RELATIVE HELICITY AND TILING TWIST

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ABSTRACT. We consider domino tilings of 3D cubiculated regions. The tilings have two invariants, flux and twist, often integer-valued, which are given in purely combinatorial terms. These invariants allow one to classify the tilings with respect to certain elementary moves, flips and trits. In this paper we present a construction associating a divergence-free vector field $\xi_{\mathbf{t}}$ to any domino tiling $\mathbf{t}$, such that the flux of the tiling $\mathbf{t}$ can be interpreted as the (relative) rotation class of the field $\xi_{\mathbf{t}}$, while the twist of $\mathbf{t}$ is proved to be the relative helicity of the field $\xi_{\mathbf{t}}$.

1. INTRODUCTION

Domino tilings of 3D regions traditionally have two invariants associated to them, flux and twist. Flux is understood as the homology class associated with a certain cycle constructed for the tiling. For a cubiculated region (a topological 3-manifold with boundary) the flux Flux($\mathbf{t}$) of a tiling $\mathbf{t}$ is an element of the first homology group of the region, defined up to an additive constant. The ambiguity can be removed by considering the *relative flux* RFlux($\mathbf{t}$) as discussed below. For tilings with zero relative flux there is an integer invariant, the *twist*, associated with the tiling and measuring the 'mutual linking of tiles' around each other. It is invariant with respect to *flips*, local moves which consist of removing two neighboring parallel dominoes and placing them back after a rotation. The twist changes under another move, a *trit*, which replaces a frame-like triple of tiles to the one pointing in the opposite way (see Section 2 and a detailed discussion in [8]).

While there have been pointed out similarities of the twist invariant with the Hopf invariant, the correspondence remained at either an intuitive level or in the continuous limit for turning tiles into vector fields via a broadly understood tiling's refinement.

In this paper we present a construction of a smooth divergence-free vector field associated to an arbitrary tiling ('5-pipe construction', see Section 5.1) so that *the twist invariant becomes, up to a factor, the relative helicity* of that vector field. Furthermore, we extend the notion of relative helicity [5] to divergence-free vector fields on arbitrary three-manifolds and not necessarily tangent to their boundaries. This allows one to compare relative helicity with twists of tilings in non-simply-connected regions (see Section 4.3). The toolbox includes an introduction of 'an

isolating shell' for a cubiculated region, the use of refinements and appropriate connectivity of the spaces of tilings. Finally, we relate *the flux invariant of a tiling to the rotation class* of the associated vector field.

In a nutshell, the results of the paper are as follows. Recall that for a divergence-free vector field ξ in a simply-connected domain $M \subset \mathbb{R}^3$ and tangent to its boundary ∂M, its helicity is the quantity $\mathrm{Hel}(\xi) = \int_M (\xi, \mathrm{curl}^{-1}\xi) \, d^3x$. For a field whose support consists of several linked tubes the helicity reduces to the mutual linking of those tubes, the self-linking and fluxes in the tubes (see [12] and Section 4.2 and Appendix).

The helicity notion can be extended to null-homologous vector fields tangent to the boundary in arbitrary three-manifolds M equipped with a volume form. If the field ξ is not tangent to the boundary ∂M, one can define only its relative helicity, i.e., the difference of helicities of two vector fields ξ and η with the same behaviour at the boundary ∂M. Namely, one can extend ξ and η by the same field to vector fields in a bigger manifold $\widetilde{M} \supseteq M$, where their helicities are well-defined, but depend on the extension. However, the difference of helicities will not depend on the extension, hence the name *relative helicity*, see Section 4.3.

On the other hand, given a 3D domino tile, consisting of two cubes (say, white and black), consider five smooth curves joining the symmetric faces at their centers and approaching those faces orthogonally, see Figure 4. Tubular neighborhoods of those curves will become supports of the five smooth divergence-free vector fields directed from the black to the white unit cube and each having flux φ. By performing this construction in each domino, to any tiling $\mathbf{t}$ of a cubiculated region $\mathcal{R}$ one associates a smooth divergence-free vector field $\xi_{\mathbf{t}}$ in the whole region. By using an 'isolating shell' for the region, one can define the relative helicity of the vector field $\xi_{\mathbf{t}}$ associated with the tiling. The construction of such an isolating shell turns out to be possible whenever the relative flux vanishes, $\mathrm{RFlux}(\mathbf{t}) = 0 \in H_1(\mathcal{R}, \partial\mathcal{R})$. Our main result is the following:

Theorem 1.1. *For two tilings $\mathbf{t}_0$ and $\mathbf{t}_1$ of the same flux and of zero relative flux in a 3D cubiculated region $\mathcal{R}$, the difference of their twists is proportional to the relative helicity of the associated vector fields $\xi_{\mathbf{t}_0}$ and $\xi_{\mathbf{t}_1}$:*

$$\mathrm{Hel}(\xi_{\mathbf{t}_1}) - \mathrm{Hel}(\xi_{\mathbf{t}_0}) = 36\varphi^2 \left(\mathrm{Tw}(\mathbf{t}_1) - \mathrm{Tw}(\mathbf{t}_0) \right) ,$$

where $\varphi \in \mathbb{R}$ is the flux in a single pipe of the vector fields $\xi_{\mathbf{t}_i}$.

