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To Vladimir Igorevich Arnold
on the occasion of his 70th birthday

SHOCK WAVES FOR THE BURGERS EQUATION AND CURVATURES OF DIFFEOMORPHISM GROUPS

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Abstract. We establish a simple relation between curvatures of the group of volume-preserving diffeomorphisms and the lifespan of potential solutions to the inviscid Burgers equation before the appearance of shocks. We show that shock formation corresponds to a focal point of the group of volume-preserving diffeomorphisms regarded as a submanifold of the full diffeomorphism group and, consequently, to a conjugate point along a geodesic in the Wasserstein space of densities. This establishes an intrinsic connection between ideal Euler hydrodynamics (via Arnold's approach), shock formation in the multidimensional Burgers equation and the Wasserstein geometry of the space of densities.

Introduction

In the famous 1966 paper [1] V. Arnold described the dynamics of an ideal fluid as a geodesic flow on the group of volume-preserving diffeomorphisms of a fixed domain equipped with the metric defined by the kinetic energy. He also showed how sectional curvature of this group enters the problem of Lagrangian stability of Eulerian fluid motions. In this paper we are concerned with the exterior geometry of the group of volume-preserving diffeomorphisms considered as an infinite-dimensional submanifold of the group of all diffeomorphisms. We study its second fundamental form and introduce the associated shape operator. The focal points of this infinite-dimensional submanifold turn out to correspond to shocks forming in potential solutions of the inviscid Burgers equation on the underlying finite-dimensional domain. This provides a common geometric framework for both Eulerian hydrodynamics of ideal fluids and the phenomenon of shocks of the inviscid Burgers equation.

More precisely, let M be a compact n-dimensional Riemannian manifold. Consider the group $\mathcal{D}^s(M)$ of Sobolev class diffeomorphisms of M along with its subgroup $\mathcal{D}^s_\mu(M)$ of diffeomorphisms preserving the Riemannian volume form μ. (As usual, if $s > n/2+1$ both groups can be considered as smooth Hilbert manifolds.) For a curve $\eta(t)$ in $\mathcal{D}^s(M)$ defined on an interval $[0,a]$ its L^2-energy is given by

$$E(\eta) = \frac{1}{2}\int_0^a \|\dot\eta(t)\|^2_{L^2}\,dt\,, \tag{0.1}$$

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where the norm $\|X\|^2_{L^2} = \langle X, X\rangle_{L^2}$ is defined by the L^2-inner product

$$\langle X, Y\rangle_{L^2} = \int_M \langle X(x), Y(x)\rangle \, dx \tag{0.2}$$

on each tangent space $T_\eta \mathcal{D}^s(M)$. The corresponding (weak) Riemannian metric on $\mathcal{D}^s(M)$ is right-invariant when restricted to the subgroup $\mathcal{D}^s_\mu(M)$ of volume-preserving diffeomorphisms, although it is not right-invariant on the whole of $\mathcal{D}^s(M)$.

Arnold [1] proved that the Euler equation of an ideal incompressible fluid occupying the manifold M,

$$\partial_t u + \nabla_u u = -\nabla p$$

for a divergence-free field u, corresponds to the equation of the geodesic flow of the above right-invariant metric on the group $\mathcal{D}^s_\mu(M)$. He also showed that Lagrangian instability of such flows, regarded as geodesic deviation, can be estimated in terms of sectional curvatures of the group of volume-preserving diffeomorphisms $\mathcal{D}^s_\mu(M)$ and provided first such curvature estimates of this group.

Below, by regarding the group of volume-preserving diffeomorphisms as a subgroup of the group of all diffeomorphisms (cf. e.g. [7]), we describe its sectional curvatures by means of the second fundamental form of the embedding $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$ and relate its principal curvatures to the distance from its nearest focal point. Recall that the distance from a submanifold N to the first focal point along a geodesic in the direction normal to this submanifold gives a lower bound on the principal curvature radius of N (in the subspace containing the normal direction):

$$\begin{aligned} \mathrm{dist}(N, \textit{nearest focal point}) &= \min \big| \textit{ curvature radius of } N \big| \\ &= 1/\max \big| \textit{ principal curvature of } N \big| . \end{aligned}$$

