# Universal Drinfeld-Sokolov Reduction and Matrices of Complex Size

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**Abstract**

We construct affinization of the algebra $\mathfrak{gl}_\lambda$ of "complex size" matrices, that contains the algebras $\hat{\mathfrak{gl}}_n$ for integral values of the parameter. The Drinfeld–Sokolov Hamiltonian reduction of the algebra $\hat{\mathfrak{gl}}_\lambda$ results in the quadratic Gelfand–Dickey structure on the Poisson–Lie group of all pseudodifferential operators of fractional order.

This construction is extended to the simultaneous deformation of orthogonal and simplectic algebras which produces self-adjoint operators, and it has a counterpart for the Toda lattices with fractional number of particles.

## 1 Introduction

As a rule quadratic Poisson structures appear as either the Poisson bracket on a Poisson–Lie group or as a result of Hamiltonian reduction from the linear bracket on a dual Lie algebra.

This paper is devoted to a relation between these two approaches to the classical $W_n$–algebras (called also Adler–Gelfand–Dickey or higher $KdV$-structures), natural infinite-dimensional quadratic Poisson structures on differential operators of $n^{th}$ order.

The noncommutative Hamiltonian reduction (see[2], [18]) for the Gelfand–Dickey structures (associated to any reductive Lie group) is known as the reduction of Drinfeld and Sokolov ([6]). They showed that those quadratic structures on *scalar* $n^{th}$ *order* differential operators on the circle can be obtained as a result of the two-step process (restriction to a submanifold and taking quotient) from the linear Poisson structure on

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*matrix first order* differential operators. The latter object is nothing but the dual space to an affine Lie algebra on the circle ([9], [19]).

On the other hand all Poisson $W_n$ algebras can be regarded as Poisson submanifolds in a certain universal Poisson–Lie group of pseudodifferential operators of arbitrary (complex) degree([12]). In such a way differential operators $DO_n = \{D^n + u_1(x)D^{n-1} + u_2(x)D^{n-2} + ... + u_n(x)\}$ for any $n$ turn out to be included as a Poisson submanifold to a one-parameter family of pseudodifferential symbols $\Psi DS_\lambda = \{D^\lambda + u_1(x)D^{\lambda-1} + u_2(x)D^{\lambda-2} + ...\}$.

At this point we bump into the following puzzle. While a natural description of the Gelfand–Dickey structures through the above Poisson–Lie group exists for symbols of every complex degree $\lambda$ , the Drinfeld–Sokolov reduction is defined essentially for diffential operators, that is for integer $\lambda = n$ and first $n$ coefficients $\{u_*(x)\}$. The latter restriction is due to the very nature of the Drinfeld–Sokolov reduction: it starts from the affine $\mathfrak{gl}_n$ algebra and to find its counterpart for complex $\lambda$ one needs to define groups of $\lambda \times \lambda$ matrices.

Actually the definition of $\mathfrak{gl}_\lambda, \lambda \in \mathbf{C}$ has been known all the time since representation theory of $\mathfrak{sl}_2$ appeared. It is simply the universal enveloping algebra of $\mathfrak{sl}_2$ modulo the relation: Casimir element is equal to $(\lambda - 1)(\lambda + 1)/2$. It was Feigin, however, who placed this object in the proper context (deformation theory) and applied it to calculation of the cohomology of the algebra of differential operators on the line, see [8].

For technical reasons we have here to replace the algebra $\mathfrak{gl}_\lambda$ with its certain extension $\bar{\mathfrak{gl}}_\lambda$. We further construct an affinization of the latter $\hat{\mathfrak{gl}}_\lambda$. This gives a family of algebras, $\lambda$ being the parameter, such that for integral $\lambda$ the algebra has a huge ideal and the corresponding quotient is the conventional affine Lie alegebra $\hat{\mathfrak{gl}}_n$. We prove the following conjecture of B. Feigin and C. Roger.

**Theorem 1.1** *The classical Drinfeld–Sokolov reduction defined on $\hat{\mathfrak{gl}}_n$ admits a one-parameter deformation to the Hamiltonian reduction on $\hat{\mathfrak{gl}}_\lambda$. As a Poisson manifold the result of the reduction coincides with the entire Poisson–Lie group of pseudodifferential operators equipped with the quadratic Gelfand–Dickey structure.*

It should be mentioned that, unlike integral $\lambda$ case, for a generic $\lambda$ we can not use the formalism of the Miura transform (cf.[16]) or embedding of scalar higher order differential operators into first order matrix ones by means of Frobenius matrices. Both the operations are main tools in the classical $\hat{\mathfrak{gl}}_n$-case.

Feigin constructed also a simultaneous deformation of the symplectic and orthogonal algebras. We show that the corresponding deformation of the Hamiltonian reduction results in the Gelfand–Dickey bracket on self-adjoint pseudodifferential symbols.

We finish with a construction of the continuous deformation of Toda lattice hierarchies.

The paper is essentially selfcontained. The Section 2 is devoted to basics in Poisson geometry and Drinfeld–Sokolov reduction. Then we outline the construction of the Poisson–Lie group of pseudodifferential operators and define the Adler–Gelfand–Dickey structures explicitly (Sect.3.1). Further we recall definition of $\mathfrak{gl}_\lambda$ and define its extension and affinization (Sect.3.2) which we beleive is of interest by itself. Remark that a similar interpolating object appears as a *sine*-algebra and the algebra of "quantum torus" (see [3]). In the Section 3.3 we construct the universal reduction of that algebra, resulting in the structure on the Poisson –Lie group. Section 4 is devoted to proofs. We conclude with discussion of $\mathfrak{sp}$-, $\mathfrak{so}$-cases of the reduction and with consideration of a deformation of the Toda lattices.

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# 2 Poisson manifolds and Drinfeld-Sokolov reduction

## 2.1 Preliminaries on Poisson geometry and Hamiltonian reduction

1. *Poisson algebras.* Symplectic manifold is a pair $(M,\omega)$, where $M$ is a manifold and $\omega$ is a symplectic structure, i.e. non-degenerate skew-symmetric 2-form on $M$. The symplectic form can be viewed as a non-degenerate fiberwise linear map

$$\omega:\ TM \to T^*M, \tag{1}$$

where $TM$ ($T^*M$) is a tangent (cotangent *resp.*) bundle over $M$. For any $f,g \in C^\infty(M)$ set

$$\{f,g\} = dg(\Omega df).$$

where $\Omega = \omega^{-1} : T^*M \to TM$. The bracket $\{.,.\}$ makes $C^\infty(M)$ a *Poisson algebra*, meaning that the following holds:

$$\text{the space } C^\infty(M) \text{ is a Lie algebra with respect to } \{.,.\} \tag{2}$$

$$\text{the Leibnitz identity } \{f, gh\} = \{f,g\}h + g\{f,h\} \text{ fulfills.} \tag{3}$$

Let $Vect_\omega(M)$ be a Lie algebra of Hamiltonian vector fields on $M$:

$$Vect_\omega(M) = \{X \in Vect(M) : \ X\omega = 0\}.$$

Then the map $\Omega \circ d : \ C^\infty(M) \to Vect_\omega(M)$ is a Lie algebra homomorphism.

2. *Hamiltonian reduction.* Let a Lie group $G$ act on a symplectic manifold $M$ by symplectomorphisms, i.e. by diffeomorphisms preserving the symplectic form $\omega$. Then there arises a morphisms of Lie algebras: $\mathfrak{g} \to Vect_\omega(M)$, where $\mathfrak{g}$ is a Lie algebra of $G$.

The action of $G$ on $M$ is called *Poisson* if in addition the latter morphism lifts to a Lie algebra morphism $\mathfrak{g} \to C^\infty(M)$. Denote by $H_a \in C^\infty(M)$, $a \in \mathfrak{g}$, the image of $a$ in $C^\infty(M)$, i.e. the Hamiltonian function corresponding to an infinitesimal action $a$.

Denote by $\mathfrak{g}^*$ a space dual to $\mathfrak{g}$. The group $G$ naturally acts on $\mathfrak{g}^*$ (via coadjoint action). For any Poisson action action of $G$ on $M$ there arises a $G-$equivariant mapping called *momentum map*:

$$p : \ M \to \mathfrak{g}^*, \ < p(x), a >= H_a(x), \text{ for } x \in C^\infty(M), a \in \mathfrak{g}.$$

Fix $a \in \mathfrak{g}^*$ and denote by $G_a \subset G$ its stabilizer. Obviously, the set $M_a = p^{-1}(a)$ is preserved by $G_a$. Assume now that, first, $M_a$ is a manifold and that, second, so is the quotient space $F_a = M_a/G_a$. One can show that $F_a$ is a symplectic manifold with respect to the symplectic form $\bar{\omega}$ defined by setting

$$\bar{\omega}(\bar{\xi}, \bar{\eta}) = \omega(\xi, \eta),$$

where $\xi$ and $\eta$ are arbitrary preimages of $\bar{\xi}$ and $\bar{\eta}$ with respect to the natural projection $TM_a \to TF_a$ (see [18]).