In order to clarify the condition of vanishing relative flux, recall that given a vector field ξ in a manifold M with a volume form μ and tangent to the boundary ∂M, the 2-form $\omega_\xi := i_\xi \mu$ is closed iff ξ is divergence-free with respect to μ. If ξ is tangent to ∂M, the restriction of the 2-form ω_ξ vanishes on the boundary ∂M. The cohomology class $[\omega_\xi] \in H^2(M, \partial M)$ can be identified with a class in $H_1(M)$ (via the Poincaré isomorphism) and is called the *rotation class* of the dynamical system ξ, see [1]. Similarly, if a divergence-free vector field ξ is not tangent to ∂M, then $[\omega_\xi] \in H^2(M) \simeq H_1(M, \partial M)$. Thus such a vector field ξ defines a *relative rotation class* as an element of $H_1(M, \partial M)$.

In Section 5.2 we prove the relation of flux $\mathrm{Flux}(\mathbf{t})$ and relative flux $\mathrm{RFlux}(\mathbf{t})$ for a tiling $\mathbf{t}$ to the (relative) rotation class of an appropriate vector field $\xi_{\mathbf{t}}$:

Theorem 1.2. *Given a tiling $\mathbf{t}$ of a 3D cubiculated region $\mathcal{R}$ its relative flux $\mathrm{RFlux}(\mathbf{t}) \in H_1(\mathcal{R}, \partial\mathcal{R})$ coincides modulo a factor with the relative rotation class*

$[\xi_{\mathbf{t}}]$ *of the vector field $\xi_{\mathbf{t}}$ obtained via the 5-pipe construction:*

$$[\xi_{\mathbf{t}}] = 6\varphi \operatorname{RFlux}(\mathbf{t}).$$

In particular, the relative rotation class of the field $\xi_{\mathbf{t}}$ vanishes iff the relative flux $\operatorname{RFlux}(\mathbf{t})$ vanishes. By normalizing the flux as $\varphi = 1/6$, consistent with the 6-pipe setting of Section 5.2, the two main theorems provide the exact matching: for any tiling its relative flux coincides with the relative rotation class of the associated vector field, while the relative helicity of the latter coincides with the tiling's twist.

Remark 1.3. There are numerous papers studying tilings in the context of quantum dimer models. In the papers [4, 7] the authors emphasized the existence of topologically nontrivial configurations called 'Hopfions' in those systems with the understanding that in a suitable continuous limit of the dimer model, those excitations would correspond to the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$. In [2] (see also [3]), Arnold proved that an asymptotic version of the Hopf invariant for a vector field is equal to the field's helicity. Theorem 1.1 shows that even without taking the continuous limit, a tiling's twist itself is already equal (modulo the flux factor) to the corresponding relative helicity.

In a sense, the present paper shows that one can regard tilings as occupying an intermediate place between divergence-free vector fields on the one hand and links with framings on the other: by associating special divergence-free vector fields to tilings, one can compute their helicities via consideration of linking and self-linking of framed cycles.

The paper is organized as follows. In Sections 2 and 3 we recall the main moves and invariants for domino tilings in purely combinatorial terms. In Section 4 we introduce relative rotation class and relative helicity of vector fields not tangent to boundary. In Section 5 we present the 5-pipe construction associating a divergence-free vector field to a domino tiling, introduce an isolating shell, and prove Theorem 1.2 on the relative rotation class. Section 6 presents two key examples which are used in the proof of the main Theorem 1.1 in Section 7. One more example manifesting the main theorem is described in Section 8. Appendix 9 derives the formula for helicity via linking and self-linking numbers of pipes.

2. Domino tilings, combinatorial flux and local moves

A cubiculated region is a connected and oriented manifold $\mathcal{R}$ (usually with boundary) decomposed into finitely many unit cubes. A simple example of cubiculated region is a *box*: $\mathcal{R} = [0, L] \times [0, M] \times [0, N] \subset \mathbb{R}^3$, LMN even; the unit cubes are $[a, a+1] \times [b, b+1] \times [c, c+1]$, $(a, b, c) \in \mathbb{Z}^3$. We assume that the interior of a cubiculated region is as in this example, so that interior edges (resp. vertices) are surrounded by four (resp. eight) unit cubes. We also assume that unit cubes are painted black and white, with adjacent cubes of opposite colors and the same number of cubes of each color. A (3D) domino is the union of two unit cubes with a common face, thus a $2 \times 1 \times 1$ rectangular cuboid. A (domino) tiling of $\mathcal{R}$ is a set of dominoes with disjoint interiors whose union is equal to $\mathcal{R}$.

We follow [10], [14] and [8] in drawing tilings of cubiculated regions by floors, as in Figure 1. Vertical dominoes (i.e., dominoes in the z direction) appear as two squares, one in each of two adjacent floors; the top square of a vertical domino (which, in the figure, appears at the right) is left unfilled for visual facility.
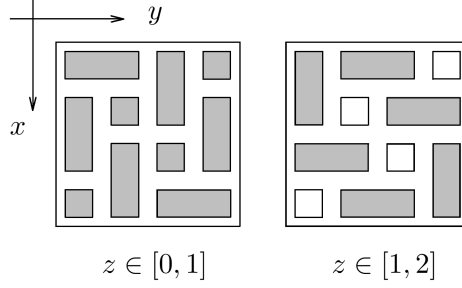
FIGURE 1. A tiling of the box $[0,4] \times [0,4] \times [0,2]$. The orientation of $\mathbb{R}^3$ is important: The z axis points upward away from the paper. Examples of dominoes in this tiling are $[0,1] \times [0,2] \times [0,1]$, $[0,2] \times [0,1] \times [1,2]$ and $[1,2] \times [1,2] \times [0,2]$. This tiling admits no flips.