Hence a lower bound on the distance to a focal point provides an upper bound for (principal) curvatures of N. This motivates our study of the Riemannian geometry of the embedding $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$ as the following sequence of implications, which we make precise below:

i) Geodesics in the full diffeomorphism group $\mathcal{D}^s(M)$ with respect to the above L^2-metric are described by solutions of the inviscid Burgers equation. The geodesics normal to the submanifold $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$ of volume-preserving diffeomorphisms are given by potential Burgers solutions, see Figure 1.

ii) The first focal point along a (potential) Burgers solution determines the moment when the shock wave develops. The geometry of the initial profile allows one to precisely estimate the lifespan of this solution.

iii) Focal points along normal geodesics are in one-to-one correspondence with conjugate points along the projection of these geodesics to the space of densities $\mathcal{P}(M)$ equipped with the Wasserstein L^2-metric.

iv) On the other hand, sectional curvatures of $\mathcal{D}^s_\mu(M)$ can be explicitly computed from the Gauss-Codazzi equations using the second fundamental form (or, the corresponding shape operator) of the embedding $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$.

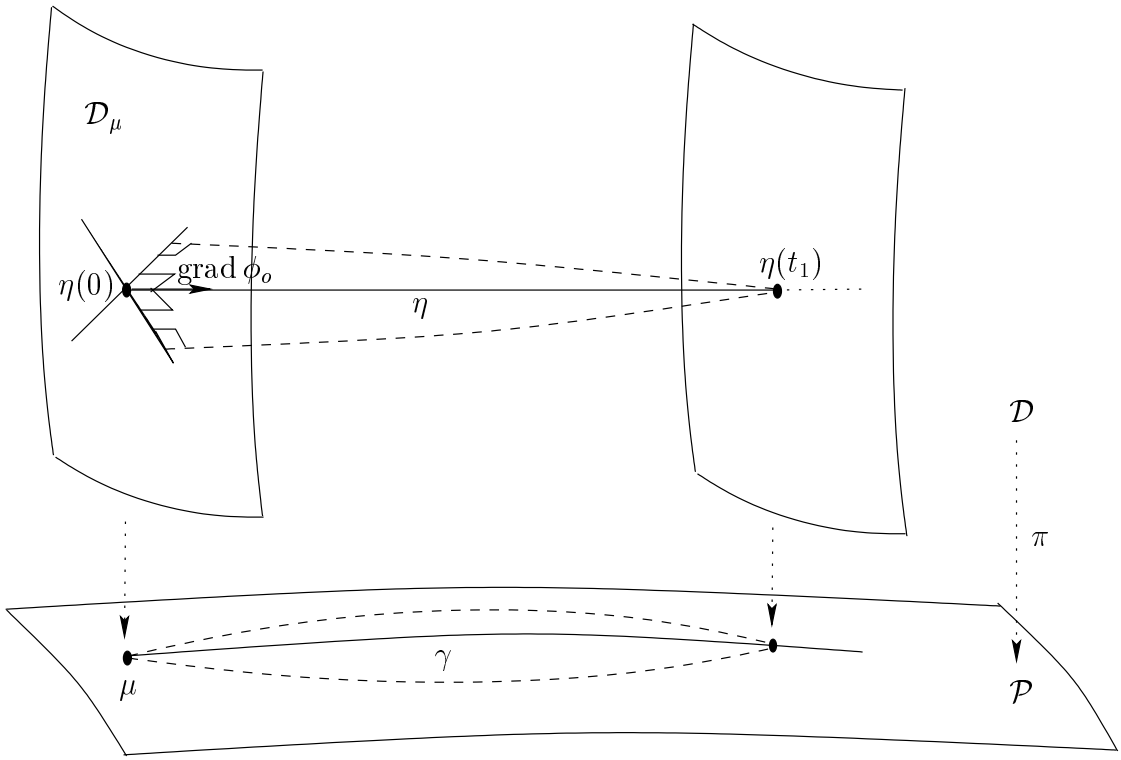

FIGURE 1. Diffeomorphism group $\mathcal{D}^s(M)$ projects to the space of densities $\mathcal{P}(M)$ with the fiber $\mathcal{D}^s_\mu(M)$; normals to $\mathcal{D}^s_\mu(M)$ are horizontal geodesics, and focal points along them correspond to conjugate points along geodesics in $\mathcal{P}(M)$.

v) The location of the first focal point along a normal geodesic can be estimated in terms of the spectral radius of the shape operator of $\mathcal{D}^s_\mu(M)$.

