Polar Bear or Penguin? Musings on Earth Cartography and Chebyshev Nets

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This column is a place for those bits of contagious mathematics that travel from person to person in the community because they are so elegant, surprising, or appealing that one has an urge to pass them on.

Contributions are most welcome.

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Let's begin with an innocent question from mathematical folklore. A hunter sets out from his tent and walks 10 kilometers due south, then 10 kilometers due west, then 10 kilometers due north, and finally 10 kilometers due east. After his journey, he finds himself back at his tent. Where could the hunter's tent be located?

The seemingly evident answer "Anywhere!" would be correct for plane geography, but cartography on Earth's surface is more intricate.

Let us first try to solve a well-known auxiliary children's problem: The same hunter went for only a three-leg trip: 10 km south, then 10 km west, then 10 km north, and arrived back at his tent. Where could his tent be located?

In some old folklore versions of this problem [1, 4, 6, 7], one adds that the hunter saw a bear and asks about the bear's color. For this three-leg trip, one evident solution is the North Pole (and then the bear would supposedly be a white polar bear). But this solution is not unique!

Indeed, consider the circle of latitude ℓ_1 exactly 10 km in length, somewhere near the South Pole. Then the hunter's tent can be anywhere on the circle of latitude m_1 that is 10 km north of ℓ_1. More generally, the tent can stand at any point of the infinite number of circles m_k, $k = 1, 2, 3, \ldots$, that are 10 km north of the parallels ℓ_k of length $10/k$ km in the vicinity of the South Pole; see Figure 1.

Therefore, the solution set to this folklore problem is the union of the North Pole and the circles of latitude $m_k, k = 1, 2, 3, \ldots$, accumulating at the parallel located 10 km away from the South Pole. (Note that this set of solutions is not closed!) So it would be natural to ask, "What was the color of the penguins encountered by the hunter?"

Let us return to our original problem of the hunter's four-leg trip. It is clear that while traveling along a meridian from north to south, the hunter covers the same distance as he does while traveling another meridian from south to north between the same two parallels, but because those

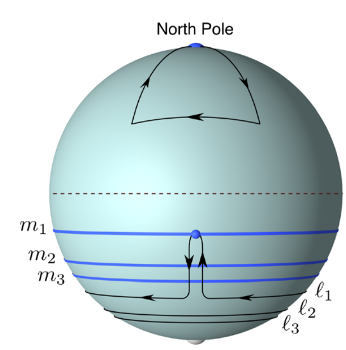

Figure 1. Two types of solutions for a three-leg trip.

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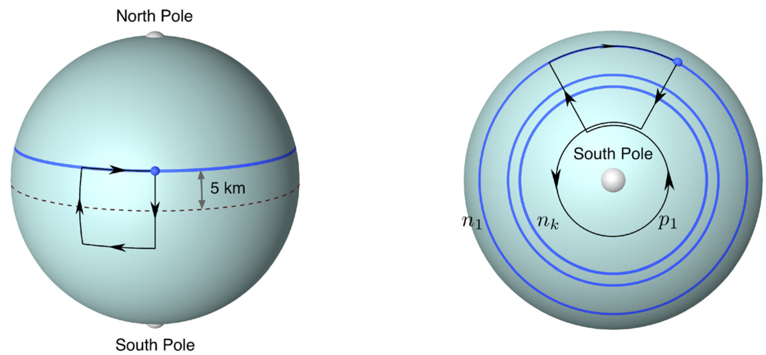

Figure 2. (a) Near-equatorial and (b) near-South-Pole solutions for the four-leg trip.

parallels may have different circumferences, he could end up beyond or short of his tent.

One natural solution is to put the tent anywhere on the circle of latitude located 5 km north of the equator; see Figure 2(a). Then the two parallels along which the hunter travels are located at the same distance from the equator and hence are of the same length. So he will return to his tent. But are there any other possible locations?

As a matter of fact, there are! One set of solutions is a variation of the folklore problem above. The hunter's trip will now have a keyhole shape. Let us start anywhere on a certain parallel n_1 south of the parallel m_1 and north of m_2, where exactly how far south n_1 is will be defined by the following condition: after traveling 10 km south along the meridian and arriving at the parallel[1] we shall call p_1, then going west the full circle and the bit extra along that parallel to make 10 km altogether, and finally returning along another meridian to the parallel n_1, the hunter would have exactly 10 km to travel to the east to arrive at his tent; see Figure 2(b). Such a parallel n_1 must exist between parallels m_1 and m_2 by the intermediate value theorem.