Definition 2.1. Given a region $\mathcal{R}$, let $\mathcal{T}(\mathcal{R})$ denote the set of domino tilings of $\mathcal{R}$. A *flip* is a local move in $\mathcal{T}(\mathcal{R})$: two parallel and adjacent (3D) dominoes are removed and placed back in a different position. A *trit* is the only local move in $\mathcal{T}(\mathcal{R})$ involving three dominoes which does not reduce to flips. The three dominoes involved are in three different directions and fill a $2 \times 2 \times 2$ box minus two opposite unit cubes.

Figure 2 shows in the $3 \times 3 \times 2$ box a trit followed by a sequence of flips (the first tiling admits no flips). We shall come back to this example in Section 8.

FIGURE 2. Five tilings of the box $[0,3] \times [0,3] \times [0,2]$. The first move is a trit, the two other moves are flips. Five more flips take us from the fourth to the fifth tiling. The first tiling has twist -1, the others have twist 0.

To apply homology theory we will use a finer complex structure $\mathcal{R}^\sharp$. The vertices of $\mathcal{R}^\sharp$ are the original vertices of $\mathcal{R}$, together with the centers of edges, faces and unit cubes of $\mathcal{R}$. Each edge of $\mathcal{R}$ is decomposed into two edges of $\mathcal{R}^\sharp$. Similarly, each square (resp. cube) of $\mathcal{R}$ is decomposed into four squares (resp. eight cubes) of $\mathcal{R}^\sharp$. Notice that the boundary $\partial \mathcal{R}$ also acquires the structure of a polyhedral complex $\partial \mathcal{R}^\sharp$.

A domino is identified with the sum of two oriented edges in $\mathcal{R}^\sharp$, from the center of the black cube to the center of the common face and from there to the center of the white cube. A domino tiling $\mathbf{t}$ of $\mathcal{R}$ is therefore identified as $\mathbf{t} \in C_1(\mathcal{R}^\sharp)$. The boundary $\partial \mathbf{t} \in C_0(\mathcal{R}^\sharp)$ is the sum of the centers of the white cubes (of $\mathcal{R}$) minus the sum of the centers of the black cubes. In particular, for any two tilings $\mathbf{t}_0, \mathbf{t}_1$ we have $\partial \mathbf{t}_0 = \partial \mathbf{t}_1$ or, equivalently, $\mathbf{t}_1 - \mathbf{t}_0 \in Z_1(\mathcal{R}^\sharp)$. We then write $[\mathbf{t}_1 - \mathbf{t}_0] \in H_1(\mathcal{R}) = H_1(\mathcal{R}; \mathbb{R})$; in the present paper we almost always use coefficients in $\mathbb{R}$.

Given an initial tiling $\mathbf{t}_0$, let $\mathcal{T}_0(\mathcal{R}) \subseteq \mathcal{T}(\mathcal{R})$ be the set of tilings $\mathbf{t}$ belonging to the same 'homology class', i.e. such that $[\mathbf{t} - \mathbf{t}_0] = 0 \in H_1(\mathcal{R})$. We will say that tilings $\mathbf{t}$ have the same *flux* as $\mathbf{t}_0$, $\text{Flux}(\mathbf{t}) = \text{Flux}(\mathbf{t}_0)$.

In the present paper we define a somewhat different object, the *relative flux* $\mathrm{RFlux}(\mathbf{t}) \in H_1(\mathcal{R}, \partial\mathcal{R})$. It takes values in relative homology and does not require the choice of a base tiling. We work in the complex $\mathcal{R}^\sharp$ and, given a tiling $\mathbf{t}$, define $\mathbf{t} \in C_1(\mathcal{R}^\sharp)$ as above. We construct $\mathbf{q}_1 \in C_1(\mathcal{R}^\sharp)$ as follows: it is the sum over all black unit cubes of all edges from the cube centers to the centers of their faces minus the same sum (of all edges from cube centers to their faces) over all white cubes. Thus, for every tiling $\mathbf{t}$ we have that $\mathbf{q}_0 = \partial(6\mathbf{t} - \mathbf{q}_1) \in C_0(\partial\mathcal{R}^\sharp)$ is the sum of $+1$ (resp. -1) times the centers of the white (resp. black) squares in $\partial\mathcal{R}$. Thus, $6\mathbf{t} - \mathbf{q}_1 \in Z_1(\mathcal{R}^\sharp, \partial\mathcal{R}^\sharp)$; we define

$$(1) \qquad \mathrm{RFlux}(\mathbf{t}) = \left[\mathbf{t} - \frac{\mathbf{q}_1}{6}\right] \in H_1(\mathcal{R}, \partial\mathcal{R}).$$

Inclusion induces a homomorphism $i : H_1(\mathcal{R}) \to H_1(\mathcal{R}, \partial\mathcal{R})$, while for any tiling $\mathbf{t}$ one has $i(\mathrm{Flux}(\mathbf{t})) = \mathrm{RFlux}(\mathbf{t}) - \mathrm{RFlux}(\mathbf{t}_0)$.

