The group and Hamiltonian descriptions of hydrodynamical systems

Boris Khesin*

April 2006

Abstract

We survey applications of the Hamiltonian approach and group theory to ideal fluid dynamics and integrable systems. In particular, we review the derivations of the Landau-Lifschitz and Korteweg-de Vries equations as Euler equations on certain infinite-dimensional groups.

1 Introduction.

In [1] V. Arnold showed that the Euler equation for an ideal incompressible fluid describes the geodesic flow with respect to a right-invariant metric on the corresponding group of volume-preserving diffeomorphisms. Furthermore, he suggested a general framework for Euler-type equations on arbitrary groups, which we recall below. This approach turned out to be surprisingly general: many interesting equations in mathematical physics admit descriptions as geodesic flows with respect to suitable one-sided invariant Riemannian metrics on appropriate groups.

This paper is an extended version of the series of lectures given at the CIME school on Topological Fluid Dynamics. The main reference for many facts discussed below is [1, 3], see also details in [8].

2 Euler equations and geodesics.

2.1 The Euler equation of an ideal incompressible fluid.

We start with the main example, the Euler equation for an ideal fluid. Consider an incompressible fluid occupying a domain M in $\mathbb{R}^n$. The fluid motion is described by a velocity field $v(t,x)$ and a pressure field $p(t,x)$ which satisfy the classical Euler equation:

$$\partial_t v + (v,\nabla)v = -\nabla p, \tag{2.1}$$

*Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3, Canada; e-mail: khesin@math.toronto.edu

where div $v = 0$ and the field v is tangent to the boundary of M. The function p is defined uniquely modulo an additive constant by the condition that v has zero divergence at any moment t.

The geodesic nature of the Euler equation can be seen from the following application of the chain rule. The flow $(t, x) \to g(t, x)$ describing the motion of fluid particles is defined by its velocity field $v(t, x)$:

$$\partial_t g(t, x) = v(t, g(t, x)),\ \ g(0, x) = x.$$

Acceleration of particles is given by

$$\partial_t^2 g(t, x) = (\partial_t v + (v, \nabla)v)(t, g(t, x))\,,$$

according to the chain rule, and hence the Euler equation is equivalent to

$$\partial_t^2 g(t, x) = -(\nabla p)(t, g(t, x)).$$

The latter form of the Euler equation says that the acceleration of the flow is given by a gradient and hence it is L^2-orthogonal to the set of volume-preserving diffeomorphisms, which satisfy the incompressibility condition

$$\det(\partial_x g(t, x)) = 1\,.$$

More precisely, the tangent space to the set of volume-preserving diffeomorphisms (at the identity) consists of divergence-free vector fields. They are L^2-orthogonal to all gradient fields. But this is one of possible definitions of a geodesic curve: it is a curve on a manifold, whose acceleration vector is at every moment orthogonal to the manifold itself. In our case, the curve is the fluid flow, understood as a family of volume-preserving diffeomorphisms, while its acceleration, given by a gradient field, is indeed orthogonal to the set of all such diffeomorphisms. Thus the fluid motion $g(t, x)$ is a geodesic line on the set of volume-preserving diffeomorphisms of the domain M with respect to the induced L^2-metric. Note that this metric is invariant with respect to reparametrizing the fluid particles, i.e. it is *right-invariant* on the set of volume-preserving diffeomorphisms (the reparametrization of independent variable is the *right* action of a diffeomorphism).

A similar equation describes an ideal incompressible fluid filling an arbitrary manifold M equipped with a volume form [1, 5]. The corresponding equation defines the geodesic flow on the group of volume-preserving diffeomorphisms of M. It turns out that the group-geodesic point of view, developed in [1] is quite fruitful for topological and qualitative understanding of the fluid motion, as well as for obtaining various quantitative results related to stability and first integrals of the Euler equation.

2.2 Geodesics on Lie groups.

Consider now the general framework proposed in [1]. Suppose that a configuration space of some physical system is a (possibly infinite-dimensional) Lie group G. For a rigid body

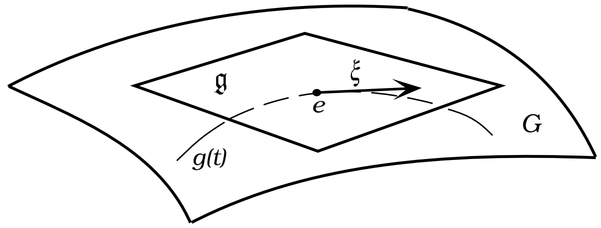

Figure 1: The vector ξ in the Lie algebra $\mathfrak{g}$ is the velocity at the identity e of a geodesic $g(t)$ on the Lie group G.

this group is $SO(3)$, while for ideal hydrodynamics it is the group $\mathrm{SDiff}(M)$ of volume-preserving diffeomorphisms for an ideal fluid filling a domain M. The tangent space at the identity of the Lie group G is the corresponding Lie algebra $\mathfrak{g}$. Fix some (positive definite) quadratic form, called the energy, on the space $\mathfrak{g}$. We consider right translations of this quadratic form to the tangent space at any point of the group (the "translational symmetry" of the energy). This way the energy defines a right-invariant Riemannian metric on the group G. The geodesic flow on G with respect to this energy metric represents the extremals of the least action principle, i.e., the motions of our physical system. (For a rigid body one has to consider left translations.)