In the next three sections we describe the geometry behind relations $i) - iii)$, respectively. The second fundamental form, the shape operator and its spectral properties in items $iv) - v)$ are described in Section 4. A detailed analysis of the shape operator will be postponed to a future publication. In the last section we recall the properties of asymptotic directions on the diffeomorphism groups discussed in [9] within the framework of the differential geometry of $\mathcal{D}^s(M)$. Finally, we mention that the connection between the Lagrangian instability of ideal fluids and negativity of curvatures of the corresponding diffeomorphism group was precised by V.Arnold in [1, 2], and further explored in [10], [14].

1. Burgers potential solutions

The geometric characterization of the onset of shock waves in terms of focal points of the embedding $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$ is based on the following observation.

Proposition 1. *Geodesics in the group $\mathcal{D}^s(M)$ with respect to the L^2-metric (0.2) correspond to solutions of the Burgers equation: each particle moves with constant velocity along a geodesic in M. Geodesics normal to the submanifold $\mathcal{D}^s_\mu(M)$ have potential initial conditions.*

Proof. First assume for simplicity that M is equipped with a flat metric. Then the statement that particles move with constant velocity along their own geodesics in M follows from the "flatness" of the L^2-metric (0.2). To see that the corresponding velocity field satisfies the Burgers equation denote the flow of fluid particles by $(t,x) \to \eta(t,x)$ and let u be its velocity field

$$\frac{d\eta}{dt}(t,x) = u(t,\eta(t,x)), \qquad \eta(0,x) = x.$$

The chain rule immediately gives

$$\frac{d^2\eta}{dt^2}(t,x) = \big(\partial_t u + Du \cdot u\big)(t,\eta(t,x))$$

and hence the Burgers equation on M

$$\partial_t u + Du \cdot u = 0$$

is equivalent to

$$\frac{d^2\eta}{dt^2}(t,x) = 0\,,$$

the equation of freely flying non-interacting particles in M.

In the general case, any Riemannian metric on M induces a unique Levi-Civita L^2-connection $\bar\nabla$ on $\mathcal{D}^s(M)$, which is determined pointwise by the Riemannian connection ∇ on the manifold M itself (see for example [7] or [12]). Then the same chain rule leads to the Burgers equation

$$\partial_t u + \nabla_u u = 0$$

equivalent to the Riemannian version of freely flying particles: $\bar\nabla_{\dot\eta}\dot\eta = 0$. The latter equation has the explicit form $\ddot\eta + \sum_{ijk} \Gamma^i_{jk}(\eta)\dot\eta^j\dot\eta^k \frac{\partial}{\partial x^i} = 0$, where Γ^i_{jk} are the Christoffel symbols of ∇ in a local coordinate system $x^1,\dots,x^n$ on M.

Finally, observe that the tangent space to the subgroup $\mathcal{D}^s_\mu(M)$ at the identity consists of divergence-free vector fields and hence the space of normals is given by gradients of H^{s+1} functions on M. Thus horizontal geodesics are the ones whose initial velocities are gradient fields $u|_{t=0} = \operatorname{grad}\phi_o$. □

2. Burgers shock waves

The following result describes a connection between focal points and formation of shocks for the Burgers equation.

