TOPOLOGY BOUNDS THE ENERGY

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Abstract. We survey several results related to topological obstructions to energy relaxation of a magnetic field in a perfectly conducting medium.[1]

1. Introduction.

In this expository note we survey several results related to topological obstructions to energy relaxation of a magnetic field in a perfectly conducting medium. Many more details and results, as well as a guide to the literature on this topic can be found in [2] or [10].

A motivation for this problem is the following model of a star. The magnetic field is supposed to be *frozen in* the perfectly conducting medium (plasma) filling the star, i.e. the topology of the field's trajectories does not change under the fluid flow. On the other hand, the magnetic energy can and does change, and the conducting fluid keeps moving (due to Maxwell's equations) until the excess of magnetic energy over its possible minimum is fully dissipated (this process is called "energy relaxation"). It turns out that mutual linking of magnetic lines may prevent complete dissipation of the magnetic energy. The problem is to describe the energy lower bounds of the magnetic field in terms of topological characteristics of its trajectories.

Main problem. Consider a divergence-free (magnetic) vector field ξ in a (simply connected) bounded domain $M \subset \mathbf{R}^3$. The *energy* of the field ξ is the square of its L_2-norm, i.e., the integral $E(\xi) = \int_M ||\xi||^2 \, d^3x$. (This definition makes sense for any compact Riemannian manifold M, possibly

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with boundary. In this case, $||\xi||^2 = (\xi, \xi)$ is the square of the Riemannian length of a vector, d^3x is replaced by the Riemannian volume form, and the field ξ is supposed to be tangent to the boundary of M.)

Let us act on the field ξ by a volume-preserving diffeomorphism h: $\xi_h := h_*\xi$. Given divergence-free field ξ, the main problem is to give a good lower bound for the energy $\inf_h E(\xi_h)$ in terms of topology of the field ξ. (Here we minimize the energy over the action of all volume-preserving diffeomorphisms h of M.)

Example 1.1 A topological obstruction to the energy relaxation can be seen in the example of a magnetic field confined to two linked solitori. Assume that the field vanishes outside those tubes and the field trajectories are all closed and oriented along the tube axes inside.

To minimize the energy of a vector field with closed orbits by acting on the field by a volume-preserving diffeomorphism, one has to shorten the length of most trajectories. (Indeed, the orbit periods are preserved under the diffeomorphism action; therefore, a reduction of the orbits' lengths shrinks the velocity vectors along the orbits.) In turn, the shortening of the trajectories implies a fattening of the solitori (since the acting diffeomorphisms are volume-preserving).

For a linked configuration, as in Fig.1, the solitori prevent each other from endless fattening and therefore from further shrinking of the orbits. Therefore, heuristically, in the volume-preserving relaxation process the magnetic energy of the field supported on a pair of linked tubes is bounded from below and cannot attain too small values.

2. Helicity bounds the energy.

To describe the first obstruction to the energy minimization we need the following notion.

Definition 2.1 *The* helicity *of the field ξ in the domain $M \subset \mathbf{R}^3$ is*

$$\mathcal{H}(\xi) = \int_M (\xi, \mathrm{curl}^{-1}\xi)\ d^3x,$$

where $(\ ,\ )$ is the Euclidean inner product, and $A = \mathrm{curl}^{-1}\xi$ is a divergence-free vector potential of the field ξ, i.e., $\nabla \times A = \xi$, $\mathrm{divA} = 0$.

One can easily see that the integral is independent of the particular choice of A (which is defined up to addition of the gradient ∇f of a harmonic function, since M is simply connected).

The word "helicity" was coined by K. Moffatt in [8] and it reveals the topological meaning of this characteristic of a vector field (see [9], [10] for the relevant historical surveys).

Theorem 2.2 [1] *For a divergence-free vector field ξ,*

$$E(\xi) \geq C \cdot |\mathcal{H}(\xi)|,$$

where C is a positive constant dependent of the shape and size of the compact domain M.

The proof is a composition of the Schwarz inequality and the Poincaré inequality, applied to the potential vector field $A = \mathrm{curl}^{-1}\xi$.

Note that the inverse (nonlocal) operator curl^{-1} sends the space of divergence-free vector fields (tangent to the boundary) on a simply connected manifold onto itself. This operator is symmetric, and its spectrum accumulates at zero on both sides. Thus the constant C can be taken equal to the largest absolute value of the operator curl^{-1} eigenvalues. The corresponding eigenfield has the minimal energy within the class of divergence-free fields obtained from this eigenfield by the action of volume-preserving diffeomorphisms. (Example: the Hopf field on S^3.)