To describe a geodesic on the *Lie group* with an initial velocity $v(0) = \xi$, we transport its velocity vector at any moment t to the identity of the group (by using the right translation). This way we obtain the evolution law for $v(t)$, given by a (non-linear) dynamical system $dv/dt = F(v)$ on the *Lie algebra* $\mathfrak{g}$ (Fig.1).

Definition 2.1 *The system on the Lie algebra $\mathfrak{g}$, describing the evolution of the velocity vector along a geodesic in a right-invariant metric on the Lie group G, is called the* Euler *(or* Euler–Arnold*)* equation *corresponding to this metric on G.*

2.3 Geodesic description for various equations.

A similar Arnold-type description as the geodesic flow on a Lie group can be given to a variety of conservative dynamical systems in mathematical physics. Below we list several examples of such systems to emphasize the range of applications of this approach. The choice of a group G (column 1) and an energy metric E (column 2) defines the corresponding Euler equations (column 3).

Group	Metric	Equation
$SO(3)$	$< \omega, A\omega >$	Euler top
$SO(3)\dot{+}\mathbb{R}^3$	quadratic forms	Kirchhoff equations for a body in a fluid
$SO(n)$	Manakov's metrics	$n-$dimensional top
$\mathrm{Diff}(S^1)$	L^2	Hopf (or, inviscid Burgers) equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa − Holm equation
Virasoro	$\dot{H}^1$	Hunter − Saxton (or Dym) equation
$\mathrm{SDiff}(M)$	L^2	Euler ideal fluid
$\mathrm{SDiff}(M)\dot{+}\mathrm{SVect}(M)$	$L^2 + L^2$	Magnetohydrodynamics
$\mathrm{Maps}(S^1, SO(3))$	H^{-1}	Landau − Lifschits equation

In some cases these systems turn out to be not only Hamiltonian, but also bi-Hamiltonian. More detailed descriptions and references can be found in the book [3].

3 Euler equations on groups as Hamiltonian systems and the binormal equation.

3.1 Hamiltonian reformulation of the Euler equations.

The differential-geometric description of the Euler equation as a geodesic flow on a Lie group has a Hamiltonian reformulation. Fix the notation

$$E(v) = \frac{1}{2}\langle v, Av\rangle$$

for the energy quadratic form on $\mathfrak{g}$ which defines the right-invariant Riemannian metric on the group. Identify the Lie algebra and its dual with the help of this quadratic form. This identification $A : \mathfrak{g} \to \mathfrak{g}^*$ (called the *inertia operator*) allows one to rewrite the Euler equation on the dual space $\mathfrak{g}^*$.

Definition 3.1 (e.g. [3]) *Let* ad^*_v *be (the negative of) the* coadjoint operator*, dual to the operator* $[v, .]$ *defining the structure of the Lie algebra* $\mathfrak{g}$*: for any given vector* v *in the Lie algebra* $\mathfrak{g}$*, the operator* $\mathrm{ad}^*_v : \mathfrak{g}^* \to \mathfrak{g}^*$ *acts on the dual space* $\mathfrak{g}^*$ *according to the formula*

$$\langle \mathrm{ad}^*_v \xi, w\rangle := -\langle \xi, [v, w]\rangle$$

for all vectors $w \in \mathfrak{g}$ *and all covectors* $\xi \in \mathfrak{g}^*$*.*

The Euler equation on $\mathfrak{g}^*$*, corresponding to the right-invariant metric* $E(v) = \frac{1}{2}\langle Av, v\rangle$ *on the group, is given by the following explicit formula:*

$$\frac{dm}{dt} = \mathrm{ad}^*_{A^{-1}m} m, \tag{3.2}$$

as an evolution of a point $m \in \mathfrak{g}^*$*.*

Below we explain the meaning of this operator ad^* in the cases of three different equations of mathematical physics.

Remark 3.2 *It turns out that the Euler equation on $\mathfrak{g}^*$ is Hamiltonian with respect to the natural Lie–Poisson structure on the dual space [1]. Moreover, the corresponding Hamiltonian function is the energy quadratic form lifted from the Lie algebra to its dual space by the same identification: $E(m) = \frac{1}{2}\langle A^{-1}m, m\rangle$, where $m = Av$. Above we have taken this Hamiltonian equation as the* definition *of the Euler equation on the dual space $\mathfrak{g}^*$, corresponding to the geodesic flow in the right-invariant metric on the group. For left-invariant metrics the right-hand-side of the equation (3.2) acquires the minus sign: $-\mathrm{ad}^*_{A^{-1}m}m$.*

3.2 Hamiltonian structure of the Landau-Lifschitz equation.

Consider a vector-valued function $L = L(\theta)$ on the circle with values in $\mathbb{R}^3$.

Definition 3.3 *The* Landau-Lifschitz equation, *called also the* Heisenberg magnetic chain equation, *is defined as the following evolution equation on the function L:*

$$\partial_t L = L \times \partial_\theta^2 L\,,$$

This equation can be viewed as the Euler equation for a certain *left-invariant* metric on the loop group $LSO(3) = C^\infty(S^1, SO(3))$ of matrix-valued functions on the circle with values in the group $SO(3)$.