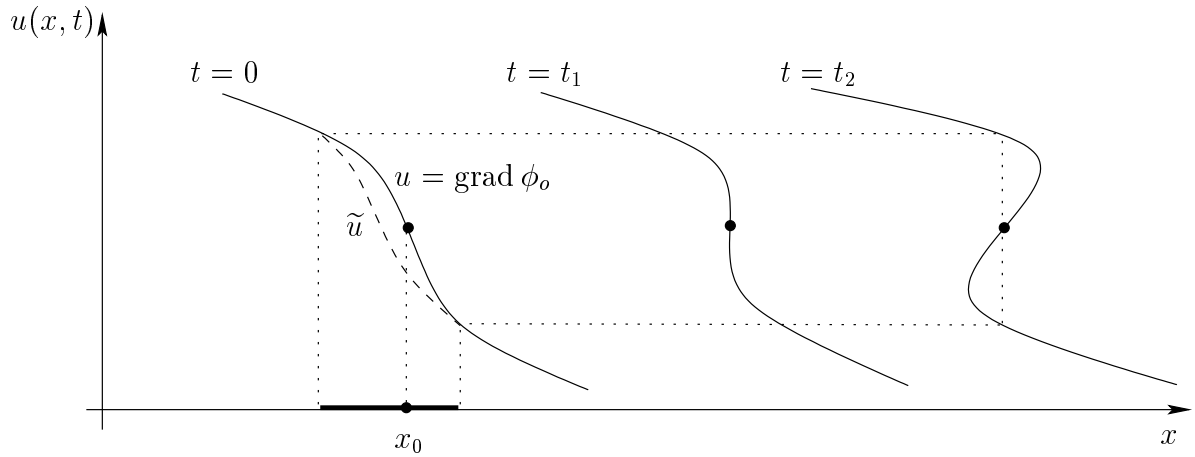

FIGURE 2. The velocity profile for a solution of the one-dimensional Burgers equation. An inflection point of the initial profile u generates the shock wave. The perturbed profile $\tilde{u}$ leads to a later formation of the shock wave.

Theorem 2. *The first focal point to the submanifold $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$ in the direction* $\operatorname{grad}\phi_o$ *is given by the moment of the shock wave formation for the Burgers equation with this initial condition $u|_{t=0} = \operatorname{grad}\phi_o$.*

Example 3. In the 1-dimensional case the shock wave solutions of the Burgers equation appear from the inflection points of the initial velocity profile, see Figure 2. Such points correspond to $u''|_{t=0} = 0$, i.e. to $\phi_o''' = 0$ for $u = \phi'$.

Proof. Before the appearence of a shock wave any solution with smooth initial data remains smooth and the shock wave is the first moment of non-smoothness. Furthermore, for a given potential ϕ_o, any such solution is given by a family of diffeomorphisms $\eta(t) : x \mapsto \exp(t \operatorname{grad}\phi_o(x))$ parameterized by time t. The loss of smoothness occurs when the potential $-t\,\phi_o$ ceases to be c-concave with respect to the distance function $c(x,y) = d^2(x,y)/2$ on the manifold. In this case $\eta(t)$ is no longer a diffeomorphism. On the other hand, as long as the potential is c-concave, the curve $\eta(t)$ remains the shortest curve joining the diffeomorphisms $\eta(0)$ and $\eta(t)$, see [11], and focal points on $\eta(t)$ cannot appear.

Finally, we note that the moment when a shock wave appears corresponds to a focal point (i.e., to a caustic), rather than to a cut (or Maxwell) point. This follows from the fact that in any shock wave the first collisions start between neighboring particles, see e.g. [3, 4]. To see why such collisions lead to the existence of a neighboring shorter geodesic, one can employ the following heuristic argument. Consider a shock wave generated by a solution to the 1-dimensional Burgers equation with the initial velocity $u = \operatorname{grad}\phi_o$, see Figure 2. Assume that this shock wave emerged at time t_1 and consider it at time $t_2 > t_1$. Consider a perturbation $\tilde{u}$ of the original

solution which at the initial moment differs from the initial velocity u on a small segment. This segment consists of exactly those particles which are glued together in the shock wave between the times $[t_1, t_2]$. Perturbations $\tilde{u}$, whose velocity profile on the (x, u)-plane (see Figure 2) lies below that of u correspond to smooth perturbations of the initial velocities of these particles, slowing them down to ensure that the shock wave develops after time t_1 but no later than t_2. Note that such a perturbation defines another solution which coincides with the original one after time t_2 but represents a shorter geodesic. Indeed, since we decreased the velocity vectors of the particles, the length of the new geodesic, being the integral of the velocity squares, can only get smaller. By arbitrarily shortening the segment $[t_1, t_2]$ this implies that the first moment t_1 of the original shock wave for the initial condition u indeed defines a focal point, rather than a cut point, since there exists a shorter geodesic nearby. □

Remark 4. In higher dimensions the shock waves are first generated from the special points of the initial potential $u|_{t=0} = \operatorname{grad} \phi_o$: singularities of type A_3 modulo certain linear and quadratic terms in local orthogonal coordinate charts, see [6]. The list of initial singularities, possible bifurcations of the shock waves and other related questions for the inviscid Burgers equations, can be found in [3, 4, 8]. A given shock wave for the Burgers equation forms at a caustic corresponding to a deeper degeneration of this potential, see e.g. [3].