Example 2.2. Already for cubiculated regions $\mathcal{R} \subset \mathbb{R}^3$, there are examples with nontrivial homology or homotopy. The solid torus

$$\mathcal{R}_0 = ([0, 12] \times [0, 12] \times [0, 4]) \smallsetminus ((4, 8) \times (4, 8) \times [0, 4])$$

is not simply connected, with $\pi_1(\mathcal{R}_0) = \mathbb{Z}$, $H_1(\mathcal{R}_0; \mathbb{R}) = \mathbb{R}$ and $H_1(\mathcal{R}_0, \partial\mathcal{R}_0; \mathbb{R}) = 0$. There exist tilings of $\mathcal{R}_0$ with different values of flux, but the relative flux is always 0.

The cube with a hole

$$\mathcal{R}_{1,0} = ([0, 12] \times [0, 12] \times [0, 12]) \smallsetminus ((4, 8) \times (4, 8) \times (4, 8))$$

is simply connected but has nontrivial relative homology: $H_1(\mathcal{R}_{1,0}, \partial\mathcal{R}_{1,0}) = \mathbb{R}$. For any tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{1,0})$ we have $\mathrm{RFlux}(\mathbf{t}) = 0$. For a different cube with hole

$$\mathcal{R}_{1,1} = ([0, 13] \times [0, 13] \times [0, 13]) \smallsetminus ((4, 9) \times (4, 9) \times (4, 9)),$$

we have $\mathrm{RFlux}(\mathbf{t}) \neq 0$ for all $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{1,1})$, since the numbers of white and black cubes in the hole surrounded by $\mathcal{R}_{1,1}$ are different, see [8].

Quotients and similar identifications give us other interesting examples of regions. The quotient

$$\mathcal{R}_2 = \mathbb{R}^3/(6\mathbb{Z})^3$$

is a cubiculated region homeomorphic to a 3-torus $\mathbb{T}^3 = (\mathbb{S}^1)^3$ (see Figures 2, 7 and 8 in [8] for examples of tilings of $\mathcal{R}_2$); since $\partial\mathcal{R}_2 = \emptyset$, we have $H_1(\mathcal{R}_2) = H_1(\mathcal{R}_2, \partial\mathcal{R}_2)$.

The quotient

$$\mathcal{R}_3 = (\mathbb{R}^2/(8\mathbb{Z})^2) \times [0, 4]$$

is a cubiculated region homeomorphic to $\mathbb{T}^2 \times [0, 1]$, so that $H_1(\mathcal{R}_3) = \mathbb{R}^2$. It admits tilings with different values of the flux. We have $H_1(\mathcal{R}_3, \partial\mathcal{R}_3) = \mathbb{R}$, while the map $i : H_1(\mathcal{R}_3) \to H_1(\mathcal{R}_3, \partial\mathcal{R}_3)$ is the zero homomorphism, $i = 0$ and $\mathrm{RFlux}(\mathbf{t}) = 0$ for all $\mathbf{t} \in \mathcal{T}(\mathcal{R}_3)$.

We may also remove a few cubes from a larger region; for instance, $\mathcal{R}_4 = \mathcal{R}_2 \smallsetminus (2, 4)^3$ is a 3-torus with a hole.

Remark 2.3. The examples above demonstrate independence of the assumptions of zero relative flux and two tilings having the same flux. Indeed, for $\mathcal{R}_2$ the relative and absolute homology groups coincide and one can have two tilings with the same non-zero flux, $[\mathbf{t}_1] = [\mathbf{t}_0] \neq 0 \in H_1(\mathcal{R}_2) = H_1(\mathcal{R}_2; \partial\mathcal{R}_2)$. On the other hand, all tilings of $\mathcal{R}_3$ have zero relative flux, $\mathrm{RFlux}(\mathbf{t}) = 0$, while realizing various fluxes in $H_1(\mathcal{R}_3) = \mathbb{R}^2$.

3. Twist of a tiling

Recall that given an initial tiling $\mathbf{t}_0$ we denote by $\mathcal{T}_0(\mathcal{R})$ the set of tilings $\mathbf{t}$ with $\mathrm{Flux}(\mathbf{t}) = [\mathbf{t} - \mathbf{t}_0] = 0 \in H_1(\mathcal{R})$. In general, the *twist* of a tiling is a map $\mathrm{Tw} : \mathcal{T}_0(\mathcal{R}) \to \mathbb{Z}/(m\mathbb{Z})$ where $m \in \mathbb{Z}$, $m \geq 0$. In the present paper we are concerned with the case $m = 0$, where the twist map is a function $\mathrm{Tw} : \mathcal{T}_0(\mathcal{R}) \to \mathbb{Z}$ well-defined up to an additive constant. In this case, one can define the twist of a tiling recursively and, if needed, using refinements as follows (we refer to [8] for a general combinatorial definition of the twist and more details).