Definition 3.4 *Consider the Lie algebra $\mathfrak{g} = L\mathfrak{so}(3)$ of smooth functions on the circle with values in 3×3 skew-symmetric matrices and the pointwise commutator. The elements of this algebra are naturally identified with vector-valued functions with values in $\mathbb{R}^3$ and whose commutator is given by the cross-product of their values:*

$$[X(\theta), Y(\theta)] = X(\theta) \times Y(\theta)\,.$$

Furthermore, regard the dual space $\mathfrak{g}^*$ as the same space of vector-valued functions with the natural pairing between the spaces:

$$\langle X(\theta), L(\theta)\rangle = \int_{S^1} (X(\theta), L(\theta))\, d\theta$$

for $X \in \mathfrak{g}$, $L \in \mathfrak{g}^*$ and where we used the Euclidean product $(\,,\,)$ in $\mathbb{R}^3$.

Theorem 3.5 *The Landau-Lifschitz equation is the Euler equation corresponding to the left-invariant H^{-1}-metric on the loop group $LSO(3) = C^\infty(S^1, SO(3))$:*

$$E(X) = \frac{1}{2}\int_{S^1} (\partial_\theta^{-1}X, \partial_\theta^{-1}X)\, d\theta\,.$$

Here ∂_θ^{-1} is the integration operator inverse to the differentiation $\partial_\theta := \partial/\partial\theta$ on the space of functions with zero mean on the circle.

Proof. By using integration by parts we rewrite the energy in the form

$$E(X) = \frac{1}{2}\int_{S^1} (\partial_\theta^{-1}X, \partial_\theta^{-1}X)\, d\theta = -\frac{1}{2}\int_{S^1} (\partial_\theta^{-2}X, X)\, d\theta\,,$$

which corresponds to the inertia operator $A = \partial_\theta^{-2}$ (also defined on finctions with zero mean). For the Euler equation we need the inverse of the inertia operator, that is even easier to define:

$$A^{-1}(X) = \partial_\theta^2 X\,.$$

Finally, note that the operator ad^* has the form of the cross product, just like the commutator in the Lie algebra:

$$\mathrm{ad}^*_v L = v \times L.$$

Indeed,

$$\langle \mathrm{ad}^*_v L, w\rangle = -\langle L, [v,w]\rangle = -\int_{S^1} (L, v\times w)\, d\theta = -\int_{S^1} (L\times v, w)\, d\theta = \langle v\times L, w\rangle\,.$$

Combining the above we obtain the Euler equation for the left-invariant H^{-1}-metric (i.e. with the minus sign, see Remark 3.2). It has the Landau-Lifschitz form:

$$\partial_t L = -\mathrm{ad}^*_{A^{-1}(L)} L = L \times \partial_\theta^2 L\,,$$

as required. □

In a similar way one can derive analogues of the Landau-Lifschitz equation for the loops in matrices of higher order. However, the 3×3 case discussed above has several equivalent reformulations and specific 3D features.

The most straightforward reformulation of the Landau-Lifschitz equation is in the form of the binormal equation. Namely, let $\gamma : S^1 \to \mathbb{R}^3$ be a closed immersed curve in $\mathbb{R}^3$ with an arc-length parameter θ.

Definition 3.6 *The* binormal *or* filament equation *is the evolution of the closed curve γ according to the following law:*

$$\partial_t \gamma = \gamma' \times \gamma''\,, \tag{3.3}$$

see Fig.2. (Here, γ' denotes the derivative $\frac{\partial}{\partial\theta}\gamma$.)

We can see that the binormal equation is equivalent to the Landau-Lifschitz equation by setting $L = \gamma'$: we obtain

$$\partial_t L = \partial_t \gamma' = \gamma'' \times \gamma'' + \gamma' \times \gamma''' = \gamma' \times \gamma''' = L \times L''\,.$$

There are several equivalent forms for the binormal equation (3.3), which we discuss in the next section. Note that if θ is not arc-length, then the corresponding Hamiltonian equation becomes

$$\partial_t \gamma = k(\theta,t)(\gamma' \times \gamma'')\,, \tag{3.4}$$

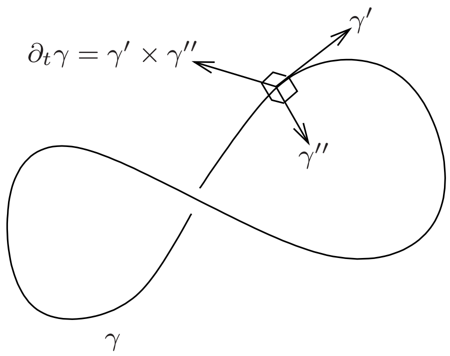

Figure 2: The curve evolution in the direction of the binormal.

where $k(\theta, t)$ is the curvature of the curve at the point θ at time t. This equation is also called da Rios or LIA (Localised Induction Approximation) equation, see a nice survey [13].

3.3 Properties of the binormal equation.

Even without the reference to the Landau-Lifschitz equation, the binormal equation is Hamiltonian with respect to a nice symplectic structure which exists on the space of knots in $\mathbb{R}^3$ with a fixed volume form μ. Consider the space $\mathcal{C}$ whose points are immersed closed curves in $\mathbb{R}^3$. A tangent vector a to the space $\mathcal{C}$ at γ is an infinitesimal variation of the curve γ, that is a normal vector field $a(\theta)$ attached to γ.