The above statement also holds for an arbitrary closed three-dimensional Riemannian manifold if one confines oneself to divergence-free fields that are *null-homologous*, i.e., have a single-valued divergence-free potential.

Remark 2.3 Note that one can give a metric-free definition of helicity as follows, see [1]. Let M be a simply connected manifold with a volume form μ, and ξ a divergence-free vector field on M. The latter means that the Lie derivative of μ along ξ vanishes: $L_\xi \mu = 0$, or, which is the same, the substitution $i_\xi \mu =: \omega_\xi$ of the field ξ to the volume form μ is a closed 2-form: $d\omega_\xi = 0$. On a simply connected manifold M the latter means that ω_ξ is exact: $\omega_\xi = d\alpha$ for some 1-form α (called a potential). (If M is not simply connected, that we have to require that the field ξ is null-homologous, i.e., that the 2-form ω_ξ is exact. If M is with boundary, we require that ξ is tangent to it.)

Definition 2.4 (=2.1′) *The helicity $\mathcal{H}(\xi)$ of a null-homologous field ξ on a three-dimensional manifold M (possibly with boundary) equipped with a volume element μ is the integral of the wedge product of the form $\omega_\xi := i_\xi \mu$ and its potential:*

$$\mathcal{H}(\xi) = \int_M \alpha \wedge d\alpha = \int_M d\alpha \wedge \alpha, \text{ where } d\alpha = \omega_\xi.$$

Exercise 2.5 *Prove the equivalence of Definitions 2.1 and 2.4 if M is a domain in $\mathbf{R}^3$.*

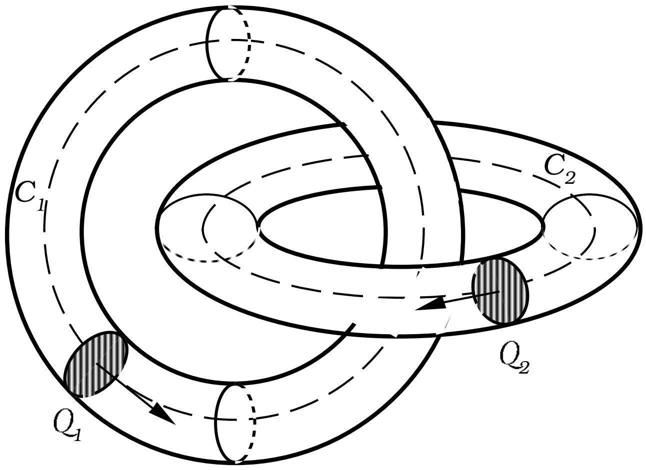

Figure 1. C_1, C_2 are axes of the tubes; Q_1, Q_2 are the corresponding fluxes.

An immediate consequence of this pure topological (metric-free) definition is the following

Theorem 2.6 [1] *The helicity $\mathcal{H}(\xi)$ is preserved under the action on ξ of a volume-preserving diffeomorphism of M.*

In this sense $\mathcal{H}(\xi)$ is a topological invariant: it was defined without coordinates or a choice of metric, and hence every volume-preserving diffeomorphism carries a field ξ into a field with the same helicity.

3. What is helicity?

Example 3.1 (=1.1′) To get familiar with helicity we first consider a magnetic (that is, divergence-free) field ξ which is identically zero except in two narrow linked flux tubes whose axes are closed curves C_1 and C_2. The magnetic fluxes of the field in the tubes are Q_1 and Q_2 (Fig.1).

Suppose further that there is no net twist within each tube or, more precisely, that the field trajectories foliate each of the tubes into pairwise unlinked circles.

Exercise 3.2 *The helicity invariant of such a field is given by*

$$\mathcal{H}(\xi) = 2\ lk(C_1, C_2) \cdot Q_1 \cdot Q_2,$$

where $lk(C_1, C_2)$ is the linking number of C_1 and C_2.

