lambda)$, where M_λ is the holonomy of the operator $\partial_x^2 + u(x) - \lambda^2$. The expansion of the function h_λ in λ produces the first integrals of the KdV equation:

$$h_\lambda \approx 2\pi\lambda - \sum_{n=1}^{\infty} c_n h_{2n-1} \lambda^{1-2n},$$

where

$$h_1 = \int_{S^1} u(x)\, dx, \quad h_3 = \int_{S^1} u^2(x)\, dx, \quad h_5 = \int_{S^1} \left(u^3(x) - \frac{1}{2}(u_x(x))^2\right) dx, \ldots$$

and $c_1 = 1/2, c_n = (2n-3)!!/(2^n n!)$ for $n > 1$. One can see that the Hamiltonian h_3 is quadratic in u and coincides with the "energy" Hamiltonian of the KdV equation, regarded as an Euler equation. (Note that the KdV Hamiltonians h_j are differential polynomials whose degree increases with j. The latter follows from the recurrence relation $\{h_{2j+1}, f\}_0 + \{h_{2j-1}, f\}_{\mathrm{LP}} = 0$ for Hamiltonians h_j (cf. Theorem 6.3) for the constant and linear Poisson brackets on vir^*.) More details on the KdV structures can be found in [GZ].

Similar computations can be done for CH and HS, the other two equations considered in this paper, cf. [BSS].

References

[Arn] V. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble) 16 (1966) 319–361;
Mathematical Methods of Classical Mechanics, Springer, Berlin, New York, 1989, xvi + 508pp.

[AK] V. Arnold, B. Khesin, Topological Methods in Hydrodynamics, Springer, New York, 1998, xv + 376pp.

[BSS] R. Beals, D.H. Sattinger, J. Szmigielski, Acoustic scattering and the extended Korteweg-de Vries hierarchy, Adv. in Math. 140 (1998) 190–206;
Inverse scattering solutions of the Hunter–Saxton equations, preprint, 2001.

[Bre] Y. Brenier, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations, Comm. Pure Appl. Math. 52 (4) (1999) 411–452.

[CH] R. Camassa, D. Holm, An integrable shallow water equation with peaked solutions, Phys. Lett. Rev. 71 (1993) 1661–1664.

[FF] A. Fokas, B. Fuchssteiner, Symplectic structures, their Bäklund transformations and hereditary symmetries, Physica D 4 (1981/82) 47–66.

[GZ] I.M. Gelfand, I. Zakharevich, Spectral theory of a pencil of third-order skew-symmetric differential operators on S^1, Funct. Anal. Appl. 23 (1989) 85–93.

[GR] L. Guieu, C. Roger, Algebre et groupe de Virasoro; aspects geometriques et algebriques, in: Cours Specialises, Societe Mathematique de France, to appear.

[GS] V. Guillemin, S. Sternberg, On collective complete integrability according to the method of Thimm, Ergodic Theory Dyn. Systems 3 (2) (1983) 219–230.

[HS] J.K. Hunter, R. Saxton, Dynamics of director fields, SIAM J. Appl. Math. 51 (6) (1991) 1498–1521.

[HZh] J. Hunter, Y. Zheng, On a completely integrable nonlinear hyperbolic variational equation, Physica D 79 (1994) 361–386.

[Kir] A. Kirillov, The orbits of the group of diffeomorphisms of the circle, and local Lie superalgebras, Funct. Anal. Appl. 15 (1981) 135–136;
The orbit method, II: infinite-dimensional Lie groups and Lie algebras, Contemp. Math, 145 (1993) 33–63.

[Kir2] A. Kirillov, Kahler structure on the K-orbits of a group of diffeomorphisms of the circle, Funct. Anal. Appl. 21 (2) (1987) 42–45.

[Kru] M. Kruskal, Nonlinear wave equations, in: Dynamical Systems, Theory and Applications, Lecture Notes in Phys., Vol. 38, Springer, Berlin, 1975, pp. 310–354.

[Ma] S. Manakov, A note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body, Funct. Anal. Appl. 10 (1976) 93–94.

[Mag] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156–1162.

[Mi] G. Misiołek, Classical solutions of the periodic Camassa–Holm equation, Geom. Funct. Anal. 12 (2002) 1080–1104.

[Ovs] V. Ovsienko, On the Denogardus great number and Hooke's law, Kvant 8 (1989) 8–16;
Kvant selecta: Algebra and Analysis, II, Math. World, vol. 15, Amer. Math. Soc., Providence, RI, 1999, pp. 153–159.

[OK] V. Ovsienko, B. Khesin, The (super) KdV equation as an Euler equation, Funct. Anal. Appl. 21 (4) (1987) 81–82.
[OK2] V. Ovsienko, B. Khesin, Symplectic leaves of the Gelfand–Dikii brackets and homotopy classes of non-degenerate curves, Funct. Anal. Appl. 24 (1) (1990) 33–40.
[Seg] G. Segal, Unitary representations of some infinite-dimensional groups, Comm. Math. Phys. 80 (1981) 301–342; The geometry of the KdV equation, in: Topological Methods in Quantum Field Theory (Trieste, 1990), or Internat. J. Modern Phys. A 6(16) (1991), 2859–2869.
[Th] A. Thimm, Integrable geodesic flows on homogeneous spaces, Ergodic Theory Dyn. Systems 1 (1981) 491–517.
[Wit] E. Witten, Coadjoint orbits of the Virasoro group, Comm. Math. Phys. 114 (1988) 1–53.
[Za] I. Zakharevich, Manuscript, in preparation.