Now observe that there are infinitely many similar solutions: the hunter could start closer to the South Pole, anywhere on an appropriate parallel n_k, so that he could travel via a meridian to the parallel p_k that has a circumference a little less than $10/k$ km, then travel west k full circles and a bit farther, before returning via a meridian to the parallel n_k at the point exactly 10 km west of his tent. In this way, the hunter could set his tent anywhere on the infinite number of parallels n_k, for which the corresponding parallels p_k, each being 10 km south of n_k, accumulate at the South Pole.

Is that all? Not yet. The problem is completely symmetric. A similar set of solutions exists near the North Pole. There, however, one makes any number of full rotations going east along the northern parallel and travels just a part of the circle of latitude going west.

To summarize: the hunter's tent could be anywhere on an infinite number of special parallels accumulating at the North Pole, on an infinite number of special parallels accumulating at the parallel 10 km away from the South Pole, and anywhere on the parallel 5 km north of the equator. If he were to set his tent anywhere else on Earth, he would miss his tent at the end of his journey.

Note that as $k \to \infty$, the four-leg solutions near the North Pole degenerate to the three-leg one in Figure 1, where the trip's northern leg "disappears." A similar phenomenon occurs near the South Pole. This kind of degeneration does not happen with the two meridian legs of the trip, since there is no "West Pole" or "East Pole" on Earth.

Indeed, west and east are relative notions, while North and South are absolute ones (just as in Martin Gardner's question, Why does the reflection in the mirror interchange left and right, but not up and down?)

And now back to the question of what animal the hunter has a chance of encountering on the way. One observes that the "polar sets" of solutions for the hunter's tent are too close to the North and South Poles to encounter any animal at all, either bears or penguins.

However, there is one type, the Galápagos penguins, which can be found at the Galápagos Islands close to the equator; see Figure 3. These are the only type of penguins that can be found in natural habitat partly in the northern hemisphere. The near-equatorial solution for the hunter's tent would nicely fit the topography of the Galápagos Islands and could lead to the hunter meeting such a penguin!

Chebyshev Nets

Let us switch gears and look at the problem from a different angle. We shall use some material, including the pictures, from Lecture 18 in [3] (reproduced by permission of the American Mathematical Society).

The Earth is covered by a net (a 2-parameter family of curves) consisting of meridians and circles of latitude. Using these two families of curves to define "forward–backward" and "right–left" directions, we asked whether going consecutively the same distance in these four directions takes one back to the starting point. The answer was a sound no, except for some very special locations.

A net of curves on a surface is called a Chebyshev net if the lengths of the opposite sides of every quadrilateral made by a pair of curves from each family of curves are equal; see Figure 4(a).

[1] Which will be south of ℓ_1 and north of ℓ_2 and therefore less than 10 km but more than 5 km in circumference.

Figure 3. The Galápagos Islands. (Wikimedia commons, public domain.)

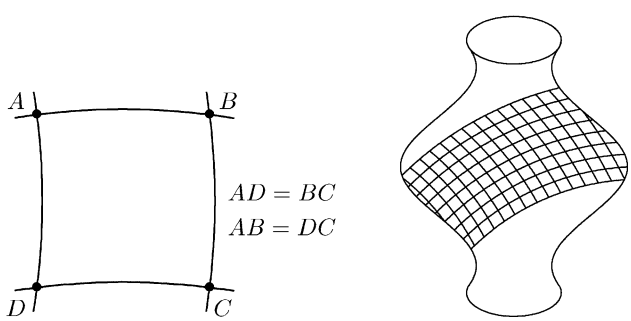

Figure 4. (a) Chebyshev net; (b) a piece of curved fabric.

Figure 5. A basketball in a mesh grocery bag.

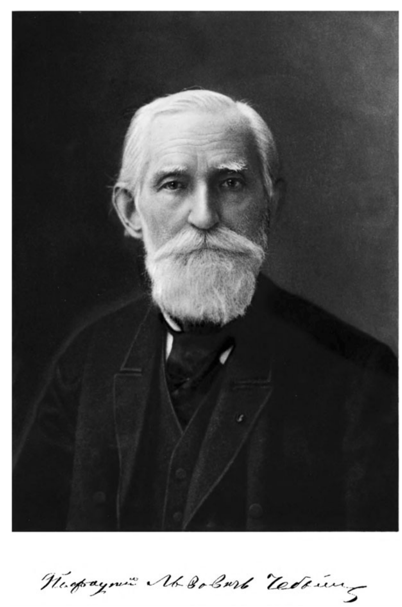

Figure 6. Pafnuty Lvovich Chebyshev, 1821–1894. (Wikimedia Commons, public domain.)