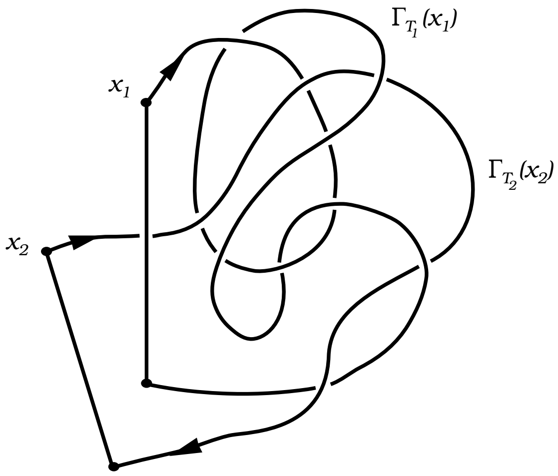

Figure 2. The long segments of the trajectories are closed by the "short paths."

Recall, that the (Gauss) *linking number* $lk(C_1, C_2)$ of two oriented closed curves C_1, C_2 in $\mathbf{R}^3$ is the signed number of the intersection points of one curve with an arbitrary (oriented) surface spanning the other curve.

V. Arnold proposed the following ergodic interpretation of helicity in the general case of any divergence-free field (when the trajectories are not necessarily closed or confined to invariant tori) as the average linking number of the field's trajectories.

Let ξ be a divergence-free field on M and $\{g^t : M \to M\}$ its phase flow. We will associate to each pair of points in M a number that characterizes the "asymptotic linking" of the trajectories of the flow $\{g^t\}$ passing through these points. Given any two points x_1, x_2 in M and two large numbers T_1 and T_2, we consider "long segments" $g^t x_1 (0 \leq t \leq T_1)$ and $g^t x_2 (0 \leq t \leq T_2)$ of the trajectories of ξ issuing from x_1 and x_2. Close these long pieces by the shortest geodesics between $g^{T_k} x_k$ and x_k. We obtain two closed curves, $\Gamma_1 = \Gamma_{T_1}(x_1)$ and $\Gamma_2 = \Gamma_{T_2}(x_2)$; see Fig.2. Assume that these curves do not intersect (which is true for almost all pairs x_1, x_2 and for almost all T_1, T_2). Then the linking number $lk_\xi(x_1, x_2; T_1, T_2) := lk(\Gamma_1, \Gamma_2)$ of the curves Γ_1 and Γ_2 is well-defined.

Definition 3.3 [1] *The* asymptotic linking number *of the pair of trajectories $g^t x_1$ and $g^t x_2$ $(x_1, x_2 \in M)$ of the field ξ is defined as the limit*

$$\lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{lk_\xi(x_1, x_2; T_1, T_2)}{T_1 \cdot T_2},$$

where T_1 and T_2 are to vary so that Γ_1 and Γ_2 do not intersect.

It turns out that this limit exists (as an element of the space $L_1(M \times M)$ of the Lebesgue-integrable functions on $M \times M$) and is independent of the system of geodesics (i.e., of the Riemannian metric), see Remark 3.5 below.

Theorem 3.4 [1] *Helicity of a divergence-free vector field ξ on a simply connected manifold M with a volume element μ is equal to the average self-linking of trajectories of this field, i.e., to the asymptotic linking number $\lambda_\xi(x_1, x_2)$ of trajectory pairs integrated over $M \times M$:*

$$\mathcal{H}(\xi) = \int\limits_M \int\limits_M \lambda_\xi(x_1, x_2)\ \mu_1 \mu_2.$$

In the case of a manifold M with boundary, all the vector fields involved are supposed to be tangent to the boundary.

Remark 3.5 In the original paper [1], instead of segments of shortest geodesics, one considered systems of "short paths" between every two points of the manifold, which satisfy some conditions to provide the existence of $\lambda_\xi(x_1, x_2)$ almost everywhere as a pointwise limit as $T_1,\ T_2 \to \infty$. Such a system of "short paths" would, generally speaking, depend on a vector field. In [13] T. Vogel suggested to use the L_1-convergence, rather than the pointwise one, and showed that in the L_1-case it is sufficient to use the system of shortest geodesics for any vector field. His approach settled in a universal way the existence question for "short paths" systems. It might also shed some light on the following long-standing problem.

Problem 3.6 [1] *Is the average self-linking number of a divergence-free vector field invariant under the action of homeomorphisms preserving the measure on the manifold?* Here, a measure-preserving homeomorphism is supposed to transform the flow of one smooth divergence-free vector field into the flow of the other, both fields having well-defined average self-linking numbers.