The described passage from a symplectic manifold $M$ to a symplectic manifold $F_a$ of dimension less than dimension of $M$ by $2\, dimG_a$ is called (noncommutative) *Hamiltonian reduction*

**Example 2.1** *$T^*M$ is canonically a symplectic manifold for any $M$. $T^*G$ is a symplectic manifold with a Poisson action of $G$ by left translations. The momentum $T^*G \to \mathfrak{g}^* \approx T_e^*G$ is given by right translations to the unit (e) of the group $G$. The result of the hamiltonian reduction with respect to the element $a \in \mathfrak{g}$ is the orbit $\mathcal{O}_a$ of $a$ in the coadjoint representation equipped with the celebrated Lie–Poisson–Kirillov–Kostant symplectic form.*

3. *Symplectic leaves.* We saw above that for any symplectic manifold $M$ its algebra of functions is a Poisson algebra. More generally, $M$ is called a Poisson manifold if $C^\infty(M)$ is a Poisson algebra with respect to a certain bracket $\{.,.\}$.

The bracket $\{.,.\}$ determines a Lie algebra morphism $C^\infty(M) \to Vect(M)$. It follows that $\{.,.\}$ can be regarded as a fiberwise linear map $\Omega: \ T^*M \to TM$, where $\Omega$ already does not necessarily come from a symplectic form (1). Corank of restriction of $\Omega$ to a fiber measures how far $\{.,.\}$ is from being induced by a symplectic structure. A notion of a symplectic form has the following substitute for a generic Poisson manifold.

The assignment $M \ni x \mapsto \Omega(T_x^*M) \subset T_xM$ defines a distribution on $M$. The integral submanifolds of this distribution are called *symplectic leaves* of $M$. One shows that each symplectic leaf is indeed a symplectic manifold (and, therefore, is also a Poisson manifold). A Poisson submanifold is a manifold being a union of symplectic leaves. The embedding of a Poisson submanifold of $M$ into $M$ is a Poisson morphism, meaning that the induced morphism of algebras of functions is a morphism of Poisson algebras.

**Example 2.2** *The dual space $\mathfrak{g}^*$ is a Poisson manifold, the bracket being defined by:*

$$\{f,g\}(x) = < [d_xf, d_xg], x >,$$

*where $f,g \in C^\infty(\mathfrak{g}^*)$, $d_xf$ signifies the value of the differential of a function at the point $x$; therefore $d_xf$, $d_xg \in \mathfrak{g}$, and so the right hand side of the equality is understood as a Lie bracket of a pair of elements of $\mathfrak{g}$. Symplectic leaves of $\mathfrak{g}^*$ are exactly orbits of the coadjoint action, see Example 2.1.*

Suppose a Lie group $G$ acts on a Poisson manifold $M$ by diffeomorphisms preserving the Poisson structure. Such an action is called *Poisson* if, first, it preserves all symplectic leaves and, second, the induced Lie algebra morphism of $\mathfrak{g}$ to $Vect(M)$ lifts to a Lie algebra morphism $\mathfrak{g} \to C^\infty(M)$. In this case one can define a momentum $p: \ M \to \mathfrak{g}^*$ so that its restrictions to the symplectic leaves are exactly momenta of the above discussion. Assuming further that for some $a \in \mathfrak{g}^*$ , $M_a$ and $F_a = M_a/G_a$ are manifolds one shows that $F_a$ is naturally a Poisson manifold and that its symplectic leaves are exactly symplectic manifolds obtained via Hamiltonian reduction applied to symplectic leaves of $M$.

**Example 2.3** *Let $\mathfrak{n}$ be a subalgebra of $\mathfrak{g}$ and $N$ be the Lie group related to $\mathfrak{n}$. A coadjoint action of $N$ on $\mathfrak{g}^*$ is Poisson, the momentum being the natural projection $\mathfrak{g}^* \to \mathfrak{n}^*$. So, $\mathfrak{g}^*/N$ is a Poisson manifold.*

An explicit calculation of the Poisson bracket of a pair of functions on $F_a$ can be carried out as follows. Let $\pi : M_a \to F_a = M_a/G_a$ be the natural projection. Let $f, g \in \mathbf{C}^\infty(F_a)$. Then $\pi^* f$, $\pi^* g$ are functions on $M_a$. Choose an arbitrary continuation of each of the functions on the entire $M$ and denote it also $\pi^* f$, $\pi^* g$. Set

$$\{f, g\}(\pi(x)) = \{\pi^* f, \pi^* g\}(x), \text{ for any } x \in M_a. \tag{4}$$

What was said above is enough to prove that, despite the obvious ambiguities in this definition, the result is uniquely determined .

## 2.2 Drinfeld-Sokolov construction

Drinfeld-Sokolov reduction is the procedure outlined in 2.1.3, especially in Example 2.3, in the case when $\mathfrak{g}$ is an affine Lie algebra and $\mathfrak{n}$ is its "nilpotent" subalgebra. To make a precise statement let us fix the following notations:

$\mathbf{C}[z, z^{-1}]$ is a ring of Laurent polynomials, $\mathbf{C}[[z]]$ is a ring of formal power series, $\mathbf{C}((z)) = \mathbf{C}[z, z^{-1}] + \mathbf{C}[[z]]$;

$\mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{n} \in \mathfrak{g}$ is the subalgebra of strictly upper triangular matrices;

$\mathfrak{a}(z) = \mathfrak{a} \otimes \mathbf{C}((z))$ for any Lie algebra $\mathfrak{a}$; $\mathfrak{a}(z)$ is called a *loop algebra*; its elements can be thought of as "formal" functions of $z \in \mathbf{C}^*$ with values in $\mathfrak{a}$;

$\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{C}$ is the corresponding affine Lie algebra, the universal central extension of $\mathfrak{g}$ by the cocycle being given by $\phi(f(z), g(z)) = Res_{z=0} Tr(f(z) dg(z))$;

$A$, $N$, $G$, $A(z)$, $N(z)$, $G(z)$, $\hat{G}$ are Lie groups related to $\mathfrak{a}, \mathfrak{n}, \mathfrak{g}, \mathfrak{a}(z)$, $\mathfrak{n}(z), \mathfrak{g}(z)$, $\hat{\mathfrak{g}}$.

The dual space $\mathfrak{g}(z)^*$ is naturally isomorphic to $\mathfrak{g}(z)$ by means of the invariant inner product ("Killing form")

$$(f(z), g(z)) = Res_{z=0} Tr f(z) g(z) z^{-1}.$$

The dual space $\hat{\mathfrak{g}}^* = \mathfrak{g}((z)) \oplus \mathbf{C}$ can be identified with the space of 1st order linear differential operators on the circle with matrix $(n \times n)$ coefficients $DO_{n\times n}$.

The correspondence $\hat{\mathfrak{g}}^* \to DO_{n\times n}$ is established by

$$(f(z), k) \mapsto -kz\frac{d}{dz} + f(z). \tag{5}$$

**Proposition 2.4** *([9], [19]) The identification above makes the coadjoint action of the group $G[z, z^{-1}]$ on $\hat{\mathfrak{g}}^*$ into the gauge action on differential operators:*

$$T(z) \cdot (f(z), k) = (-kzT(z)'T(z)^{-1} + T(z)f(z)T(z)^{-1}, k),$$

*here and elsewhere "$'$ "means the application of the operator $d/dz$.*

**Remark 2.5** *The operators above can be viewed as differential operators on the circle $z = \frac{1}{\sqrt{-1}} \exp(\sqrt{-1}\tau)$:*

$$k\frac{d}{d\tau} + f(\frac{1}{\sqrt{-1}} \exp(\sqrt{-1}\tau)). \tag{6}$$

*Solutions to differential equations with matrix coefficients are vector functions. Natural action of $G[z, z^{-1}]$ on solutions induces the gauge action of $G$ on differential operators.*

Further we fix a hyperplane in $\hat{\mathfrak{g}}$ by fixing a cocentral term: $\hat{\mathfrak{g}}_1^* = \{(f(z), 1),\ f(z) \in \mathfrak{g}\}$. Obviously, $\hat{\mathfrak{g}}_1^* \subset \hat{\mathfrak{g}}^*$ is a Poisson submanifold: the Lie-Poisson bracket on the dual space $\hat{\mathfrak{g}}_1^*$ admits restriction to the hyperplane.

The coadjoint action of the subgroup $\mathfrak{n}$ on $\hat{\mathfrak{g}}_1^*$ is Poisson. Consider its momentum map

$$p : \hat{\mathfrak{g}}_1^* \to \mathfrak{n}^*.$$

If the space $\mathfrak{n}^*$ is identified with lower triangular matrices then the momentum map $p$ is nothing but the projection of (functions with values in) matrices onto their lower-triangular parts.

To start Hamiltonian reduction we need to fix a point in the image of the momentum map. Regard the (lower triangular) matrix

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{7}$$

as an element of $\mathfrak{n}^*$. The preimage $p^{-1}(\Lambda)$ is a manifold. It is, in fact, an affine subspace: $p^{-1}(\Lambda) = z\,d/dz + \Lambda + \mathfrak{b}((z))$, where $\mathfrak{b} \in \mathfrak{g}$ is the subalgebra of upper triangular matrices.