On a surface with a Chebyshev net, a hunter will return to his tent no matter where the tent is located if he travels along the curves of the net. Loosely speaking, a surface admits a Chebyshev net if one can tightly attach to it a fishing net, as in Figure 5.

Pafnuty Chebyshev (Figure 6) came up with this notion, motivated by an applied problem: how to cut fabric in a more economical way (there was a high demand for army uniforms due the Crimean War). Chebyshev presented his results in a talk entitled "Sur la coupe des vêtements" in 1878, at a session of the Association française pour l'avancement des sciences in Paris. We refer to the appendix of E. Ghys's paper [5] and to [2] for the intriguing history of this work.

In this model, flat fabric is woven out of two families of unstretchable threads that make a rectangular grid. When one drapes a piece of fabric over a curved surface, the rectangles are distorted to curved parallelograms, but their opposite sides are still of equal lengths; see Figure 4(b).

Let us describe a geometric construction of a surface with a Chebyshev net. Start with two curves, a and b, in space. For every pair of points A on a and B on b, let C be the midpoint of the segment AB. The locus of points C is a surface. This surface is composed of two families of curves: those obtained by fixing point A (the first family) and those obtained by fixing point B (the second family). This is a Chebyshev net.

Indeed, consider two pairs of points A and A' on curve a, and B and B' on curve b; see Figure 7. The midpoints K, L, M, N of the four segments lie on the surface. The pairs K, L and M, N lie on two curves from one family, and the pairs K, N and L, M lie on two curves from the other family. One has

$$KL = \frac{1}{2}AA' = NM,$$
$$KN = \frac{1}{2}BB' = LM,$$

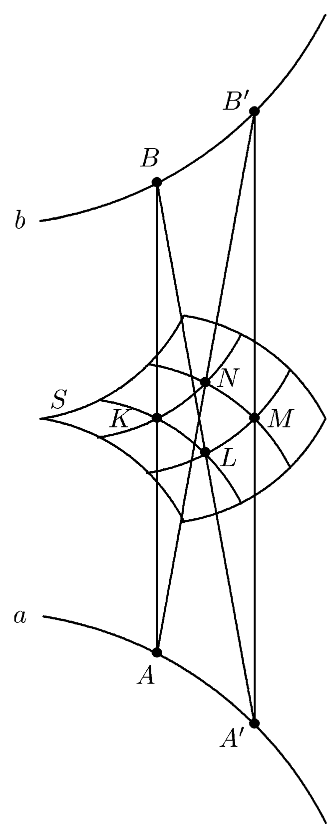

Figure 7. A construction of a translation surface with a Chebyshev net. (Image from Wikimedia commons, public domain.)

and hence curve NM is obtained from curve KL by parallel translation through the vector KN, and likewise for curves LM and KN. It follows that the lengths of the opposite sides of the curvilinear quadrilateral $KLMN$ are equal.

The surface, the locus of points C in the above construction, is the half-scaled result of parallel translation of curve a along curve b. Such surfaces are called translation surfaces. Note that the seed curves a and b could intersect and even lie in the same plane, in which case the translation surface would be the same plane equipped with a Chebyshev net.

Examples of translation surfaces include the hyperbolic paraboloid $z = x^2 - y^2$ and the circular paraboloid $z = x^2 + y^2$, the respective curves a and b being the parabolas $(2x, 0, 2x^2)$ and $(0, 2y, \mp 2y^2)$; see Figure 8.

As for general surfaces, Chebyshev nets are locally described by a remarkable partial differential equation.

Let s and t be arc-length parameters along the curves of the two families that define a net. Then (s, t) is a local coordinate system on the surface. Let $\omega(s,t)$ be the angle between the curves, and let $K(s, t)$ be the curvature of the surface. The net is a Chebyshev net if and only if ω satisfies the equation

$$\frac{\partial^2 \omega}{\partial s\, \partial t} = -K \sin \omega.$$

If $K = -1$, this is the famous sine–Gordon equation, a completely integrable partial differential equation of soliton type.