Definition 3.7 ([9]) *The* Marsden-Weinstein symplectic structure *is the following 2-form ω_{MW} on the space $\mathcal{C}$. The value of the symplectic form ω_{MW} on two tangent vectors a and b to the space $\mathcal{C}$ at γ is given by the oriented volume of the collar spanned along γ by the vectors $a(\theta)$ and $b(\theta)$:*

$$\omega_{MW}(a, b) := \int_{S^1} \mu(a(\theta), b(\theta), \gamma'(\theta))d\theta = \int_\gamma \iota_b \iota_a \, \mu \, ,$$

see Fig. 3.

Define the following Hamiltonian function H on the space of immersed closed curves by assigning to a path γ its length:

$$H(\gamma) = \int_{S^1} \sqrt{(\gamma'(\theta), \gamma'(\theta))} \, d\theta \, .$$

Theorem 3.8 (e.g. [4]) *The Hamiltonian equation corresponding to the length Hamiltonian function H with respect to the Marsden-Weinstein symplectic structure is given by the binormal equation.*

Notice that the binormal vector $\gamma' \times \gamma''$ is orthogonal to γ', and hence the evolution of the curve γ does not change the length of γ, i.e. the Hamiltonian itself is indeed a first integral of the binormal equation.

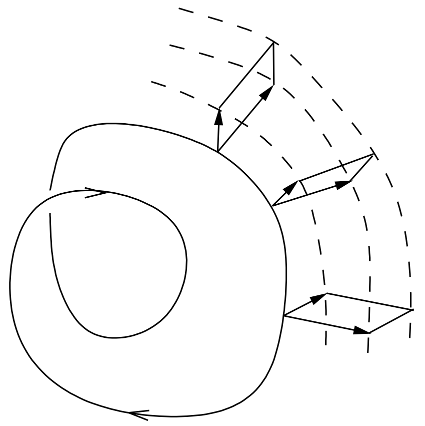

Figure 3: The symplectic form on two variations of the curve is the volume of the collar spanned by these variations.

Proof. In order to see why the length functional indeed gives the binormal equation we notice that for the functional $H(\gamma)$ in the arc-length parametrization the variational derivative (i.e. the "gradient" of H in the space of curves) is $\delta H/\delta\gamma = -(\text{length of }\gamma)^{-1}\cdot\gamma''$. This can be thought of as a normal to γ vector field. In turn, the Marsden-Weinstein symplectic structure at γ can be regarded as the symplectic structure averaged over all two-dimensional planes normal to γ. Then to obtain the corresponding Hamiltonian field v_H, which is the skew-gradient of H, we have to rotate the above gradient field $\delta H/\delta\gamma$ by $\pi/2$ in all such normal planes. Given θ, the corresponding rotation operator in the plane normal to the tangent vector $\gamma'(\theta)$ is $J_\gamma(*) = \gamma' \times *$. Thus we obtain:

$$v_H = J_\gamma(\delta H/\delta\gamma) = -(\text{length of }\gamma)^{-1}\cdot(\gamma'\times\gamma'').$$

□

The Marsden-Weinstein symplectic structure is closely related to the Hamiltonian description of the ideal incompressible fluid filling $\mathbb{R}^3$, see the discussion in Section 5.2.

Remark 3.9 *The binormal equation $\partial_t\gamma = \gamma'\times\gamma''$ can be rewritten in the Frenet frame for the curve γ as evolution equations for the curvature $k(\theta,t)$ and torsion $\tau(\theta,t)$, see [14]. By passing to the new functions "velocity" $v := \tau$ and "energy density" $\rho := k^2$ one obtains the equations of 1-dimensional barotropic fluid*

$$\begin{cases}\partial_t v = -(v,\nabla)v - \nabla h(\rho)/\rho \\ \partial_t\rho + \operatorname{div}(\rho\cdot v) = 0\,,\end{cases}$$

for a specific function $p = h(\rho)$, relating density and pressure.

Another important feature of the binormal equation was observed by Hasimoto [6]. He found the transformation

$$\psi(\theta,t) = k(\theta,t)\exp\left(i\cdot\int_0^\theta \tau(\eta,t)d\eta\right)\,,$$

which sends the binormal equation to the non-linear Schrödinger equation:

$$-i\,\partial_t\psi = \psi'' + \frac{1}{2}|\psi|^2\psi$$

for a complex-valued wave function $\psi : S^1 \to \mathbb{C}$. The latter equation is known to be a completely integrable (bi-Hamiltonian) infinite-dimensional system, and hence so is the binormal or Landau-Lifschitz equations.

4 The KdV-type equations as Euler equations.

4.1 The Virasoro algebra and the KdV equation.

Definition 4.1 *The* Virasoro algebra $vir = Vect(S^1) \oplus \mathbb{R}$ *is the vector space of pairs which consist of a smooth vector field on the circle and a number. This space is equipped with the following commutation operation:*

$$[(f(x)\frac{\partial}{\partial x},\ a),\ (g(x)\frac{\partial}{\partial x},\ b)] = \left((f'(x)g(x) - f(x)g'(x))\frac{\partial}{\partial x},\ \int_{S^1} f'(x)g''(x)\,dx\right),$$

for any two elements $(f(x)\partial/\partial x,\ a)$ and $(g(x)\partial/\partial x,\ b)$ in vir.