To perform the second step of the reduction we notice that the quotient space $p^{-1}(\Lambda)/N[z,z^{-1}]$ is also a manifold. Indeed, one can show that each $N[z,z^{-1}]$–orbit contains one and only one element of the form

$$z\frac{d}{dz} + \begin{pmatrix} b_1(z) & b_2(z) & b_3(z) & \dots & b_{n-1}(z) & b_n(z) \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

The space of first order differential operators of this form ( the corresponding matrices are sometimes called Frobenius matrices) is in 1-1 correspondence with (ordinary scalar) differential operators $DO_n$ of order $n$ on the circle:

$$z\frac{d}{dz} + \begin{pmatrix} b_1(z) & b_2(z) & b_3(z) & \dots & b_{n-1}(z) & b_n(z) \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \leftrightarrow \frac{d^n}{d\tau^n} + \bar{b}_1(\tau)(\frac{d^{n-1}}{d\tau^{n-1}}) + \cdots + \bar{b}_n(\tau),$$

where $\bar{b}_k(\tau) = b_k(\exp(\sqrt{-1}\tau)/\sqrt{-1}),\ \tau \in \mathbf{R}$.

The Hamiltonian reduction of Sect.2.1.3 of the Kirillov–Kostant structure on $\hat{\mathfrak{g}}_1^*$ equips $DO_n$ with a structure of a Poisson manifold.

On the other hand the space $DO_n$ is known to carry a Poisson structure - the celebrated "second Gelfand-Dickey structure".

**Theorem 2.6** *[6] The Poisson structure on $DO_n$ obtained as a result of the Hamiltonian reduction described above is the second Gelfand-Dickey structure.*

It is appropriate at this point to give a definition of the second Gelfand-Dickey bracket in the way Adler, Gelfand and Dickey did it, i.e. by explicit formulas. We will, however, refrain from doing so and instead describe a generalization of this construction to the case when $n$ is an arbitrary complex number.

# 3 Differential operators of complex order and matrices of complex size.

## 3.1 Poisson-Lie group of pseudodifferential symbols

In this section we describe the main underlying structure, the Gelfand–Dickey bracket on the group of pseudodifferential symbols of complex powers following [12] (see also [7] for related questions).

Points of the Poisson manifold under consideration are classical pseudodifferential symbols, i.e. formal Laurent series of the following type:

$$G = \{D^{\lambda} + \sum_{k=-\infty}^{-1} u_k(z) D^{\lambda+k} | u_k \in \mathbf{C}((z)), \lambda \in \mathbf{C}\}.$$

This expression is to be understood as a convenient written form for a semi-infinite sequence of functions $\{u_k\}$.

This (infinite dimensional) manifold can be equipped with a group structure, where product of two such symbols is a generalization of the Leibnitz rule $D \circ f(z) = f(z)D + f'(z)$ (that explains the meaning of the symbol $D = d/dz$). For an arbitrary (complex) power of $D$ one has:

$$D^{\lambda} \circ f(z) = f(z) D^{\lambda} + \sum_{\ell \geq 1} \binom{\lambda}{\ell} u^{(\ell)}(x) D^{\lambda-\ell} \tag{8}$$

where $\binom{\lambda}{\ell} = \frac{\lambda(\lambda-1)\cdots(\lambda-\ell+1)}{\ell!}$. The number $\lambda$ is called the order of a symbol. It is easy to see that every coefficient of the product of two symbols is a differential polynomial in coefficients of the factors.

**Definition.** The (quadratic generalized) Gelfand-Dickey Poisson structure on $G = \{L = (1 + \sum_{k=-\infty}^{-1} u_k(x) D^k) D^{\lambda}\}$ is defined as follows:

a) The value of the Poisson bracket of two functions at the given point is determined by the restrictiong of these functions to the subset $\Psi DS_{\lambda}$ of symbols of fixed order $\lambda = const$.

b) The subset $\lambda = const$ is an affine space, so we can identify the tangent space to this subset with the set of operators of the form $\delta L = (\sum_{k=-\infty}^{-1} \delta u_k D^k) \circ D^{\lambda}$.

We can also identify the cotangent space with the space of operators of the form $X = D^{-\lambda} \circ DO$, where $DO$ is a purely differential operator (i.e. polynomial in $D$) using the following pairing:

$$F_X(\delta L) := < X, \delta L >= Tr(\delta L \circ X).$$

Here the product $\delta L \circ X$ is a symbol of an integer order $\sum p_k(z) D^k$, and its trace $Tr$ is defined as the residue at $z = 0$ of $p_{-1}(z)$.

c) Now it is sufficient to define the bracket on linear functionals, and

$$\{F_X, F_Y\}\Big|_L = F_Y(V_{F_X}(L)), \tag{9}$$

where $V_{F_X}$ is the following Hamiltonian mapping $F_X \mapsto V_{F_X}(L)$ (from the cotangent space $\{X\}$ to the tangent space $\{\delta L\}$):

$$V_{F_X}(L) = (LX)_+ \ L - L(XL)_+ \ . \tag{10}$$

**Remark 3.1** *Usually this definition is given only in the case when $\lambda$ is a fixed positive integer and $L$ is a differential operator ($L_+ = L$, here and above "+" means taking differential part of a symbol of integral order), cf.[1], [5]. The set $DO_n$ of purely differential operators of order $n$ is a Poisson submanifold in the Poisson "hyperplane" $\Psi DS_{\lambda=n}$ of all pseudodifferential symbols of the same order. Indeed, for any operator $L = D^n + u_{-1}(z)D^{n-1} + ... + u_{-n}(z)$ and an arbitrary symbol $X = DO \circ D^{-n}$ the corresponding Hamiltonian vector $V_{F_X}(L) = (LX)_+ \ L - L(XL)_+$ is a differential operator of the order $n-1$ and hence all Hamiltonian fields leave the submanifold $DO_n$ of such operators $L$ invariant.*

*Exactly those Poisson submanifolds arise as a result of Hamiltonian reduction in the Drinfeld-Sokolov construction. For an arbitrary (noninteger) $\lambda$ one has no counterparts of "purely differential operators" (what would be the differential part of $u(z)D^{1/2}$ ?) and of the corresponding Poisson submanifolds.*

As we noted, regardless of $\lambda$, there is a natural homeomorphism between $\Psi DS_\lambda$ and semi-infinite sequence of coefficients $\{u_i(z)\}$:

$$\Psi DS_\lambda \approx \prod_{i \geq 1} \mathbf{C}((z)) \tag{11}$$

The Poisson structure on the group induces a family $\{.,.\}_\lambda$ of Poisson structures on $\prod_{i\geq 1}\mathbf{C}((z))$, with polynomial dependence on $\lambda$ (combine formulas (8 - 10)).

The following filtration of the space $\prod_{i\geq 1}\mathbf{C}((z))$ will be used later.

Represent it as $\prod_{i=1}^{k}\mathbf{C}((z))\times\prod_{i\geq k+1}\mathbf{C}((z))$ and let $i_k$ be the projection on the first factor. This gives an embedding of the spaces of functions

$$i_k^*:\ Fun(\prod_{i=1}^{k}\mathbf{C}((z)))\hookrightarrow Fun(\prod_{i\geq 1}\mathbf{C}((z))).$$

Set $W_k=i_k^*(Fun(\prod_{i=1}^{k}\mathbf{C}((z))))$. The sequence $\{W_k\}$ forms the filtration

$$W_1\subset W_2\subset\cdots\subset Fun(\prod_{i=1}^{k}\mathbf{C}((z))),\ \cup_{i\geq 1}W_i=Fun(\prod_{i=1}^{k}\mathbf{C}((z))). \tag{12}$$

**Proposition 3.2** *This filtration satisfies the condition:*

$$\{W_i,W_j\}_\lambda\subset W_{i+j}. \tag{13}$$

*Proof.* Reformulating the statement one needs to extract from the definition above that for $L_0\in\Psi DS_\lambda$ the bracket

$$\{a(z)D^{i-1-\lambda},\ b(z)D^{j-1-\lambda}\}_\lambda(L_0+L_1)$$

does not depend on $L_1$ if $deg\ L_1\leq\lambda-i-j,\ L_0\in\Psi DS_\lambda$.

It follows from the explicit formulas (9-10) :

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\{A,B\}(L_0+\epsilon L_1)=Tr((L_0A)_+\ L_1B-L_0(AL_1)_+B+(L_1A)_+\ L_0B-L_1(AL_0)_+B).$$

Calculation of the degrees shows that the right hand side vanishes under the condition $deg\ L_1\leq\lambda-i-j$. $\Box$

It should be mentioned that an analogous fact is basic for "quantum" counterparts of the Gelfand–Dickey Poisson structures (called quantum $W$–algebras).