The most degenerate caustic points correspond to the endpoints of the spectrum for the shape operator $Sh_{\operatorname{grad}\phi_o}$, which we describe below. The lifespan of a solution before the shock formation provides an estimate for the spectrum of this operator: a *lower bound* for the lifespan, and hence for the distance to the first focal point gives the *upper bound* for the (principal) curvatures of the submanifold $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$.

3. Focal points in diffeomorphisms and conjugate points in densities

Next we would like to show that conjugate points along geodesics in the Wasserstein space of densities are in 1-1 correspondence with focal points on the group of diffeomorphisms. Namely, consider the fibration $\pi : \mathcal{D}^s(M) \to \mathcal{P}(M)$, where π is the projection of diffeomorphisms onto the space of (normalized) densities on M. Thus, two diffeomorphisms η_1 and η_2 belong to the same fiber if and only if $\eta_1 = \eta_2 \circ \varphi$ for some volume-preserving diffeomorphism φ. (Moreover, this projection can be defined for more general maps by tracing how they transport the given standard density form on M.) The map π is a Riemannian submersion [13] and the above L^2-metric on $\mathcal{D}^s(M)$ induces the Wasserstein (or, Kantorovich–Rubinstein) metric in the space of densities $\mathcal{P}(M)$.

The following general observation holds. Consider a projection $\pi : \mathcal{D} \to \mathcal{P}$ between two (possibly weak) Riemannian manifolds which is a Riemannian submersion and where $\mathcal{D}$ an infinite-dimensional Lie group. Let γ be a geodesic starting at a

point γ_o in $\mathcal{P}$ and let η be a horizontal lift of this geodesic to $\mathcal{D}$, i.e. a horizontal geodesic in $\mathcal{D}$ whose initial point lies in the fiber $\pi^{-1}(\gamma_o)$ and whose projection to $\mathcal{P}$ is γ. Assume, in addition, that the metric restricted to $\pi^{-1}(\gamma_o)$ is right invariant. (In our case, the fiber $\pi^{-1}(\gamma_o)$ is the group $\mathcal{D}^s_\mu(M)$ of volume-preserving diffeomorphisms for γ_o).

Proposition 5. *The points along the geodesic $\gamma \subset \mathcal{P}$ conjugate to the initial position γ_o are in one-to-one correspondence with the focal points of the fiber $\mathcal{D}_\mu = \pi^{-1}(\gamma_o)$, regarded as a submanifold in $\mathcal{D}$, along a horizontal geodesic η in $\mathcal{D}$. Moreover, the multiplicities of the conjugate points in $\mathcal{P}$ coincide with the multiplicities of the corresponding focal points in $\mathcal{D}$.*

Recall that conjugate points and their generalizations, the focal points, arise as singularities of the Riemannian exponential map. In a general infinite-dimensional space two types of such points, called epi-conjugate (epi-focal) and mono-conjugate (mono-focal) points, can be found. Roughly speaking, geometric significance of the former has to do with covering properties of the exponential map, while the latter are responsible for the minimizing properties of geodesics.

Proof. The statement follows from the submersion property of π. Since geodesics in the base manifold have unique lifts to horizontal geodesics in $\mathcal{D}$ we can identify the geodesics emanating from γ_o that are close to γ and which come together near a conjugate point in $\mathcal{P}$ with the horizontal geodesics emanating from $\pi^{-1}(\gamma_o)$, that are close to $\pi^{-1}(\gamma)$ and come together near the focal point in $\mathcal{D}$.

Furthermore, the argument works for both mono- and epi-conjugate (or focal) points, since the exponential map is nondegenerate along the fibers. Indeed, shifts along the fibers correspond to right multiplication by an element of $\pi^{-1}(\gamma_o)$, while the metric in $\mathcal{D}$ is $\pi^{-1}(\gamma_o)$-invariant. □

Remark 6. Geodesics in the Wasserstein space $\mathcal{P}(M)$ of (smooth) densities are projections of the horizontal geodesics in the diffeomorphism group $\mathcal{D}^s(M)$. For a given potential $-\phi_o$ they are the diffeomorphisms $\eta(t) : x \mapsto \exp(t \operatorname{grad} \phi_o(x))$ parameterized by t. According to the theorem on polar decomposition on manifolds [11] every non-degenerate map $\eta \in \mathcal{D}^s(M)$ has a unique decomposition $\eta = gr \circ vp$ into a "gradient map" $gr(x) := \exp_x(\operatorname{grad}\phi_o)$ for a c-concave potential $-\phi_o$ and a volume-preserving map vp. Consequently, the projection of the family of diffeomorphisms $\eta(t) : x \mapsto \exp(t \operatorname{grad}\phi_o(x))$ remains a shortest geodesic in $\mathcal{P}$ (in the space of non-degenerate maps) as long as the potential $-t\,\phi_o$ remains c-concave.