The bilinear skew-symmetric expression $c(f,g) := \int_{S^1} f'(x)g''(x)dx$ is called the Gelfand–Fuchs 2-cocycle.

There exists a Virasoro group, an extension of the group of smooth diffeomorphisms of the circle, whose Lie algebra is the Virasoro algebra vir. Fix the L^2-energy quadratic form in the Virasoro Lie algebra:

$$E(f(x)\frac{\partial}{\partial x},\ a) = \frac{1}{2}\left(\int_{S^1} f^2(x)\ dx + a^2\right).$$

As before, we equip the Virasoro group with a (right-invariant) Riemannian metric and consider the corresponding Euler equation, i.e., the equation of the geodesic flow generated by this metric on the Virasoro group.

Theorem 4.2 [12] *The Euler equation corresponding to the geodesic flow (for the above right-invariant L^2-metric) on the Virasoro group is a one-parameter family of the Korteweg–de Vries (KdV) equations:*

$$\partial_t u + u\partial_x u + c\partial_x^3 u = 0; \quad \partial_t c = 0$$

on a time-dependent function u on S^1. Here c is a (constant) parameter, the "depth" of the fluid.

Proof. The dual space vir^* can be identified with the set of pairs

$$\left\{(u(x)(dx)^2, c) |\ u(x) \quad \text{is a smooth function on} \quad S^1,\ c \in \mathbb{R}\right\}.$$

Indeed, it is natural to contract the quadratic differentials $u(x)(dx)^2$ with vector fields on the circle, while the constants are to be paired between themselves:

$$\langle (u(x)(dx)^2, c), (v(x)\frac{\partial}{\partial x},\ a) \rangle = \int_{S^1} u(x) \cdot v(x)\ dx + a \cdot c.$$

The coadjoint action of a Lie algebra element $(f\,\partial/\partial x, a) \in vir$ on an element $(u(x)(dx)^2, c)$ of the dual space vir^* is

$$\mathrm{ad}^*_{(f\,\partial/\partial x, a)}(u(dx)^2, c) = -(2(\partial_x f)u + f\partial_x u + c\partial_x^3 f,\ 0).$$

It is obtained from the identity

$$\langle \mathrm{ad}^*_{(f\frac{\partial}{\partial x},\ a)}(u(dx)^2, c), (g\frac{\partial}{\partial x},\ b) \rangle = -\langle (u(dx)^2, c), [(f\frac{\partial}{\partial x},\ a),\ (g\frac{\partial}{\partial x},\ b)] \rangle\,,$$

which holds for every pair $(g\frac{\partial}{\partial x},\ b) \in vir$.

The quadratic energy functional E on the Virasoro algebra determines the "tautological" inertia operator $A : vir \to vir^*$, which sends a pair $(u(x)\partial/\partial x, c) \in vir$ to $(u(x)(dx)^2, c) \in vir^*$.

The corresponding Euler equation for the right-invariant metric on the group (according to the general formula (3.2)) is given by

$$\frac{\partial}{\partial t}(u(dx)^2, c) = \mathrm{ad}^*_{A^{-1}(u(dx)^2, c)}(u(dx)^2, c).$$

Making use of the explicit formula for the Virasoro coadjoint action ad^* for

$$(f\,\partial/\partial x, a) = A^{-1}(u(dx)^2, c) = (u\,\partial/\partial x, c),$$

we obtain the required Euler equation:

$$\partial_t u = -2(\partial_x u)u - u\partial_x u - c\partial_x^3 u = -3u\partial_x u - c\partial_x^3 u, \quad \partial_t c = 0.$$

The coefficient c is preserved in time, and the function u satisfies the KdV equation. □

4.2 Similar equations and conservation laws.

If we change the metric on the Virasoro group, other interesting equations arise from the same scheme. The Euler equation on the Virasoro group with respect to the right-invariant H^1-metric gives the Camassa-Holm equation:

$$\partial_t u - \partial_{xxt} u = -3u\partial_x u + 2(\partial_x u)\partial_{xx} u + u\partial_{xxx} u + c\partial_x^3 u,$$

see [10]. Similarly, the homogeneous $\dot{H}^1$-metric gives the Hunter-Saxton equation (an equation in the Dym hierarchy):

$$\partial_{xxt}u = -2(\partial_x u)\partial_{xx}u - u\partial_{xxx}u,$$

see [7].

Remark 4.3 *It turns out that all these three equations (KdV, CH, and HS) are bi-Hamiltonian systems, and hence admit an infinite family of conservation laws. The corresponding Hamiltonian (or Poisson) structures are naturally related to the Virasoro algebra.*

For instance, for the KdV equation these conserved quantities can be expressed in the following way. Consider the KdV equation on $(u(x)(dx)^2,\ c)$ as an evolution of Hill's operator $c\frac{d^2}{dx^2}+u(x)$. The monodromy $M(u)$ of this operator is a 2×2-matrix with $\det M(u) = 1$. Look at the following function of the monodromy for a family of Hill's operators:

$$h_\lambda(\frac{d^2}{dx^2} + u(x) - \lambda^2) := \log(\mathrm{trace}\, M(u - \lambda^2)),$$

where $M(u-\lambda^2)$ is the monodromy of the operator $\frac{d^2}{dx^2} + u(x) - \lambda^2$.