Given a cubiculated region $\mathcal{R}$, the *refinement* $\mathcal{R}'$ of $\mathcal{R}$ is obtained by decomposing each unit cube in $\mathcal{R}$ into $5 \times 5 \times 5$ smaller cubes. Given a tiling $\mathbf{t}$ of $\mathcal{R}$, its refinement $\mathbf{t}'$ is a tiling of $\mathcal{R}'$: each domino d in $\mathbf{t}$ is decomposed into $5 \times 5 \times 5$ smaller dominoes, all parallel to d. We write $\mathcal{R}^{(0)} = \mathcal{R}$ and $\mathcal{R}^{(k+1)} = (\mathcal{R}^{(k)})'$ and define $\mathbf{t}^{(k)} \in \mathcal{T}_0(\mathcal{R}^{(k)})$ in a similar manner.

Theorem 3.1 ([8])**.** *Consider a cubiculated region $\mathcal{R}$ and two tilings $\mathbf{t}_0, \mathbf{t}_1$ of $\mathcal{R}$. If $\mathrm{Flux}(\mathbf{t}_1) = \mathrm{Flux}(\mathbf{t}_0)$ then there exist k such that the tilings $\mathbf{t}_0^{(k)}$ and $\mathbf{t}_1^{(k)}$ can be joined by a finite sequence of flips and trits.*

Define the set of all refinements of $\mathcal{T}_0(\mathcal{R})$ by

$$\mathcal{T}_0(\mathcal{R}^{(*)}) = \bigsqcup_{k \in \mathbb{N}} \mathcal{T}_0(\mathcal{R}^{(k)}),$$

where $\mathcal{T}_0(\mathcal{R}^{(k)}) \subseteq \mathcal{T}(\mathcal{R}^{(k)})$ is the subset of tilings $\mathbf{t}$ with $[\mathbf{t} - \mathbf{t}_0^{(k)}] = 0 \in H_1(\mathcal{R})$. One can interpret $\mathcal{T}_0(\mathcal{R}^{(*)})$ as the set of vertices of an infinite graph: two vertices $\mathbf{t}_0$ and $\mathbf{t}_1$ are connected by an edge if the tilings $\mathbf{t}_0$ and $\mathbf{t}_1$ differ by a flip, a trit or by one step of a refinement. The graph $\mathcal{T}_0(\mathcal{R}^{(*)})$ is connected, as follows directly from Theorem 3.1.

The following result gives us a recursive definition of twist in $\mathcal{T}_0(\mathcal{R})$. It is proved in [8] in different terms and (in certain cases before that) in [11]. Below we provide a different proof.

Theorem-Definition 3.2 (Cf. [8])**.** *Consider a cubiculated region $\mathcal{R}$, with an initial tiling $\mathbf{t}_0$. If the relative flux of $\mathbf{t}_0$ is zero then there exists a function $\mathrm{Tw} : \mathcal{T}_0(\mathcal{R}^{(*)}) \to \mathbb{Z}$ (unique up to an additive constant) with the following properties:*

 (1) *If two tilings $\mathbf{t}_a, \mathbf{t}_b \in \mathcal{T}_0(\mathcal{R}^{(*)})$ are joined by a flip then $\mathrm{Tw}(\mathbf{t}_b) = \mathrm{Tw}(\mathbf{t}_a)$.*
 (2) *If $\mathbf{t}_a$ and $\mathbf{t}_b$ are joined by a trit then $\mathrm{Tw}(\mathbf{t}_b) = \mathrm{Tw}(\mathbf{t}_a) \pm 1$; sign is given by the orientation of the trit.*
 (3) *If $\mathbf{t}'$ is the refinement of $\mathbf{t} \in \mathcal{T}_0(\mathcal{R}^{(*)})$ then $\mathrm{Tw}(\mathbf{t}') = \mathrm{Tw}(\mathbf{t})$.*

The function Tw is called the twist function.

We say that a trit from $\mathbf{t}_a$ to $\mathbf{t}_b$ is *positive* (respectively, *negative*) if $\mathrm{Tw}(\mathbf{t}_b) - \mathrm{Tw}(\mathbf{t}_a) = +1$ (respectively, -1).

Remark 3.3. If $\mathrm{RFlux}(\mathbf{t}_0) \neq 0$, there is $m > 0$ such that the twist function becomes a map $\mathrm{Tw} : \mathcal{T}_0(\mathcal{R}) \to \mathbb{Z}/(m\mathbb{Z})$, which, in addition to the properties listed above satisfies the following. There exist a finite sequence of tilings $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_\ell = \mathbf{t}_0$

in $\mathcal{T}_0(\mathcal{R}^{(k)})$ (for some $k \geq 0$) and a sequence of integers $t_0, t_1, \ldots, t_\ell$ such that for every i, $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ are joined by a flip or trit, $\mathrm{Tw}(\mathbf{t}_i) = t_i \bmod m$, and $t_\ell = t_0 + m$.

One should stress that all known definitions of the twist of a tiling involve a significant combinatorial part and are relatively complicated, cf. [8]; one of the main aims of the present paper is to give an alternative geometric description.

4. ROTATION CLASS AND HELICITY OF VECTOR FIELDS

4.1. **Relative rotation class.** Let M be a manifold with boundary ∂M and volume form μ. Consider a divergence-free vector field ξ on M and not necessarily tangent to its boundary.