**Remarks 3.3** *(i) The Lie group structure of $G$ is compatible with the Gelfand–Dickey structure and makes the group into a Poisson–Lie one (see [12]). Its Lie algebra $\mathfrak{g}=\{\sum_{k=1}^{\infty}u_k(z)D^{-k}+\lambda\cdot\ log\,D|u_k\in\mathbf{C}[z,z^{-1}],\ \lambda\in\mathbf{C}\}$ is a Lie bialgebra. The formal expression $log\,D$ can be regarded as the velocity vector to the one-parameter subgroup $D^\lambda$:*

$$d/d\lambda\Big|_{\lambda=0}\ D^\lambda=log\,D\circ D^\lambda\Big|_{\lambda=0}=log\,D.$$

*and the commutation relation for the log $D$ and any symbol can be extracted from (8)*

$$[log\, D, f(z)D^k] = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} f^{(k)}(z)D^{n-k}.$$

*The dual space to $\mathfrak{g}$ is also a Lie bialgebra. It is nothing but the unique central extension $\hat{DO}$ of the Lie algebra of all differential operators $DO = \{\sum_{j=1}^{n} a_j(z)D^j\}$ on the circle ([11]), known also under the name of $W_{1+\infty}$. The 2-cocycle describing this extension can be given in terms of the outer derivation $[log\, D, *]$:*

$$c(A,B) = \int res\,([log\, D, A] \circ B) \tag{14}$$

*where $A$ and $B$ are arbitrary differential operators (see [15]). The restrictions of this cocycle to the subalgebra of vector fields gives exactly the Gelfand–Fuchs cocycle*

$$c(u(z)D, v(z)D) = \frac{1}{6}\int u''(z)v(z)'dz$$

*which defines the Virasoro algebra.*

*(ii) Introduction of fractional power $D^\lambda$ is a particular (Heisenberg algebra) case of formalism of fractional powers of Lie algebra generators, see [17]. This formalism has been used for purposes of representation theory. It would be interesting to find its Poisson interpretation for other Lie algebras.*

## 3.2 Definition of $\mathfrak{gl}_\lambda$, $\lambda \in \mathbf{C}$, and its extensions

### 3.2.1 What is a matrix of complex size?

Recall that $\mathfrak{sl}_2$ is a Lie algebra on generators $e, h, f$ and relations $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$. There are different ways to embed $\mathfrak{sl}_2$ in $\mathfrak{gl}_n$ and, hence, there are different structures of an $\mathfrak{sl}_2-$module on $\mathfrak{gl}_n$. Generically, however, the structure of an $\mathfrak{sl}_2-$module on $\mathfrak{gl}_n$ is independent of the embedding (see [13]) and is given by

$$\mathfrak{gl}_n = V_1 \oplus V_3 \oplus \cdots V_{2n-1}, \tag{15}$$

where $V_i$ stands for the irreducible $i-$dimensional $\mathfrak{sl}_2-$module. The image of a generic embedding of $\mathfrak{sl}_2$ in $\mathfrak{gl}_n$ is called a principal $\mathfrak{sl}_2-$triple. We do not discuss what exactly the genericity condition is and confine to mentioning that an example of a generic embedding is provided by sending

$$f \mapsto \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \tag{16}$$

and continuing this map on the entire $\mathfrak{sl}_2$.

In view of the decomposition ( 15) it is natural to ask whether the space $\oplus_{i\geq 0} V_{2i+1}$ admits a Lie algebra structure consistent with the existing structure of an $\mathfrak{sl}_2-$module on it. A construction of a 1-parameter family of such structures is as follows.

The universal enveloping algebra $U(\mathfrak{sl}_2)$ is a Lie algebra with respect to the operation $[a,b] = ab - ba$. The element $C = ef + fe + \frac{1}{2}h^2$ generates the center of $U(\mathfrak{sl}_2)$. The quotient

$$U(\mathfrak{sl}_2)/(C - \frac{1}{2}(\lambda-1)(\lambda+1))U(\mathfrak{sl}_2),\ \lambda \in \mathbf{C}$$

is naturally a Lie algebra containing $\mathfrak{sl}_2$. The fact that its $\mathfrak{sl}_2-$module structure is given by ( 15) is a consequence of much more general results of [14]. We point out that in our case:

$$\text{the elements } h^i e^j,\ h^i f^j \text{ form a basis of the algebra, and} \tag{17}$$

$$\text{the component } V_{2i+1} \text{ is generated as an } \mathfrak{sl}_2\text{-module by } e^i, \tag{18}$$

B.Feigin classified Lie algebra structures on $\oplus_{i\geq 0} V_{2i+1}$; in particular he proved that under a certain natural assumption there are no families of Lie algebra structures on $\oplus_{i\geq 0} V_{2i+1}$ different from the above mentioned, see [8]. One proves (see Remark 3.5 below) that if $\lambda$ is not integral the quotient is the sum of $\mathbf{C}$ and a simple (infinite dimensional) algebra, and if $\lambda = \pm n,\ n \in \{1, 2, ...\}$ then

$$U(\mathfrak{sl}_2)/(C - \frac{1}{2}(\lambda-1)(\lambda+1))U(\mathfrak{sl}_2)$$

contains an ideal and the quotient is isomorphic with $\mathfrak{gl}_n$. For this reason the algebra

$$U(\mathfrak{sl}_2)/(C - \frac{1}{2}(\lambda-1)(\lambda+1))U(\mathfrak{sl}_2)$$

is denoted by $\mathfrak{gl}_\lambda$ for an arbitrary complex $\lambda$. Note that our notations are inconsistent in the sense that $\mathfrak{gl}_\lambda$ is obviously different from the conventional $\mathfrak{gl}_n$ if $\lambda = n$. It is unfortunate but seems unavoidable; we will denote the finite-dimensional algebra by $\mathfrak{gl}_n$ in the sequel.

### 3.2.2 $\mathfrak{gl}_\infty$ and extensions of $\mathfrak{gl}_\lambda$.

Here we define 2 extensions of $\mathfrak{gl}_\lambda$, both being related to $\mathfrak{gl}_\infty$ and one of them incorporating a formal variable.

Fix once and for all an infinite dimensional space $\mathcal{V}$ with a basis $\{v_i,\ i=0,1,2,3,...\}$. This space carries a 1-parameter family of $\mathfrak{sl}_2-$module structures determined by:

$$fv_i = v_{i+1},\ hv_i = (\lambda-1-2i)v_i,\ ev_i = i(\lambda-i)v_{i-1},\ \lambda\in\mathbf{C}. \tag{19}$$

This, of course, makes $\mathcal{V}$ into a Verma module $M(\lambda-1)$. Hence there arises the map

$$U(\mathfrak{sl}_2)\to\mathfrak{gl}_\infty, \tag{20}$$

where $\mathfrak{gl}_\infty$ is the algebra of all linear transformations of $\mathcal{V}$. Direct calculations show that $Cv_i = \frac{1}{2}(\lambda-1)(\lambda+1)v_i$ for any $i$. Therefore (20) factors through to the map

$$\mathfrak{gl}_\lambda\to\mathfrak{gl}_\infty. \tag{21}$$

**Lemma 3.4** *The map (21) is an embedding.*

*Proof.* The map (21) is a morphism of $\mathfrak{sl}_2-$ modules. Therefore it is enough to prove that each irreducible component of $\mathfrak{gl}_\lambda$ is not annihilated by (21). But this is obvious: $V_{2i+1}$ is generated by $e^i$ (see (18)) and $e^i$ is a non-trivial operator on $M(\lambda-1)$. $\square$

From now on we will identify $\mathfrak{gl}_\lambda$ with its image in $\mathfrak{gl}_\infty$. The passage from a specific $\lambda$ to a formal parameter $t$ makes (20) into the map

$$U(\mathfrak{sl}_2)\to\mathfrak{gl}_\infty\otimes\mathbf{C}[t]. \tag{22}$$

$M(\lambda-1)$ is irreducible unless $\lambda=1,2,3,...$ and if the latter condition is satisfied then it contains the unique proper submodule $I_\lambda$ spanned by $v_\lambda, v_{\lambda+1},...$ the corresponding quotient being $V_\lambda$. This along with the definitions implies that the image of $U(\mathfrak{sl}_2)$ under ( 22) consists of matrices $A=(a_{ij}),\ i,j\geq 0$ satisfying the following conditions:

(i) for any matrix $A=(a_{ij}),\ i,j\geq 0$ there exists a number $N$ such that $a_{ij}=0$ if $i>j+N$;

(ii) for any fixed $n$, $a_{i,i+n}$ is a polynomial in $i$;

(iii) if $\lambda = 1, 2, 3, ...$ then $a_{ij}(\lambda) = 0$ once $i < \lambda$ and $j \geq \lambda$. (We naturally identify matrix elements with polynomials in $t$.) In other words, in this case $A$ has the following block form:

$$\begin{pmatrix} B & 0 \\ * & * \end{pmatrix}, \ B \in \mathfrak{gl}_n.$$

**Remark 3.5** *The property (iii) explains why, under the integrality condition $\lambda = n$, $\mathfrak{gl}_\lambda$ "contains" the usual $\mathfrak{gl}_n$: $\mathfrak{gl}_n \approx \mathfrak{gl}_\lambda / J$, where $J = \{A \in \mathfrak{gl}_\lambda : \ Im(A) \subset I_\lambda\}$.*

Denote by $\mathfrak{gl}$ the subalgebra of $\mathfrak{gl}_\infty \otimes \mathbf{C}[t]$ satisfying the property (i) above and the following weekened version of (ii) and (iii):

$(*)$ for any matrix $A \in \mathfrak{gl}$ the properties (ii) and (iii) can only be violated in a finite number of rows.