Now, the expansion of the function h_λ in λ as $\lambda \to \infty$ produces the first integrals of the KdV equation:

$$h_\lambda \approx 2\pi\lambda - \sum_{n=1}^{\infty} c_n h_{2n-1}\lambda^{1-2n},$$

where

$$h_1 = \int_{S^1} u(x)\, dx, \quad h_3 = \int_{S^1} u^2(x)\, dx, \quad h_5 = \int_{S^1} \left(u^3(x) - \frac{1}{2}(u_x(x))^2\right) dx, \ \ldots$$

and c_n are constants. Note that the trace of a monodromy is a so called Casimir function for the Virasoro algebra, i.e an invariant with respect to the action of variable changes on Hill's operators. The general theory states that the coefficients in the Casimir expansion provide a hierarchy of conserved charges for a bi-Hamiltonian systems, see more details on this and other equations in [7].

5 Hamiltonian structure of the Euler equations for an incompressible fluid.

5.1 The Euler hydrodynamics as a Hamiltonian equation.

The geodesic properties of the Euler equation can also be described within the Hamiltonian framework. The latter turns out to be useful to establish numerous conservation laws for the ideal fluid.

The Lie algebra $\mathfrak{g} = \mathrm{SVect}(M)$ for an ideal incompressible fluid filling a compact manifold M with a volume form μ consists of divergence-free vector fields on M. The main example for us will be a domain M in the Euclidean space $\mathbb{R}^n$.

Elements of the corresponding dual space $\mathfrak{g}^*$ can be thought of as 1-forms u on M defined mudulo function differentials. In other words, such 1-forms can be regarded as linear functionals on vector fields by means of the pairing

$$< u, \xi >= \int_M u(\xi)\,\mu$$

for any divergence-free field ξ. One can see that if u is an exact 1-form, i.e. if u is the differential df of a smooth function f, the corresponding pairing with a divergence-free field is necessarily zero. This is why the dual space $\mathfrak{g}^*$ is actually the quotient $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$ of all 1-forms on M modulo exact 1-forms. Elements of this quotient are called cosets and denoted by $[u] := \{u + df \mid u \in \Omega^1(M),\ f \in C^\infty(M)\}$.

The energy quadratic form on vector fields is their L^2-energy

$$E(\xi) = \int_M (\xi, \xi)\,\mu\,,$$

with respect to the fixed Riemannian metric $(\ ,\ )$ on the manifold M. The corresponding inertia operator

$$A : \mathfrak{g} = \mathrm{SVect}(M) \to \mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$$

sends a vector field ξ to the coset $[u]$ of the 1-form u which is obtained from the field ξ by "raising indices" with help of the metric: $u(\,.\,) = (\xi,\,.\,)$. Since the Riemannian metric is non-degenerate, so is the map $A : \xi \mapsto [u]$. In $\mathbb{R}^n$ with the standard Euclidean metric, the inertia operator A maps a vector field $\xi = \sum \xi_i \partial/\partial x_i$ to the 1-form $u_\xi = \sum \xi_i dx_i$.

It turns out that the operator ad^*_ξ of the coadjoint action has a natural geometric description.

Proposition 5.1 *The coadjoint action of* $S\mathrm{Vect(M)}$ *on* $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$ *is given by the negative of the Lie derivative of the 1-form u (or of its coset* $[u]$*):*

$$\mathrm{ad}^*_\xi\ :\ [u] \mapsto -[L_\xi u]$$

for any divergence free vector field ξ *on* M*.*

Proof. The commutator in the Lie algebra $\mathfrak{g} = \mathrm{SVect}(M)$ is (minus) the commutator of two vector fields on M:

$$[\xi, \eta]_{\mathfrak{g}} = -[\xi, \eta]_M = -L_\xi \eta.$$

Then by definition of the coadjoint operator

$$\langle \mathrm{ad}^*_\xi u, \eta\rangle = -\langle u, [\xi, \eta]_{\mathfrak{g}}\rangle = \langle u, L_\xi \eta\rangle = \int_M u(L_\xi \eta)\,\mu = -\int_M (L_\xi u)(\eta)\,\mu = -\langle L_\xi u, \eta\rangle\,,$$

where in the transfer of the Lie derivative L_ξ from the vector field η to the 1-form u we used that the field ξ is divergence-free, and hence the term containing the Lie derivative $L_\xi \mu$ vanishes. The above implies that $\mathrm{ad}^*_\xi = -L_\xi$.

Note also that the Lie derivative operator ad^* are well-defined on the cosets, since changes of coordinates commute with taking d. □

Now we can combine the above expressions of the coadjoint and inertia operators to obtain the Hamiltonian form of the Euler hydrodynamics equation. The Euler equation (3.2) for the right-invariant metric has the form

$$\frac{d}{dt}[u] = \mathrm{ad}^*_\xi[u] = -L_\xi[u]\,, \tag{5.5}$$

where $[u]$ is the coset of the 1-form u related to the divergence-free field ξ with the help of the metric: $u(\,.\,) = (\xi,\,.\,)$.