Definition 4.1 ([1]). Consider the 2-form $\omega_\xi := i_\xi \mu$. It is closed, since ξ is divergence-free, and hence it defines the cohomology class $[\omega_\xi] \in H^2(M)$. By the Lefschetz isomorphism (the Poincaré isomorphism for manifolds with boundary) $H^2(M) \simeq H_1(M, \partial M)$. The relative homology class defined by $[\omega_\xi]$ in $H_1(M, \partial M)$ is called a *relative rotation class* of the field ξ.

If $\partial M = \emptyset$ or ξ is tangent to ∂M, then $\omega_\xi|_{\partial M} = 0$ and hence the corresponding *rotation class* of the vector field ξ is an element in $H_1(M) \simeq H^2(M, \partial M)$.

Definition 4.1 holds for M of any dimension, although our main application is in 3D.

4.2. **Helicity of fields and pipes.** Now we assume that $M \subset \mathbb{R}^3$ is a three-dimensional domain and the field ξ is divergence-free, tangent to ∂M, and has zero rotation class, i.e. it is *exact* or *null-homologous*).

Definition 4.2 ([12]). The *helicity* of a null-homologous field ξ in a domain $M \subset \mathbb{R}^3$ is the number
$$\mathrm{Hel}(\xi) := \int_M (\xi, \mathrm{curl}^{-1}\xi)\, d^3x,$$
where the vector field $\mathrm{curl}^{-1}\xi$ is a divergence-free vector potential of the field ξ (which exists since ξ is exact), i.e., $\nabla \times (\mathrm{curl}^{-1}\xi) = \xi$ and $\mathrm{div}(\mathrm{curl}^{-1}\xi) = 0$.

Let a divergence-free field ξ be confined to two narrow linked flux tubes. Its helicity can be found explicitly as follows. Suppose that the tube cores are closed curves C_1 and C_2, the fluxes of the field in the tubes are Flux_1 and Flux_2 (see Figure 3). The curves C_i are oriented so that $\mathrm{Flux}_i = Q_i \geq 0$. Assume also that there is no net twist within each tube or, more precisely, that the field trajectories foliate each of the tubes into pairwise unlinked circles and the periods of those trajectories are equal. One can show that the helicity invariant of such a field is given by
$$\mathrm{Hel}(\xi) = 2\, lk(C_1, C_2) \cdot \mathrm{Flux}_1 \cdot \mathrm{Flux}_2,$$
where $lk(C_1, C_2)$ is the (Gauss) *linking number* of C_1 and C_2, which explains the term "helicity" coined in [12], as the measure of coiling one curve about the other. Recall that the number $lk(C_1, C_2)$ for two oriented closed curves is the signed number of the intersection points of one curve with an arbitrary oriented surface spanning the other curve.

Suppose now that a divergence-free field ξ is confined to several narrow linked oriented tubes with cores given by closed curves C_i and fluxes $\mathrm{Flux}_i \geq 0$. We still
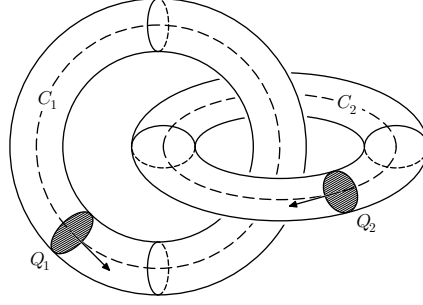
FIGURE 3. Two linked tubes: The solid tori are thin tubular neigh-
borhoods of the curves C_i. A few cross-sections are shown, while
the vector field is transversal to these sections.

assume that the field trajectories foliate each of the tubes into topological circles.
However we do not assume any longer that these field trajectories (circles) inside
the same tubes are pairwise unlinked, while their periods can now be arbitrary. The
corresponding linking of circles inside the ith tube (given by the linking number of
the core curve C_i with any satellite curve) is called the self-linking $slk(C_i)$ of the
curve C_i in the corresponding pipe.

Proposition 4.3. *The helicity of such a field ξ is*

$$\text{(2)} \qquad \text{Hel}(\xi) = 2 \sum_{i<j} lk(C_i, C_j) \cdot \text{Flux}_i \cdot \text{Flux}_j + \sum_i slk(C_i)(\text{Flux}_i)^2.$$

We recall the proof of this folklore formula in Appendix, and will use it for
computations related to tilings. Note that self-linking cannot be defined for an
isolated closed curve (since there is no canonical choice of a satellite curve), but
only for a curve with a framing, which delivers such a choice. For a field trajectory
in a pipe, neighboring trajectories provide the framing.

Remark 4.4. While helicity was defined above by using the Riemannian metric
on M, it is actually a topological characteristic of a divergence-free vector field,
depending only on the choice of a volume form on the manifold. Namely, consider
a manifold M (possibly with boundary) with a volume form μ, and let ξ be a
null-homologous vector field on M (tangent to the boundary). The divergence-free
condition means that the Lie derivative of μ along ξ vanishes: $L_\xi \mu = 0$, or, which
is the same, the substitution $i_\xi \mu =: \omega_\xi$ of the field ξ into the 3-form μ is a closed
2-form: $d\omega_\xi = 0$. If ξ is moreover null-homologous, then ω_ξ is actually an exact
2-form: $\omega_\xi = d\alpha$ for some 1-form α, called a potential. (On a simply connected
manifold M any divergence-free vector field is null-homologous.)