Let us now return to the sphere. Chebyshev outlined a proof that there exists a Chebyshev net that covers a hemisphere. Recently, Étienne Ghys [5] substantially improved this global result and proved that the whole sphere can be covered by a Chebyshev net. This solution is illustrated in Figure 9.

Figure 9(a) shows a piece of flat fabric that drapes the entire sphere. The threads of this fabric are not shown; they are the vertical and horizontal lines. The center is placed at the north pole; the four corners meet at the south pole. The boundaries of the diamond-shaped fabric become the arcs of orthogonal meridians meeting at the south pole, shown in yellow in Figure 9(b). The curves shown in Figure 9(a) are the preimages of the meridians and the circles of latitude, and the heavy blue curve is the preimage of the equator.

Figure 9(b) depicts the Chebyshev net in the southern hemisphere, with the south pole at the center. The net has four singularities, and it is not extended to the yellow seam.

Using this net for navigation, the hunter will return to his tent no matter where his tent is located on the sphere (off the yellow arcs).

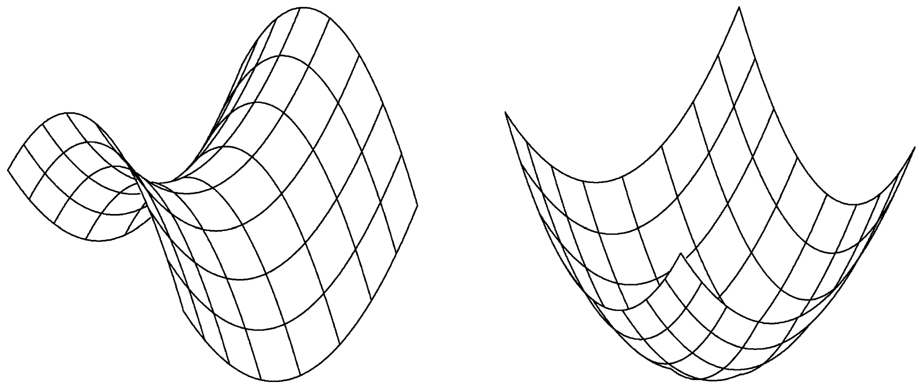

Figure 8. Two translation surfaces: (a) a hyperbolic paraboloid; (b) a circular paraboloid.

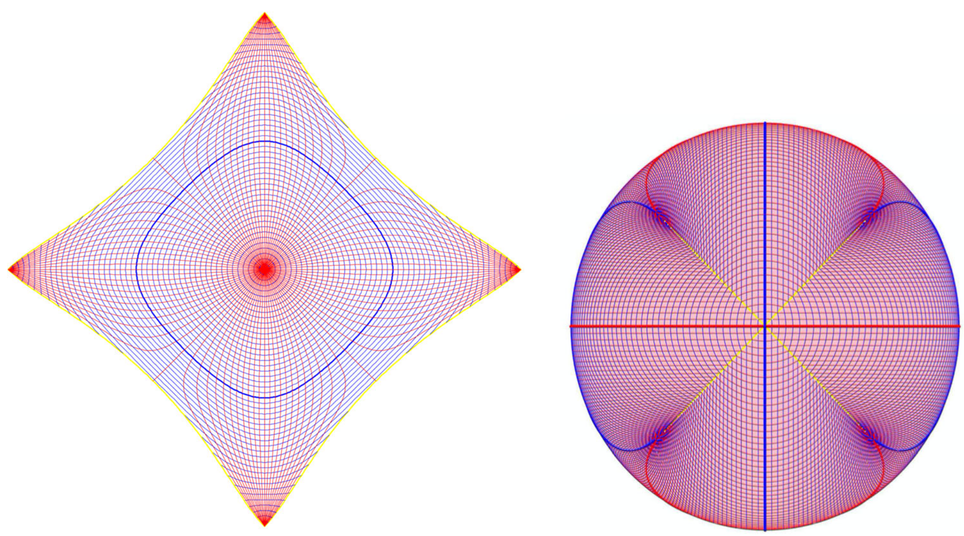

Figure 9. A Chebyshev net that covers the sphere. (a) The flat fabric; (b) the net in the southern hemisphere. (Reproduced from [5] by the kind permission of the editors of *L'Enseignement mathématique*.)

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