Passing from the equation on the cosets $[u]$ to the equation on 1-forms u themselves we get the same equality as above, which holds only modulo adding the differential of a function:

$$\partial_t u = -L_\xi u - d\tilde{p}$$

for some time-dependent function $\tilde{p}$. Now we can return to vector fields by lowering indices (i.e. by applying the inverse inertia operator A^{-1} to both sides of the latter equation), and we obtain

$$\partial_t \xi = -(\xi, \nabla)\xi - \nabla p\,,$$

the original Euler equation (2.1) for an ideal incompressible fluid on M, where p is interpreted as the pressure function. (Here we use the fact that "raising indices" sends the covariant derivative $(\xi, \nabla)\xi$ to the Lie derivative $L_\xi u$ up to a full differential, see e.g. [3].)

This description of the dual space and the Euler hydrodynamics equation allows one to describe certain first integrals of the fluid motion practically without calculations.

Remark 5.2 *In the case of a 3-dimensional manifold M, consider the functional*

$$I(u) = \int_M u \wedge du\,.$$

One can see that for any smooth function f,

$$I(u + df) = \int_M (u + df) \wedge d(u + df) = \int_M u \wedge du = I(u) =: I([u])\,,$$

i.e. this functional is well defined on the space of cosets $\Omega^1(M)/d\Omega^0(M)$. Since the integral I is defined in a coordinate-free way on this space, it is invariant under the action of the group SDiff(M) *on 1-forms by coordinate changes. The latter implies that the integral I is a Casimir function, i.e., in particular, it is a first integral of the Euler equation for whatever Riemannian metric on M we choose.*

Furthermore, if the 1-form u and a divergence-free field ξ are related by means of the Riemannian metric $u(.) = (\xi, .)$, a short direct calculation allows one to rewrite $I(u)$ in the form

$$I(\xi) = \int_M (\xi, \mathrm{curl}\ \xi)\ \mu\,.$$

The latter integral has a natural geometric meaning of helicity *of the vector field* curl ξ*: The helicity of a vector field has a topological interpretation via "average linking number" of the trajectories of this field (see [11, 2] for details).*

Similarly, in the case of a two-dimensional domain $M \subset \mathbb{R}^2$, one can interpret geometrically the invariance of the enstrophy integrals

$$I_k(\xi) = \int_M (\text{curl }\xi)^k d^2x\,,$$

where curl $\xi := \partial\xi_1/\partial x_2 - \partial\xi_2/\partial x_1$ *is the* vorticity function *on* $\mathbb{R}^2$*.*

5.2 The space of knots and the dual of the Lie algebra of divergence-free vector fields.

The geometric characterization of the dual to the Lie algebra of divergence-free vector fields allows one to define a symplectic structure on the space of knots in 3D.

Consider a (possibly knotted) embedded closed curve γ in Euclidean space $\mathbb{R}^3$. Associate to this curve the following linear functional on the space $\mathrm{SVect}(\mathbb{R}^3)$ of divergence-free fields in $\mathbb{R}^3$, that is an element of the dual space $\mathrm{SVect}(\mathbb{R}^3)^*$. Take a compact oriented surface $S \subset \mathbb{R}^3$ such that the oriented boundary of S coincides with the knot γ: $\partial S = \gamma$. Such a surface is called a *Seifert surface* for γ, see Fig.4. Then to any vector field ξ in $\mathbb{R}^3$ we can associate its flux through the surface S. If the vector field ξ is divergence-free, the flux does not depend on the choice of the surface S, so that the curve γ defines a functional on the Lie algebra $\mathrm{SVect}(\mathbb{R}^3)$:

$$\langle\gamma, \xi\rangle = Flux(\xi)|_S = \int_S \iota_\xi \mu\,, \tag{5.6}$$

where $\mu = d^3x$ is the standard volume form in $\mathbb{R}^3$.

Remark 5.3 *Although the functional on the Lie algebra* $\mathrm{SVect}(\mathbb{R}^3)$ *defined by the curve γ does not lie in the smooth part of the dual of* $\mathrm{SVect}(\mathbb{R}^3)$*, we can associate to it (the coset of) singular, rather than smooth, 1-forms on $\mathbb{R}^3$. Namely, given a Seifert surface S we consider the "δ-type" 1-form u_S supported on S, whose integral over any closed curve σ in $\mathbb{R}^3$ counts (with appropriate multiplicities) the intersections of σ with the surface S. Note that although the 1-form u_S depends on the choice of the surface S, its coset $[u_S]$ does not:*

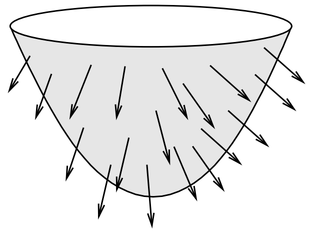

Figure 4: A Seifert surface for the unknot defines the flux functional on vector fields.

the choice of another Seifert surface $\widetilde{S}$ changes the 1-form u_S by a full differential. This coset $[u_S]$ belongs to a completion *of the dual space $\mathfrak{g}^* = \Omega^1(\mathbb{R}^3)/d\Omega^0(\mathbb{R}^3)$.*

Note that the latter space of all 1-forms modulo exact 1-forms is isomorphic to the space $Z^2(\mathbb{R}^3)$ of all closed 2-forms in $\mathbb{R}^3$: The exterior derivative d takes any such coset of 1-forms to a closed 2-form without any loss of information, since $H^1(\mathbb{R}^3) = 0$. Geometrically, we associate to the curve γ a singular closed 2-form ω_γ in $\mathbb{R}^3$ which is the δ-type 2-form (called the de Rham current) supported on γ. One can easily see that $d[u_S] = \omega_\gamma$ as currents, which exactly manifests the relation between a knot and its Seifert surface.