Definition 4.5 ([2])**.** The *helicity* $\text{Hel}(\xi)$ of a null-homologous field ξ on a three-
dimensional manifold M equipped with a volume element μ is the integral of the
wedge product of the form $\omega_\xi := i_\xi \mu$ and its potential:

$$\text{Hel}(\xi) = \int_M d\alpha \wedge \alpha, \text{ where } d\alpha = \omega_\xi.$$

As discussed, this generalizes Definition 4.2.

An immediate consequence of this purely topological (i.e. metric-free) definition is the following Arnold's theorem: The helicity $\mathrm{Hel}(\xi)$ is preserved under the action on ξ of a volume-preserving diffeomorphism of M. In this sense $\mathrm{Hel}(\xi)$ is a topological invariant: it can be defined without coordinates or a choice of metric, and hence every volume-preserving diffeomorphism carries a field ξ into a field with the same helicity. The physical significance of helicity is due to the fact that it appears as a conservation law in both fluid mechanics and magnetohydrodynamics: Kelvin's law implies the invariance of helicity of the vorticity field for an ideal fluid motion.

4.3. **Relative helicity for vector fields.** First we recall the definition of relative helicity for vector fields (elaborating the definitions in [5] and [3]). Suppose that a domain in the space $\mathbb{R}^3$ (or a closed oriented manifold M^3) is split into two regions A and B separated by a boundary surface S. Assume further that two divergence-free vector fields ξ and η in A coincide on the boundary S and have the same extension ζ into the region B. Call the extended fields in M respectively $\tilde{\xi}$ and $\tilde{\eta}$. Abusing notation we will denote them as the sums $\tilde{\xi} = \xi + \zeta$ and $\tilde{\eta} = \eta + \zeta$, where ξ, η and ζ are regarded as the (discontinuous) vector fields in the entire manifold M with supports $\mathrm{supp}\,\xi, \mathrm{supp}\,\eta \subseteq A$ and $\mathrm{supp}\,\zeta \subseteq B$. (Alternatively one can modify the fields ξ, η, ζ in a narrow neighborhood of the boundary S to avoid discontinuity; this will not affect the argument below.)

Assume that both the extended fields $\tilde{\xi}$ and $\tilde{\eta}$ are null-homologous in M and tangent to ∂M, while we do not impose any restrictions on the topology of the sets A, B, or M.

Definition 4.6. The difference $\Delta\,\mathrm{Hel}(\xi, \eta) := \mathrm{Hel}(\tilde{\xi}) - \mathrm{Hel}(\tilde{\eta})$ measures the *relative helicity* of the fields ξ and η in A.

Theorem 4.7. *The relative helicity* $\Delta\,\mathrm{Hel}(\xi, \eta) = \mathrm{Hel}(\tilde{\xi}) - \mathrm{Hel}(\tilde{\eta})$ *is independent of their common extension* ζ *in the region* B.

Proof. Define the (closed) two-forms α, β, and ω by substituting the vector fields ξ, η, and ζ into the volume form μ on M: $i_\xi\mu = \alpha$, $i_\eta\mu = \beta$, and $i_\zeta\mu = \omega$. Then one has to show that the difference

$$\Delta\,\mathrm{Hel} = \mathrm{Hel}(\tilde{\xi}) - \mathrm{Hel}(\tilde{\eta}) = \int_M (\alpha + \omega) \wedge d^{-1}(\alpha + \omega) - \int_M (\beta + \omega) \wedge d^{-1}(\beta + \omega)$$

does not depend on ω. Note that the 2-forms $\alpha + \omega$ and $\beta + \omega$ are exact in M (because of the assumption that the fields $\tilde{\xi}$ and $\tilde{\eta}$ are null-homologous) and taking d^{-1} of them makes sense.

Assume first that the regions A and B are both simply-connected. One readily obtains

$$\Delta\,\mathrm{Hel} = \int_M \alpha \wedge d^{-1}\alpha - \int_M \beta \wedge d^{-1}\beta + \int_M (\alpha - \beta) \wedge d^{-1}\omega + \int_M \omega \wedge d^{-1}(\alpha - \beta).$$

Here d^{-1} applied to a discontinuous 2-form is a continuous 1-form (the "form-potential"). The terms in $\Delta\,\mathrm{Hel}$ containing ω are $\int_M (\alpha - \beta) \wedge d^{-1}\omega + \int_M \omega \wedge d^{-1}(\alpha - \beta)$, and we want to show that their contribution vanishes.

Integrating by parts one of the terms, we come to $2\int_M (\alpha - \beta) \wedge d^{-1}\omega$, which, in turn, is equal to $2\int_A (\alpha - \beta) \wedge d^{-1}\omega$, since $\mathrm{supp}(\alpha - \beta) \subseteq A$.

On the other hand, in the domain A the 1-form $d^{-1}\omega$ is the differential of a function, $d^{-1}\omega = dh$. Indeed, it is closed (the differential $d(d^{-1}\omega) = \omega$ vanishes in A due to the condition on $\operatorname{supp}\zeta = \operatorname{supp}\omega \subseteq B$), and hence it is exact in the simply connected region A. Hence,

$$2\int_A (\alpha - \beta) \wedge d^{-1}\omega = 2\int_A (\alpha - \beta) \wedge dh = 2\int_S h(\alpha - \beta) = 0,$$

where the last equality is due to the assumption on the identity of the fields ξ and η on the boundary S. This proves that $\Delta\operatorname{Hel}$ is unaffected by the choice of the extension ζ.