The algebra $\mathfrak{gl}$ is one of the algebras we wanted to define. Definition of the other is based on the following general notion which will be of use later.

Let $W$ be a vector space and $A$ a subset of $W \otimes \mathbf{C}[t]$. The image of $A$ in $W$ under the evaluation map induced by projection $\mathbf{C}[t] \to /(t-\epsilon)\mathbf{C}[t] \approx \mathbf{C}$, $\epsilon \in \mathbf{C}$, will be denoted by $A_\epsilon$ and called *specialization.*

We now define $\bar{\mathfrak{gl}}_\lambda$ to be a specialzitaion of $\mathfrak{gl}$ when $t = \lambda$. The following is an alternative description of $\bar{\mathfrak{gl}}_\lambda$ (it will not be used in the sequel): $\bar{\mathfrak{gl}}_\lambda$ is obtained from $\mathfrak{gl}_\lambda$ by, first, allowing infinite series of the form $\sum_{i \geq 0} a_i e^i$, $a_i \in \mathbf{C}[h]$, (see (17)) and, second, extending the result by the ideal of operators with finite-dimensional image.

### 3.2.3 Affinization and Coadjoint Representation of $\bar{\mathfrak{gl}}_\lambda$

1. *Trace.* The following simple and crucial construction was communicated to us by J.Bernstein. Observe that for any $A = (a_{ij}) \in \bar{\mathfrak{gl}}_\lambda$ the sum

$$P(A, N) = \sum_{i=0}^{N-1} a_{ii}$$

is a polynomial in $N$ ( this is a consequence of $(*)$). Set

$$Tr\, A = P(A, \lambda). \tag{23}$$

It follows that both $\bar{\mathfrak{gl}}_\lambda$ and the loop algebra $\bar{\mathfrak{gl}}_\lambda(z) = \bar{\mathfrak{gl}}_\lambda \otimes \mathbf{C}((z))$ carry an invariant non-degenerate inner product defined by

$$(A, B) = Tr\, AB,$$

$$(A(z), B(z)) = Res_{z=0} Tr\, A(z)B(z)z^{-1}dz\ (resp.).$$

Observe that the restriction of the trace to $\mathfrak{gl}_\lambda$ and $\mathfrak{gl}_\lambda(z)$ is degenerate if $\lambda$ is a positive integer.

2. *Affinization and coadjoint representation* The loop algebra $\bar{\mathfrak{gl}}_\lambda(z)$ admits a central extension determined by the cocycle

$$< A(z), B(z) >= Res_{z=0}\, Tr\, A(z)' B(z)\, dz. \tag{24}$$

This provides the central extension $\hat{\mathfrak{gl}}_\lambda = \bar{\mathfrak{gl}}_\lambda(z) \oplus \mathbf{C} \cdot c$.

Using trace we make identifications $(\bar{\mathfrak{gl}}_\lambda)^* \approx \bar{\mathfrak{gl}}_\lambda$, $(\bar{\mathfrak{gl}}_\lambda(z))^* \approx \bar{\mathfrak{gl}}_\lambda(z)$, $(\hat{\mathfrak{gl}}_\lambda)^* \approx \bar{\mathfrak{gl}}_\lambda(z) \oplus \mathbf{C}$ and extract subspaces $(\hat{\mathfrak{gl}}_\lambda)^*_k \subset (\hat{\mathfrak{gl}}_\lambda)^*$, $k \in \mathbf{C}$, where $(\hat{\mathfrak{gl}}_\lambda)^*$ is all functionals equal to $k$ on the central element $c$. The third identification implies that elements of $(\hat{\mathfrak{gl}}_\lambda)^*_k$ are pairs $(A(z), k)$, $A(z) \in \bar{\mathfrak{gl}}_\lambda(z)$. It follows from the definitions that

$$ad^*_{x(z)}(A(z), k)) = ([X(z), A(z)] - kzX(z)', k) \tag{25}$$

3. *Nilpotent subalgebra and subgroup.* Let $\mathfrak{n}_\lambda \subset \bar{\mathfrak{gl}}_\lambda$ be a subalgebra of strictly upper triangular matrices and $\mathfrak{n}_\lambda(z)$ the corresponding loop algebra. Set $N_\lambda(z) = id \oplus \mathfrak{n}_\lambda(z)$, where $id$ stands for the identity operator.

**Lemma 3.6** *$N_\lambda(z)$ is a group and the map*

$$\exp: \ \mathfrak{n}_\lambda(z) \to N_\lambda(z)$$

*is a homeomorphism.*

*Proof* is an easy exercise.

Exponentiating ( 25) one obtains that the coadjoint action of the group $N(z)$ is given by

$$Ad^*_{X(z)}((A(z), k)) = (-zkX(z)'X(z)^{-1} + X(z)A(z)X(z)^{-1}, k). \tag{26}$$

## 3.3 Drinfeld-Sokolov reduction on $\hat{\mathfrak{gl}}_\lambda$

The general theory (see sect. 2.1.3 and Lemma 26) give the following:

(i) $(\hat{\mathfrak{gl}}_\lambda)^*$ and $(\mathfrak{n}_\lambda(z))^*$ are Poisson manifolds;

(ii) action of $N_\lambda(z)$ on $(\hat{\mathfrak{gl}})^*$ is Poisson;

(iii)the natural projection (momentum)

$$p_\lambda : (\hat{\mathfrak{gl}}_\lambda)^* \to (\mathfrak{n}_\lambda(z))^*,$$

is Poisson and $N_\lambda(z)$-equivariant.

Analogously to what we did above (see sect. 2.2), cosider the matrix

$$f = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \end{pmatrix}$$

as an element of $(\mathfrak{n}_\lambda(z))^*$. ( To justify the notation observe that from the $\mathfrak{sl}_2$—point of view the matrix above is simply the image of $f \in \mathfrak{sl}_2$ in $\mathfrak{gl}_\infty$.) Restrict the momentum $p_\lambda$ to $(\hat{\mathfrak{gl}}_\lambda)^*_1$. It is obvious that, firstly, $p_\lambda^{-1}(f) \approx f + \mathfrak{b}_\lambda(z)$, where $\mathfrak{b}_\lambda(z)$ is the sublagebra of uppertriangular matrices, and, secondly, that $N_\lambda(z)$ is the stabilizer of $f$. Hence there arises the quotient space $p_\lambda^{-1}(f)_\lambda / N_\lambda(z)$, $\lambda \in \mathbf{C}$.

**Proposition 3.7** *(i) Each $N_\lambda(z)-$orbit in $p_\lambda^{-1}(f)$ contains one and only one Frobenius matrix, i.e. an element of the form*

$$\begin{pmatrix} b_1(z) & b_2(z) & b_3(z) & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ . & . & . & . \end{pmatrix}.$$

*(ii) Elements of $N_\lambda(z)$ have no fixed points on $p_\lambda^{-1}(f)$.*

*(iii) For any $\lambda$ the quotient space $p_\lambda^{-1}(f)/N_\lambda(z)$ is isomorphic to the direct product $\prod_{i\geq 1} \mathbf{C}((z))$ equipped with the topology of projective limit.*

Again the general theory says that $p_\lambda^{-1}(f)/N_\lambda(z)$ is a Poisson manifold with the Poisson structure being reduced from the Kirillov–Kostant structure on $(\hat{\mathfrak{gl}}_\lambda)^*$. It is naturally isomorphic to $\Psi DS_\lambda$ as a topological space, the isomorphism being independent of $\lambda$ (c.f.

**Theorem 3.8** *The spaces $p_\lambda^{-1}(f)/N_\lambda(z)$ (equipped with the reduced Poisson structure) and $\Psi DS_\lambda$ (equipped with the quadratic Gelfand–Dickey structure) are isomorphic as Poisson manifolds for any $\lambda$.*

**Remark 3.9** *The (finite dimensional)* $\mathfrak{gl}_n$*-quotient of the algebra* $\mathfrak{gl}_\lambda$ *(for integral* $\lambda = n$*) over the maximal ideal* $J$ *corresponds precisely (via affinization and the classical Drinfeld–Sokolov construction) to Poisson submanifolds* $DO_n$ *of purely differential operators in the affine space* $\Psi DS_\lambda$*. Indeed, functions vanishing on a Poisson submanifold form an ideal in the Lie algebra of functions on the entire Poisson manifold. The corresponding quotient is nothing else but the Poisson algebra of functions on the submanifold.*

**Remark 3.10** *The first (linear) Gelfand–Dickey structure is defined by the formula* $V_A(L) = (LA - AL)_+$*. Unlike the second (quadratic) structure above the first one exists not on the entire group of* $\Psi DS$*, but only on the subspaces of integral degree* $\lambda$ *([6]). Drinfeld–Sokolov reduction represents the linear Poisson structure on scalar differential operators of* $n^{th}$ *order as the reduction of a constant Poisson structure on* $\hat{\mathfrak{gl}}_n^*$ *(i.e. on first order matrix differential operators). This constant Poisson structure on the dual space is obtained by the freezed argument principle applied to the Kirillov–Kostant structure at the point* $(0, e_{1n}) \in \hat{gl}_n^*$*. Here 0 is the coefficient at* $z\frac{d}{dz}$*, and* $e_{1n}$ *is the current on* $S^1$ *with the only nonvanishing entry equal 1 at (1,n)-place.*