This way a knot γ in $\mathbb{R}^3$ can be seen as an element of the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g} = \mathrm{SVect}(\mathbb{R}^3)$ of divergence-free vector fields on $\mathbb{R}^3$. From this viewpoint, the coadjoint orbit through γ is the equivalence class of the knot γ under various (volume-preserving) isotopies. This leads to the curious observation that all knot invariants *constitute a part of Casimirs, i.e. the coadjoint invariants of the whole Lie group* $\mathrm{SDiff}(\mathbb{R}^3)$*, see e.g. [2, 3].*

One can show that the Marsden-Weinstein symplectic structure on the space of embedded curves containing γ discussed in Section 3.3 coincides with the natural Kirillov–Kostant symplectic structure on the coadjoint orbit of the group $\mathrm{SDiff}(\mathbb{R}^3)$ passing through the curve γ, where the curve is regarded as an element of the dual space to $S\mathrm{Vect}(\mathbb{R}^3)$.

The Euler dynamics on such curves γ is called the *dynamics of vorticity lines* in $\mathbb{R}^3$. Although this dynamics is governed by the non-local interaction law in terms of vorticity functions, the Helmholtz equation, one can truncate this equation by keeping only local terms. This leads to the Localized Induction Approximation (LIA), or the binormal equation. This truncation turns out to be integrable. For $n = 2$ the corresponding local approximation recover the *dynamics of point vortices*, much studied in the literature.

The same construction works for any three-dimensional manifold equipped with a volume form. In higher dimensions, a similar construction allows one to define a symplectic structure on the space of closed immersed codimension 2 submanifolds of M. One can also prove its equivalence with the Kirillov–Kostant symplectic structure on ("singular") coadjoint orbits for $\mathrm{SDiff}(M)$, which are linear functionals on divergence-free vector fields in M, represented by "fluxes through" these submanifolds.

Acknowledgments

This paper presents the lecture notes for author's CIME minicourse given in the Summer 2001. I am thankful to the organizers of this CIME school, and in particular to Professor Renzo Ricca. I am also grateful to the MPI in Bonn, where these notes were completed, and to Robert Wendt for the figures to the notes. The present work was partially sponsored by an NSERC research grant.

References

[1] Arnold, V.I. (1966) Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier* **16**, 316–361.

[2] Arnold, V.I. (1973) The asymptotic Hopf invariant and its applications. *Proc. Summer School in Diff. Equations at Dilizhan,* Erevan (in Russian); English transl.: *Sel. Math. Sov.* **5** (1986), 327–345.

[3] Arnold, V.I. & Khesin, B.A. (1998) Topological methods in hydrodynamics. *Applied Mathematical Sciences, vol. 125, Springer-Verlag, New York*, pp. xv+374.

[4] Calini, A. (2000) Recent developments in integrable curve dynamics. *In Geom. Approaches to Diff. Equations*; Australian Math. Soc. Lect. Notes Ser., **15**, Cambridge University Press, 56–99.

[5] Ebin, D. & Marsden, J. (1970) Groups of diffeomorphisms and the notion of an incompressible fluid. *Ann. of Math. (2)* **92**, 102–163.

[6] Hasimoto, R. (1972) A soliton on a vortex filament, *J. Fluid Mechanics* **51**, 477–485.

[7] Khesin, B. & Misiołek, G. (2003) Euler equations on homogeneous spaces and Virasoro orbits. *Advances in Math.* **176**, 116–144.

[8] Khesin, B. & Wendt, R. (2007) The geometry of infinite-dimensional groups. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*, Springer-Verlag, to appear.

[9] Marsden, J.E., Weinstein, A. (1983) Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids, *Physica D* **7**, 305–323.

[10] Misiołek, G. (1998) A shallow water equation as a geodesic flow on the Bott-Virasoro group. *J. Geom. Phys.* **24:3**, 203–208; Classical solutions of the periodic Camassa-Holm equation. *Geom. Funct. Anal.* **12:5** (2002), 1080–1104.

[11] Moffatt, H.K. (1969) The degree of knottedness of tangled vortex lines, *J. Fluid. Mech.* **106**, 117–129

[12] Ovsienko, V.Yu. & Khesin, B.A. (1987) Korteweg-de Vries super-equation as an Euler equation. *Funct. Anal. Appl.* **21:4**, 329–331.

[13] Ricca, R.L. (1996) The contributions of Da Rios and Levi-Civita to asymptotic potential theory and vortex filament dynamics, *Fluid Dynam. Res.* **18:5**, 245–268.

[14] Turski, L.A. (1981) Hydrodynamical description of the continuous Heisenberg chain, *Canad. J. Phys.* **59:4**, 511–514.