4. The shape operator and focal points

We next turn to the shape operator of the embedding $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$. The eigenvalues of this operator are traditionally called the principal curvatures and the corresponding eigenvectors of unit norm are the principal directions. In our case, at

each point in $\mathcal{D}^s_\mu(M)$ there is a family of shape operators parametrized by vectors normal to the submanifold.

Definition 7. The *shape operator* of the submanifold $\mathcal{D}^s_\mu(M)$ at the identity is the operator $Sh_{\mathrm{grad}\psi}$ on $T_{\mathrm{id}}\mathcal{D}^s_\mu(M)$ defined by

$$\langle Sh_{\mathrm{grad}\psi}(w), v\rangle_{L^2} = \langle S_{\mathrm{id}}(w,v), \mathrm{grad}\psi\rangle_{L^2},$$

where $\psi \in H^{s+1}(M)$, w and v are in $T_{\mathrm{id}}\mathcal{D}^s_\mu(M)$ and S_{id} is the second fundamental form of $\mathcal{D}^s_\mu(M)$. By right invariance we similarly define the shape operator at any point η in $\mathcal{D}^s_\mu(M)$.

Remark 8. Recall that the (weak) Riemannian metric on $\mathcal{D}^s(M)$ induces on both the group $\mathcal{D}^s(M)$ and its subgroup $\mathcal{D}^s_\mu(M)$ the unique Levi-Civita connections $\bar\nabla$ and $\tilde\nabla = P_\eta\bar\nabla$, where $P_\eta := R_\eta \circ P_{\mathrm{id}} \circ R_{\eta^{-1}}$ is the Hodge projection P_{id} onto divergence-free vector fields on M conjugated with the right translation by η (see [7] for more details).

The second fundamental form S of the submanifold $\mathcal{D}^s_\mu(M) \subset \mathcal{D}^s(M)$ is the difference of the two connections, which at the identity is given by

$$S_{\mathrm{id}}(w,v) = Q_{\mathrm{id}}\nabla_w v\,,$$

where $Q_{\mathrm{id}}(w) := w - P_{\mathrm{id}}(w) = \mathrm{grad}(\Delta^{-1}\mathrm{div}\, w)$ is the Hodge projection onto the gradient fields on M. The second fundamental form, as well as the curvature tensors of the connections on $\mathcal{D}^s(M)$ and $\mathcal{D}^s_\mu(M)$, are bounded multilinear operators satisfying the Gauss-Codazzi equations, see [12].

The above gives an explicit formula for the shape operator at the identity

$$Sh_{\mathrm{grad}\psi}(w) = -P_{\mathrm{id}}\nabla_w \mathrm{grad}\psi\,.$$

One can check that the shape operator is bounded and self-adjoint.

The next result provides an estimate of the distance to the first focal point along a normal geodesic in terms of the spectral radius of $Sh_{\mathrm{grad}\phi_o}$.

Theorem 9. *Suppose that M is a compact manifold of non-negative sectional curvature. Let $\eta(t)$ be the geodesic normal to $\mathcal{D}^s_\mu(M)$ at η_o with $\dot\eta(0) = \mathrm{grad}\phi_o \circ \eta_o$ and let $\lambda = \|Sh_{\mathrm{grad}\phi_o}\|_{L(\mathcal{H})}$. Then $\eta(t)$ contains a focal point in the interval $0 < t \le 1/\lambda$.*

In this case our strategy to show existence of focal points along geodesics normal to the fibre $\mathcal{D}^s_\mu(M)$ is to obtain estimates on the second variation of the L^2-energy (0.1). The latter is given by the following formula

$$E''(\eta)(W,W) = \int_0^a \Big(\|\bar\nabla_{\dot\eta}W\|^2_{L^2} - \langle \bar R(W,\dot\eta)\dot\eta, W\rangle_{L^2}\Big)dt - \langle Sh_{\mathrm{grad}\phi_o}(w_o), w_o\rangle_{L^2}$$

for any vector field W along η such that $W(0) = w_o \circ \eta$ is tangent to the fibre $\mathcal{D}^s_\mu(M)$ and $W(a) = 0$. Here $\bar R$ is the curvature of the connection $\bar\nabla$ on the group $\mathcal{D}^s(M)$. We discuss this theorem and its proof in detail in a future publication.