*One can literally repeat the arguments for the Hamiltonian reduction from* $\hat{\mathfrak{gl}}_\lambda$*. Then the finite matrix* $e_{1\ n}$ *is to be replaced by an infinite matrix, an element of* $\hat{\mathfrak{gl}}_\lambda^*$ *with the only nonvanishing entry at the same place (1,n). This entry is singled out by the block structure of* $\mathfrak{gl}_\lambda$ *for integer* $\lambda = n$*. However, for a generic* $\lambda$ *no such element is specified, and no linear Poisson structure exists on the spaces* $\Psi DS_\lambda$ *after reduction.*

# 4 Proofs

## 4.1 Affinization and Coadjoint Representation of $\mathfrak{gl}$

Proposition 3.7 says what canonical form of a matrix under the action of the group $N_\lambda(z)$ is. In our case, as it sometimes happens, it is easier to find a canonical form of a family of matrices than to do so with a single matrix. In order to realize this program we need to extend some of the above introduced notions to the case of the algebra $\mathfrak{gl}$.

### 4.1.1 Affinization

The algebra $\mathfrak{gl}$ (incorporating the formal variable $t$) admits the trace: observe that for any $A = (a_{ij})$ the expression

$$P(A, N) = \sum_{i=0}^{N-1} a_{ii}$$

is a polynomial on $N$ and set

$$Tr\, A = P(A, t).$$

It follows that both $\mathfrak{gl}$ and the loop algebra $\mathfrak{gl}(z) = \mathfrak{gl} \otimes \mathbf{C}((z))$ carry an invariant non-degenerate $\mathbf{C}[t]-$valued inner product defined by

$$(A, B) = Tr\, AB,$$

$$(A(z), B(z)) = Res_{z=0} Tr\, A(z)B(z)z^{-1}dz\ (resp.).$$

The loop algebra $\mathfrak{gl}(z)$ admits the central extension determined by the cocycle

$$< A(z), B(z) >= Res_{z=0}\, Tr\, A(z)'B(z)\, dz. \tag{27}$$

This provides the central extension $\hat{\mathfrak{gl}} = \mathfrak{gl}(z) \oplus \mathbf{C}[t]$.

### 4.1.2 Coadjoint Representation

Let as usual $\mathbf{C}[[t]]$ be the ring of formal power series, $\mathbf{C}((t^{-1})) = \mathbf{C}[t, t^{-1}] + \mathbf{C}[[t^{-1}]]$. $\mathbf{C}((t^{-1}))$ is a $\mathbf{C}[t]$-module and $\mathbf{C}[t] \subset \mathbf{C}((t^{-1}))$ its $\mathbf{C}[t]$-submodule. We identify $t^{-1}\mathbf{C}[[t^{-1}]]$ with the quotient $\mathbf{C}((t^{-1}))/\mathbf{C}[t]$. This equips $t^{-1}\mathbf{C}[[t^{-1}]]$ with a $\mathbf{C}[t]$-module structure coming from $\mathbf{C}((t^{-1}))/\mathbf{C}[t]$.

Existence of a nondegenerate invariant $\mathbf{C}[t]$-valued inner product on $\mathfrak{gl}$ gives that

$$\mathfrak{gl}^* \approx t^{-1}\mathbf{C}[[t^{-1}]]/\mathbf{C}[t] \otimes_{\mathbf{C}[t]} \mathfrak{gl}. \tag{28}$$

The isomorphism is established by assigning to the pair $(g(t), A(t)),\ g(t) \in t^{-1}\mathbf{C}[[t^{-1}]], A(t) \in \mathfrak{gl}$ a functional by the formula

$$< g(t)A(t), B(t) >= Res_{t=0} g(t) Tr A(t)B(t).$$

Similarly, $\mathfrak{gl}(z)^* \approx t^{-1}\mathbf{C}[[t^{-1}]] \otimes_{\mathbf{C}[t]} \mathfrak{gl}((z))$ and $(\hat{\mathfrak{gl}})^* \approx t^{-1}\mathbf{C}[[t^{-1}]] \otimes_{\mathbf{C}[t]} \mathfrak{gl}((z)) \oplus t^{-1}\mathbf{C}[[t^{-1}]]$, where element $(0, g(t))$ sends $(A(t, z), h(t))$ to $Res_{t=0} g(t)h(t)$.

For any $g(t) \in t^{-1}\mathbf{C}[[t^{-1}]]$ set

$$(\hat{\mathfrak{gl}})^*_{g(t)} = \{(g(t)A(t,z), g(t)),\ A(t,z) \in \mathfrak{gl}((z)).$$

It is tempting to say that the dual space $\hat{\mathfrak{gl}}^*_{g(t)}$ for a fixed $g(t)$ is in one-to-one correspondence with $\mathfrak{gl}((z))$. At least there is a map

$$\mathfrak{gl}((z)) \to \hat{\mathfrak{gl}})^*_{g(t)},\ A(t,z) \mapsto (g(t)A(t,z), g(t)). \tag{29}$$

Properties of this map, however, essentially depend on the properties of $g(t)$. Call an element of $t^{-1}\mathbf{C}[[t^{-1}]]$ *rational* if it is equal to Laurent expansion at $\infty$ of a rational function of $t$; otherwise an element of $t^{-1}\mathbf{C}[[t^{-1}]]$ is called *irrational* .

**Lemma 4.1** *(i) If $g(t)$ is irrational then the map ( 29) is a one-to-one correspondence.*

*(ii) Let $g(t) = p(t)/q(t)$ for some mutually prime $p(t), q(t) \in \mathbf{C}[t]$. Then the map is a surjection with "kernel" equal to the set of all matrices with entries divisible by $q(t)$.*

*Proof.* View elements of $(\hat{\mathfrak{gl}})^*_{g(t)}$ as matrices with coefficients in $g(t)\mathbf{C}[t]/(g(t)\mathbf{C}[t] \cap \mathbf{C}[t])$ ( see ( 28)). Such a matrix determines the zero functional if and only if all its entries are equal to 0, Lemma follows. □

The definitions imply that the coadjoint action of $\mathfrak{gl}(z)$ preserves affine subspaces $(\hat{\mathfrak{gl}})^*_{g(t)}$. Lemma 4.1 implies that the space $(\hat{\mathfrak{gl}})^*_{g(t)}$ is always identified with $\mathfrak{gl}((z))$ in the sense that in the case (ii) elements of $\mathfrak{gl}((z))$ have to be viewed as matrices with entries in the quotient ring $\mathbf{C}[t]/q(t)\mathbf{C}[t]$. Having this in mind one obtains that

$$ad^*_{X(t,z)}(A(t,z)) = [X(t,z), A(t,z)] - zX(t,z)',\ A(t,z) \in (\hat{\mathfrak{gl}})^*_{g(t)}. \tag{30}$$

The specialization map $\hat{\mathfrak{gl}} \to \hat{\mathfrak{gl}}_\lambda$ induces emebddings

$$(\hat{\mathfrak{gl}}_\lambda)^*_k \hookrightarrow (\hat{\mathfrak{gl}})^*.$$

Direct calculations show that in fact

$$(\hat{\mathfrak{gl}}_\lambda)^*_1 \hookrightarrow (\hat{\mathfrak{gl}})^*_{1/(t-\lambda)}, \tag{31}$$

where $1/(t-\lambda)$ is viewed as a series $\sum_{i\geq 0} \lambda^i/t^{i+1}$.

The left dual to ( 31) is as follows

$$(\frac{1}{t-\lambda}A(t,z), \frac{1}{t-\lambda}) \mapsto A(\lambda, z). \tag{32}$$

Indeed, Lemma 4.1 implies that $\hat{\mathfrak{gl}}^*_{1/t-\lambda}$ is in one-to-one correspondence with $\mathfrak{gl}((z))$ modulo the relation $t \approx \lambda$; this produces the desired evaluation map.

In exactly the same way as in the $\bar{\mathfrak{gl}}_\lambda-$case one defines the nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{gl}$, the corresponding loop algebra $\mathfrak{n}(z)$, the corresponding group $N(z) = id \oplus \mathfrak{n}(z)$ and proves that this group is exponential.