The above problem is a counterpart of the homotopy invariance of the classical Hopf invariant for maps $\pi : S^3 \to S^2$. The classical Hopf invariant is always an integer, and it is equal to the linking of the preimages of two generic points in S^2. One can also give an integral definition of the Hopf invariant for such a map. This invariant is equal to the helicity of a vector field tangent to the levels of π (so that all orbits of this field are closed).

4. Energy estimates.

As we have seen above, a nonzero helicity (or average linking of the trajectories) of a field ξ provides a lower bound for the energy. Note that the heuristic argument mentioned in the Introduction is more general in the following sense. It demonstrates that there exists a lower bound for the energy for a field which has at least one linked pair of solitori as in the example above. However, the helicity of such a field might turn out to be zero, if, e.g., it has another ("mirror") pair of solitori linked in the opposite direction which makes vanish the total averaged self-linking of trajectories of the vector field. This shows that one needs a more subtle energy estimates, where, in particular, the contribution of any nontrivially linked "tube of trajectories" into the energy bound could not be canceled out.

Apparently, one of the the best results in this direction is as follows.

Theorem 4.1 [5] *Suppose a vector field ξ in $\mathbf{R}^3$ has an invariant torus T forming an nontrivial knot of type K. Then*

$$E(\xi) \geq \left(\frac{16}{\pi \cdot \mathrm{Vol}(T)}\right)^{1/3} \cdot Q^2 \cdot (2 \cdot \mathrm{genus}(K) - 1),$$

where $Q = \mathrm{Flux}\xi$ is the flux of ξ through a crossection of T, $\mathrm{Vol}(T)$ is the volume of the solid torus, and $\mathrm{genus}(K)$ is the genus of the knot K.

Recall, that for any knot its *genus* is the minimal number of handles of a spanning (oriented) surface for this knot. For an unknot the genus is 0, since one can take a disk as a spanning surface. For a nontrivial knot one has $\mathrm{genus}(K) \geq 1$ and, therefore, the above energy is bounded away from zero: $E(\xi) > 0$.

Remark 4.2 Note that there are no restrictions on the behavior of the divergence-free field inside this invariant torus, and hence this result has a wide range of applicability. In particular, it is sufficient for the field to have at least one closed linked trajectory of the *elliptic* type. The latter means that its Poincaré map has two eigenvalues of modulus 1. Then the KAM theory implies that a generic elliptic orbit is confined to a set of nested invariant tori. Hence any such orbit ensures that the energy of the corresponding field has a non-zero lower bound [7].

Problem 4.3 *The question remains whether the presence of* any *nontrivially linked closed trajectory* (of any type: hyperbolic, non-generic, etc.) *or the presence of chaotic behavior of trajectories for a field could provide a positive lower bound for the energy (even if the averaged linking of all trajectories totals zero) and therefore could prevent a relaxation of the field to arbitrarily small energies.*

Remark 4.4 In [5] one can find an extension of Theorem 4.1 from knots to links, i.e., explicit energy estimates for the fields having nontrivially linked invariant tubes. The paper also contains many sharper results in terms of asymptotic crossing numbers for knots, links, and field trajectories, as well as in terms of conformal modulus of solid tori. (We refer to [5] or [2] for necessary definitions.)

One should mention, that in certain cases one can give somewhat better estimates by assuming more from the vector field. In particular, following [6] we suppose that a vector field ξ not only has an invariant torus confining a knot (or a link) K but it is *strongly modeled* on K, cf. Example 3.1. The latter means that a tubular neighborhood T of K can be identified in a volume-preserving way with (possibly, several copies of) the product $D^2 \times S^1$ of the disk and circle, and this identification can be carried in such a way that the field becomes $\partial/\partial\phi$, i.e., constant and oriented along the circles S^1 in the product.

Theorem 4.5 [6] *The energy of a vector field ξ strongly modeled on a nontrivial knot (or on a nontrivial indecomposable link) K in $\mathbf{R}^3$ satisfies the inequality*

$$E(\xi) > \left(\sqrt{6/125}/\pi^2\right)^{4/3} \cdot \mathrm{Vol}(T)^{5/3} \approx 0.00624 \cdot \mathrm{Vol}(T)^{5/3}.$$

where $\mathrm{Vol}(T)$ *is the volume of the neighborhood, where we modeled the field.*

Note that for a field, strongly modeled in a tube T, the flux Q is proportional to the area of the crossection of T, and hence to its volume $\mathrm{Vol}(T)$, since the length of S^1 is fixed. This shows consistency of Theorems 4.1 and 4.5: both right-hand-sides have the order of $\mathrm{Vol}(T)^2/\mathrm{Vol}(T)^{1/3} = \mathrm{Vol}(T)^{5/3}$.