We point out that the conclusion of Theorem 9 holds provided that the geodesic η is defined on the interval $[0, 1/\lambda]$. Furthermore, this geodesic does not minimize the L^2-distance between $\mathcal{D}^s_\mu(M)$ and any $\eta(t)$ with $t > 1/\lambda$ and hence the focal point is in fact mono-focal.

Another corollary of the same formula for the second variation is as follows.

Proposition 10. *Suppose now that M has non-positive sectional curvature. Consider any initial vector* $\mathrm{grad}\phi_o$ *with a convex function ϕ_o. Then there can be no focal points on the corresponding geodesic.*

Indeed, for such a manifold M and any W one obtains

$$E''(\eta)(W,W) \geq \int_0^a \|\bar{\nabla}_{\dot\eta} W\|^2_{L^2}\, dt + \int_M \langle \nabla_{w_o} \mathrm{grad}\, \phi_o, w_o\rangle\,.$$

In particular, if ϕ_o is convex then its Hessian $\langle \nabla_{w_o} \mathrm{grad}\, \phi_o, w_o\rangle$ is non-negative and so $E''(\eta)(W,W) \geq 0$, implying the absence of focal points between $\eta(0)$ and any other $\eta(a)$.

5. Asymptotic directions

Asymptotic directions for the group of volume-preserving diffeomorhisms also appear naturally in the context of the exterior geometry of this group. Recall, that a vector tangent to a Riemannian submanifold is *asymptotic* if the (vector-valued) second fundamental form evaluated on it is zero. The geodesics issued in the direction of this vector, one in the submanifold and the other in the ambient manifold, have a second order of tangency. (Note that in general two geodesics with a common tangent will have only a simple, i.e. first order, tangency.) Asymptotic vectors for the submanifold of volume-preserving diffeomorhisms among all diffeomorhisms are given at the tangent space to the identity by vector fields X satisfying $\mathrm{div}\, \nabla_X X = \mathrm{div}\, X = 0$, see [5, 9]. Such fields rarely exist. For instance, for diffeomorhism groups of surfaces one has the following sufficient condition for non-existence of such directions.

Theorem 11. [9] *If M is a compact closed surface of nowhere zero curvature, then $\mathcal{D}_\mu(M)$ has no asymptotic directions.*

Proof. Consider the square length function $f := g(X,X)$ on M. At its maximum point x_0 we get from $df(x_0) = 0$ that the Jacobi matrix DX is degenerate at x_0. This implies that $\mathrm{tr}(DX)^2(x_0) = -2\det[DX(x_0)] = 0$ for a divergence-free field X.

Now employing the identity

$$\mathrm{div}\nabla_X X = r(X,X) + \mathrm{tr}(DX)^2\,,$$

where r stands for the Ricci curvature of the metric g, which holds for any divergence-free vector field X on M, we find that

$$\mathrm{div}\, \nabla_X X(x_0) = K(x_0)\, g(X,X)(x_0)\,,$$

since in two dimensions the Ricci curvature r and the Gaussian curvature K coincide. However, for an asymptotic field X this implies that

$$0 = K(x_0)\, g(X, X)(x_0)$$

contradicting the assumption $K \neq 0$ on M, and in particular at the point x_0. □

Remark 12. Vanishing of the second fundamental form on asymptotic vectors implies vanishing of the projections of this form to any gradient direction. In particular, such vectors must be asymptotic simultaneously for all the projections of the second fundamental form described above. This implies the following sufficient condition for the non-existence of asymptotic directions: if at least in one of the gradient projections the second fundamental form is sign-definite, then there are no asymptotic directions. The latter sufficient condition can also be given in terms of gradient solutions of the Burgers equation: in such a gradient direction all shocks develop only as t changes in the one direction (say, increases), and do not develop as t changes in the other direction.

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