Exponentiating ( 30) one obtains that the coadjoint action of the group $N(z)$ is given by

$$Ad^*_{X(t,z)}(A(t,z)) = -zX'X^{-1} + XA(t,x)X^{-1},\ X \in N[z,z^{-1}],\ A(t,x) \in (\hat{\mathfrak{gl}})_{g(t)}, \tag{33}$$

where, as always, if the assumption of Lemma 4.1 (ii) is satisfied, then all matrix entries are considered modulo the relation $q(t) \approx 0$. In particular, when $q(t) = t - \lambda$ one obtains the coadjoint action of $N_\lambda[z, z^{-1}]$ on $(\hat{\mathfrak{gl}}_\lambda)^*_1$. This also means that the embedding ( 31) is $N(z)$-equivariant, where $N(z)$ operates on $(\hat{\mathfrak{gl}}_\lambda)^*_1$ via the evaluation map $N(z) \to N_\lambda(z)$

#### 4.1.3 Conversion of a matrix to the Frobenius form – Proof of Proposition 3.7

Fix $g(t) \in t^{-1}\mathbf{C}[[t^{-1}]]$. Consider the natural projection

$$p : (\hat{\mathfrak{gl}})^* \to (\mathfrak{n}(z))^*$$

and denote its restriction to the subspace $\hat{\mathfrak{gl}}^*_{g(t)}$ by $p_{g(t)}$. As above, cosider the matrix

$$f = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \end{pmatrix}$$

as an element of $(\mathfrak{n}(z))^*_{g(t)}$. The following is a natural generalization of Proposition 3.7.

**Proposition 4.2** *(i) Each $N(z)-$orbit in $p^{-1}_{g(t)}(f)$ contains one and only one Frobenius matrix*

$$\begin{pmatrix} b_1(t,z) & b_2(t,z) & b_3(t,z) & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ . & . & . & . \end{pmatrix},$$

*where if $g(t)$ is rational then $b_i(t,z)$ is understood as an element of an appropriate quotient ring.*

*(ii) Elements of $N(z)$ have no fixed points on $p^{-1}_{g(t)}(f)$.*

Items (i) and (ii) of Proposition 3.7 follow from Proposition 4.2 as an easy consequence of properties of maps ( 31, 32). Item (iii) follow from (i) and (ii) because all Frobenius matrices by definition belong to $p_\lambda^{-1}(f)$. The rest of this section is devoted to proving Proposition 4.2.

Suppose for simplicity that $g(t)$ is irrational. Element $X \in N(z)$ converts $A \in \mathfrak{gl}(z)$ to the Frobenius form if and only if the following equation holds

$$-zX' + XA = BX \tag{34}$$

for some Frobenius matrix $B$, see ( 33). We will show that for any $A \in \mathfrak{gl}(z)$, the equation ( 34) can be effectively solved for unknown $X$ and $B$ and that the solution is unique. This is achieved by the following recurrent "diagonalwise" process.

Let $X = (x_{ij}(t,z))$, $A = (a_{ij}(t,z))$ and $B$ be as above. (Note that all matrix entries are "functions" of $z$ and $t$.) By definition, $x_{ii}(t,z) = 1$. Therefore ( 34) for diagonal entries gives

$$x_{01}(t,z) = -b_1(t,z) + a_{00}(t,z),\ a_{i\,i}(t,z) + x_{i\,i+1}(t,z) = x_{i-1\,i}(t,z),\ i \geq 1.$$

This implies that

$$x_{i\,i+1}(t,z) = -\sum_{j=0}^{i} a_{j\,j}(t,z) - b_1(t,z),\ i \geq 1.$$

The condition $(*)$ in the definition of $\mathfrak{gl}$ means that $x_{n-1\,n}(n,z) = 0$ for all sufficiently large positive integers $n$. So,

$$b_1(n,z) = \sum_{j=0}^{n-1} a_{jj}(t,z).$$

But the sum in the last expression is a polynomial in $n$, due to the definition of $\mathfrak{gl}$ , and this uniquely determines $b_1(t,z)$ as a polynomial in $t$.

Suppose we have found $b_1(t,z), \ldots b_{n-1}(t,z)$ and $x_{ij}(t,z)$, $0 \leq i < \infty$, $i < j \leq i+n-1$ for some $n > 1$, so that $x_{i\,i+k}(t,z)$ is a polynomial $k$ for all sufficiently large $k$. Equation ( 34) implies

$$-zx_{0\,n-1}(t,z)' + a_{0\,n-1}(t,z) + x_{01}a_{1\,n-1}(t,z) + \cdots + x_{0\,n-1}a_{n-1\,n-1}(t,z) + \mathbf{x_{0\,n}}$$

$$= b_1(t,z)x_{0\,n-1} + b_2(t,z)x_{1\,n-1} + \cdots + b_{n-1}x_{n-2\,n-1} + b_n,$$

$$-zx_{i\,n+i-1}(t,z)' + a_{i\,n+i-1}(t,z) + x_{i\,i+1}a_{i+1\,n+i-1}(t,z)\cdots x_{i\,n+i-1}a_{n+i-1\,n+i-1}(t,z) + \mathbf{x_{i\,n+i}}$$
$$= x_{i-1\,n+i-1},\ i > 0.$$

As above we see that solving the $i$-th equation for $x_{i\,n+i}$ we obtain

$$x_{i\,n+i}(t,z) = q(i,t,z) + b_n(t,z)$$

for some polynomial $q(i,t,z)$ and all sufficiently large $i$. Again the definition of $\mathfrak{gl}$ implies that $x_{i\,n-i}(n+i,z) = 0$ for all sufficiently large positive integers $n$, and hence $b_n(t,z) = -q(t-n,t,z)$.

The described process shows that for any $A \in p^{-1}_{g(t)}(f)$ there is at most one element of $N(z)$ converting it to the canonical form. It is easy to see that the infinite matrix $(x_{ij})$ calculated above is an element of $N(z)$. Proposition 3.7 follows in the case of irrational $g(t)$.

As to the rational $g(t)$ case, observe that, although the value of an element of the quotient ring at a point does not make sense, vanishing of an element of the quotient ring at a point does make perfect sense in our case (see again property $(*)$ in the definition of $\mathfrak{gl}$), and so, the same conversion process completes the proof. $\square$

**Remark 4.3** *We point out another consequence of the conversion process. For any $\lambda$, $\mathfrak{gl}_n$ embeds naturally into $\bar{\mathfrak{gl}}_\lambda$ by means of matrices with only finite number of non-zero columns and rows. This gives rise to the embedding of the algebra of loops in the upper triangular matrices and of the corresponding loop group. It is easy to see that this embedding is equivariant if $\lambda = N$ for all sufficiently large positive integers $N$. In particular it induces an embedding of the quotients $DO_n \hookrightarrow \Psi DS_\lambda$. Note that the last embedding is not Poisson if $\lambda > n$.*

### 4.1.4 Proof of Theorem 3.8

**A.** Filtrations of $\prod_{i\geq 1}\mathbf{C}((z))$ and $p_\lambda^{-1}(f)$. Recall that there is a filtration (c.f.( 12))

$$W_1 \subset W_2 \subset \cdots \subset Fun(\prod_{i=1}^{k}\mathbf{C}((z))),\ \bigcup_{i\geq 1} W_i = Fun(\prod_{i=1}^{k}\mathbf{C}((z))).$$

Similarly one represents $p^{-1}(f)_\lambda$ as $p^{-1}(f)^k \times p^{-1}(f)^{-k}$, $k \geq 0$, where $p^{-1}(f)^{\pm k}$ is the set of matrices $f + (a_{ij})$, where $a_{ij} = 0$ if $j \leq i+k-1$ ($j > i+k-1$ respectively). Again let $j_k$ be the projection on the first factor and set $U_k = j_k^*(Fun(p^{-1}(f)^k)),\ k \geq 1$. The result is the following filtration

$$U_1 \subset U_2 \subset \cdots \subset Fun(p^{-1}(f)),\ \bigcup_{i\geq 1} U_i = Fun(p^{-1}(f)).$$

Consider the projection $\pi :\ p^{-1}(f)_\lambda \to \prod_{i\geq 1} \mathbf{C}((z))$. The group action is compatible with the filtration and therefore

$$\pi^*(W_i) \subset U_i.$$

**B.** Proof of Theorem 3.8. Let $\{,.,\}$ be the Poisson bracket on $(\hat{\mathfrak{gl}}_\lambda)^*$, $\{.,.\}^\sim_\lambda$ be the Poisson bracket on $\prod_{i\geq 1} \mathbf{C}((z))$ obtained as a result of hamiltonian reduction, $\{.,.\}_\lambda$ be the second Gelfand-Dickey structure on $\prod_{i\geq 1} \mathbf{C}((z))$. We have to show that $\{.,.\}^\sim_\lambda = \{.,.\}_\lambda$ for all $\lambda$.

Let $f \in W_i,\ g \in W_j$. Recall that $\{f,g\}^\sim_\lambda,\ \{f,g\}_\lambda$ are polynomials on $\lambda$. Therefore it is enough to prove that

$$\{f,g\}^\sim_N = \{f,g\}_N$$

for all sufficiently large $N$.