Remark 4.6 The rotation field in the three-dimensional ball is an example of an opposite type: all its trajectories are *pairwise unlinked*. It was suggested by A. Sakharov and Ya. Zeldovich, and proved by M. Freedman (see [4]), that this field can be transformed by a volume-preserving diffeomorphism to a field whose energy is less than any given ϵ.

The idea of Freedman's proof is to split the ball into two parts. By stretching the main part of the ball into a long snake and putting it back, one shrinks most of the field's orbits, and hence makes the corresponding energy arbitrarily small. After that one controls the energy gain on the small part, needed to make the first operation smooth. A very similar idea works in Shnirelman's proof of the finiteness of the diameter of the group of volume-preserving diffeomorphisms of a three-dimensional cube (see [11]). It also consists of two steps: the first one is a combinatorial problem for small cubes (quite different from the above), while the second step is a (very similar to Freedman's) control of the stretching in the layer between the small cubes.

A recent very powerful techniques of *generalized flows* and their approximations by the regular flows provided a natural framework for existence theorems in ideal hydrodynamics, as well as they allowed one to give a much simpler estimates for the diameter and other geometric characteristics of the group of volume-preserving diffeomorphisms [3], [12]. The notion of a generalized flow (or, a polymorphism) means that every fluid particle does not necessarily have a unique position at any moment, but moves simultaneously to every other point of the media with certain probability.

Problem 4.7 *It would be very interesting to apply the techniques of generalized flows to energy estimates discussed above.*

A dream would be to define an action of a "volume-preserving polymorphism" on a divergence-free vector field and to estimate the minimal energy of its image under the action of all such polymorphisms. Then one could hope to use an approximation-type theorem for polymorphisms to show that the energy infimum of a vector field acted upon by diffeomorphisms coincides with the energy minimum for the action of polymorphisms.

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References

1. Arnold, V.I. (1973) The asymptotic Hopf invariant and its applications. *Proc. Summer School in Diff. Equations at Dilizhan,* Erevan (in Russian); English transl.: *Sel. Math. Sov.* **5** (1986), 327–345.
2. Arnold, V.I. & Khesin, B.A. (1998) Topological methods in hydrodynamics. *Applied Mathematical Sciences, vol. 125, Springer-Verlag, New York*, pp. xv+374.
3. Brenier, Y. (1989) The least action principle and the related concept of generalized flows for incompressible perfect fluids. *J. Amer. Math. Soc.* **2:2**, 225–255.
4. Freedman, M.H. (1999) Zeldovich's neutron star and the prediction of magnetic froth. *Proceedings of the Arnoldfest, Fields Institute Communications* **24** (ed. E.Bierstone, *et al.*), pp. 165–172.

5. Freedman, M.H. & He, Z.-X. (1991) Divergence-free fields: energy and asymptotic crossing number. *Annals of Math.* **134:1**, 189–229.
6. Freedman, M.H. & He, Z.-X. (1991) Links of tori and the energy of incompressible flows. *Topology* **30:2**, 283–287.
7. Kozlovsky, O. *Personal communication.*
8. Moffatt, H.K. (1969) The degree of knottedness of tangled vortex lines. *J. Fluid. Mech.* **35**, 117–129.
9. Moffatt, H.K. (1981) Some developments in the theory of turbulence. *J. Fluid Mech.* **106**, 27–47; Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. *Part 1: J. Fluid Mech.* **159** (1985), 359–378; *Part 2: J. Fluid Mech.* **166** (1986), 359–378.
10. Moffatt, H.K. and Tsinober, A. (1992) Helicity in laminar and turbulent flow. *Annual Review of Fluid Mechanics* **24**, 281–312.
11. Shnirelman, A. (1985) The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid. *Matem. Sbornik* **128:1**, 82–109; English transl.: *Math. USSR, Sbornik* **56:1** (1987), 79–105.
12. Shnirelman, A. (1994) Generalized fluid flows, their approximation and applications. *Geom. and Funct. Analysis* **4:5**, 586–620.
13. Vogel, T. (2000) On the asymptotic linking number. *München University preprint*, pp. 1–9.