By definition,

$$\{U_i, U_j\} \subset U_{i+j}.$$

Recall also ( see ( 13) ) that

$$\{W_i, W_j\}_\lambda \subset W_{i+j}.$$

The last formula along with compatibility of $\pi$ with the filtrations implies that

$$\{W_i, W_j\}^\sim_\lambda \subset W_{i+j}.$$

It follows that functions $\pi^* f,\ \pi^* g$ and their commutators are uniquely determined by their restrictions to $\mathfrak{gl}_n(z) \in (\hat{\mathfrak{gl}})^*$ (see Remark 4.3) for sufficiently large $n$. Let $x \in \mathfrak{gl}_n(z) \in (\hat{\mathfrak{gl}})^*$. One has

$$\begin{aligned} &\{f,g\}^\sim_N(\pi x) \\ &= \{\pi^* f, \pi^* g\}_N(x) && (35) \\ &= \{\pi^* f|_{\mathfrak{gl}_n}, \pi^* g|_{\mathfrak{gl}_n}\}_N \\ &= \{f,g\}_N(\pi x), && (36) \end{aligned}$$

where (35) follows from ( 4), ( 36) follows Remark 4.3, Theorem 2.6 and ( 4). Theorem 3.8 has been proved. □

# 5 Two more examples of deformation of Poisson structures

Algebra $\mathfrak{gl}_n$ plays a universal role in mathematics for the reason that almost any algebra maps into it. The algebra $\bar{\mathfrak{gl}}_\lambda$ is expected to play a similar role in the deformation theory. Here are a few examples.

## 5.1 Deformation of Drinfeld-Sokolov reduction on orthogonal and symplectic algebras

In this section we construct 2 involutions: one on the space of $\Psi DS_\lambda$, another on the algebra $\mathfrak{gl}_\lambda$, such that the Hamiltonian reduction sends a certain invariant subspace of one to a certain invariant subspace of another.

1. *Gelfand–Dickey* $\mathfrak{sp}, \mathfrak{so}$*-brackets.*

To describe the Gelfand–Dickey structures corresponding to the Lie algebras $\mathfrak{sp}$ and $\mathfrak{so}$ we introduce the following involution $*$ on the set $\Psi DS_\lambda$ of pseudodifferential symbols:

$$(\sum_{k=-\infty}^{0} u_k(z)D^{\lambda+k})^* = \sum_{k=-\infty}^{0} (-1)^k D^{\lambda+k} u_k(z)$$

**Definition.** A pseudodifferential symbol $L$ is called *self-adjoint* if $L^* = L$.

The set of self-adjoint pseudodifferential symbols $\Psi DS_\lambda^{SA}$ can be equipped with the quadratic Poisson structure in the same way as the set $\Psi DS_\lambda$. Having restricted the space of linear functionals to self-adjoint symbols one can use the same Adler–Gelfand–Dickey formula ( 10).

We would like to emphasize that the traditional definition of the $\mathfrak{sp}_{2n}$- ( $\mathfrak{so}_{2n+1}$-) Gelfand–Dickey brackets confines to the case of self-adjoint (skew self-adjoint) genuine differential operators of order $2n$ ($2n+1$, resp.).

2. *The simultaneous deformation of the algebras* $\mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}$.

Define the antiinvolution $\sigma$ of $\mathfrak{sl}_2$ to be the multiplication by $-1$. Observe that $\sigma$ preserves the Casimir element $C = ef + fe + \frac{1}{2}h^2$. Therefore $\sigma$ uniquely extends to an antiinvolution of $\mathfrak{gl}_\lambda$, which will also be denoted by the same letter $\sigma$. It is easy to see that the eigenspace of $\sigma$ related to the eigenvalue $-1$ is a subalgebra. Denote it by $\mathfrak{po}_\lambda$. The family of algebras $\mathfrak{po}_\lambda$, $\lambda \in \mathbf{C}$ is a deformation of both the families $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n+1}$: if $\lambda = 2n$ ($\lambda = 2n+1$) the algebra $\mathfrak{po}_\lambda$ contains $\mathfrak{sp}_{2n}$ ($\mathfrak{so}_{2n+1}$) as a quotient, see [8].

**Remark 5.1** *The algebra $\mathfrak{gl}_\lambda$ is a direct sum of $\mathfrak{sl}_2$-submodules $\oplus_{i\geq 0} V_{2i+1}$. The involution $\sigma$ acts trivially on the subspace $\oplus_{i\geq 0} V_{4i+1}$ and it acts by multiplication by $-1$ on the subspace $\oplus_{i\geq 0} V_{4i+3}$.*

*A direct calculation shows that if the embedding of $\mathfrak{sl}_2$ into $\mathfrak{gl}_n$ is given by the image of $f$ ( 16):*

$$f \mapsto \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

*then the involution $\sigma$ acts on $\mathfrak{gl}_n$ the following way. It transposes a matrix with respect to the "second diagonal" (not the main diagonal, but the other one), and changes the sign of all entries that are situated on every other shortened diagonal counting from the main one: $\sigma(a_{i\ j}) = (-1)^{i-j} a_{n-j-1\, n-i-1}$. Unlike the definition of $\sigma$ we used previously, the latter can not be carried over to the case of $\mathfrak{gl}_\lambda$.*

One can extend the notions of trace, affinization, nilpotent subalgebra etc. to the case of the algebra $\mathfrak{po}_\lambda$.

**Theorem 5.2** *The Hamiltonian reduction of the Kirillov–Kostant Poisson structure on the algebra $\mathfrak{po}_\lambda$ results in the quadratic Gelfand–Dickey structure on the space of pseudodifferential symbols $\Psi DS_\lambda^{SA}$.*

*Proof.* This is an equivariant version of the main theorem 3.8. Again, the result holds in virtue of the finite dimensional analog proved for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$ by Drinfeld and Sokolov ( [6]) and polynomial dependence on $\lambda$. □

**Remark 5.3** *Note that in the approach above it is possible to treat the cases of self-adjoint and skew self-adjoint operators on the same footing.*

## 5.2 Deformation of the Toda lattice

Recall the construction of the classical nonperiodic Toda lattice. Let $E_{ij}$ be the matrix whose only non-zero entry is situated at the intersection of the $i-$th row and $j-$th column and is equal to 1. Let $\mathfrak{b}_n \subset \mathfrak{gl}_n$ be the subalgebra of upper-triangular matrices and $\mathfrak{b}_n^- \subset \mathfrak{gl}_n$ the algebra of lower-triangular matrices. Identify $\mathfrak{b}_n^*$ with $\mathfrak{b}_n^-$ by means of the trace. Set

$$\Lambda_n = \sum_{i=1}^{n-1} E_{i+1\,i} \in \mathfrak{b}_n^*.$$

Let $\mathcal{O}_{\Lambda_n}$ be the orbit of $\Lambda_n$ in the coadjoint representation.

The dynamical system on $\mathcal{O}_{\Lambda_n}$ related to the flow generated by the function $H_2(A) = Tr(A+\Theta)^2$ where $\Theta = \sum_{i=1}^{n-1} i(n-i+1)E_{i\,i+1}$ is called Toda lattice. Note that the element $\Theta$ can be replaced by any linear combination of $E_{i\,i+1}$ with non-zero coefficients; our choice is motivated by the generalization to the $\mathfrak{g}\bar{\mathfrak{l}}_\lambda-$case, see below. The function $H_2$ includes in the family $H_i,\ 2 \leq i \leq n$, where $H_i(A) = Tr(A+\Theta)^i$. Functions $H_i$ are $\mathfrak{gl}_n$-invariant, they Poisson commute as functions on $\mathfrak{gl}_n^* \approx \mathfrak{gl}_n$ and, moreover, their restrictions to $\mathfrak{b}_n^-$ also Poisson commute, see [1]. Calculations show that $dim\mathcal{O}_{\Lambda_n} = 2n-2$. So, the Hamiltonian of the Toda lattice has been included into the family of Poisson commuting functions, the number of functions being equal to half the dimension of the phase space. This proves complete integrability of the Toda lattice. (In fact one also has to establish the independence of the functions, see [1].)

This all immediately carries over to the case of $\mathfrak{g}\bar{\mathfrak{l}}_\lambda,\ \lambda \in \mathbf{C}$:

Let $\mathfrak{b}_\lambda \subset \mathfrak{g}\bar{\mathfrak{l}}_\lambda \subset \mathfrak{gl}_\infty$ be the subalgebra of upper triangular matrices, $\mathfrak{b}_\lambda^- \approx \mathfrak{b}_\lambda^*$ subalgebra of lower triangular matrices.

**Proposition 5.4** *The subalgebra $\mathfrak{b}_\lambda$ can be exponentiated to a Lie group.*

*Proof.* Cf. Lemma 3.6 and the Campbell-Hausdorf formula.

Set

$$\Lambda = \sum_{i=1}^{\infty} E_{i+1i} \in \mathfrak{b}_\lambda^*.$$

(We denoted this element $f$ in sect.3.) Denote by $\mathcal{O}_\Lambda$ the corresponding orbit.

Take the element $\Theta = e \in \mathfrak{g}\bar{\mathfrak{l}}_\lambda$. Now we have infinitely many invariant functions $H_i(A) = Tr(A+\Theta)^i,\ i \geq 2$, their restrictions to $\mathcal{O}_\Lambda$ are again independent and Poisson commute (this is an obvious corollary of the corresponding finite dimensional result). Therefore we have exhibited a family of infinite dimensional integrable dynamical systems, "containing" classical Todda lattices at the points $\lambda = n \in \mathbf{Z}$, and being approximated by the latter as $n \to \infty$.

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