Now consider the case of arbitrary domains A and B. In the latter case, the 2-form ω is not exact, but only closed (and such that $\alpha + \omega$ and $\beta + \omega$ are exact, i.e. $[\alpha + \omega] = [\beta + \omega] = 0 \in H^2(M)$). To emphasize the ambiguity in the choice of ω we represent it as $\omega = \omega_0 + \gamma$, where we fix some 'reference' closed 2-form ω_0, while an arbitrary 2-form γ is exact, and both ω_0 and γ have their support in B. Now the difference we are studying is

$$\operatorname{Hel}(\tilde\xi) - \operatorname{Hel}(\tilde\eta) = \int_M (\alpha + \omega_0 + \gamma) \wedge d^{-1}(\alpha + \omega_0 + \gamma) - \int_M (\beta + \omega_0 + \gamma) \wedge d^{-1}(\beta + \omega_0 + \gamma),$$

and we would like to show that it is independent of the choice of an exact 2-form γ. We claim that this is evident once we introduce new forms $\bar\alpha = \alpha + \omega_0$ and $\bar\beta = \beta + \omega_0$ and rewrite the difference as

$$\operatorname{Hel}(\tilde\xi) - \operatorname{Hel}(\tilde\eta) = \int_M (\bar\alpha + \gamma) \wedge d^{-1}(\bar\alpha + \gamma) - \int_M (\bar\beta + \gamma) \wedge d^{-1}(\bar\beta + \gamma),$$

thus mimicking the expression above for simply-connected regions. Indeed, now it boils down to the same computation leading to the form

$$2\int_A (\bar\alpha - \bar\beta) \wedge d^{-1}\gamma = 2\int_A (\alpha - \beta) \wedge d^{-1}\gamma = 0.$$

The latter expression vanishes for the same reason as above: $\alpha - \beta$ has the support in A and it is exact (since both $\alpha + \omega$ and $\beta + \omega$ are exact), while $d^{-1}\gamma$ is closed in A. This concludes the proof that relative helicity is well-defined in the general case. $\square$

5. Pipes, fluxes, and shells

In this section we present the constructions taking us from the context of cubiculated regions and domino tilings to that of vector fields and helicities.

5.1. The five pipes construction.
In the first construction, for a domino tile $d \subset \mathbb{R}^3$, we construct a specific divergence-free vector field ξ_d in d. The vector field ξ_d is confined to five narrow flux tubes. The tubes are the boundaries of tubular neighborhoods of smooth curves approximating the polygonal lines. This is therefore the situation discussed in Section 4.2. If the domino is $d = [0,2] \times [0,1] \times [0,1]$, the tubes are thin neighborhoods of the following five oriented polygonal lines,

also shown in Figure 4:

$$(0, \tfrac{1}{2}, \tfrac{1}{2}) \longrightarrow (2, \tfrac{1}{2}, \tfrac{1}{2}),$$

$$(\tfrac{1}{2}, 0, \tfrac{1}{2}) \to (\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}) \to (\tfrac{3}{2}, \tfrac{1}{4}, \tfrac{1}{2}) \to (\tfrac{3}{2}, 0, \tfrac{1}{2}),$$

$$(\tfrac{1}{2}, \tfrac{1}{2}, 0) \to (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}) \to (\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{4}) \to (\tfrac{3}{2}, \tfrac{1}{2}, 0),$$

$$(\tfrac{1}{2}, 1, \tfrac{1}{2}) \to (\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) \to (\tfrac{3}{2}, \tfrac{3}{4}, \tfrac{1}{2}) \to (\tfrac{3}{2}, 1, \tfrac{1}{2}),$$

$$(\tfrac{1}{2}, \tfrac{1}{2}, 1) \to (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{4}) \to (\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{3}{4}) \to (\tfrac{3}{2}, \tfrac{1}{2}, 1).$$

Notice that the figure is symmetric under rotations by $\frac{\pi}{2}$ around the central line, and also under reflection in the plane $y = \frac{1}{2}$. The narrow tubes are assumed to be likewise symmetric.
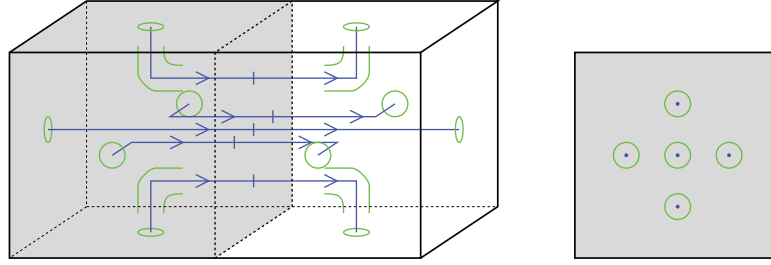

FIGURE 4. A domino, five polygonal lines inside it and the respective smooth tubes approximating the lines. Tubes are only very partially drawn in order to keep the figure simple. Lines are always oriented from black to white. At the right, the central square separating the two unit cubes of the domino, crossed by five lines and five tubes.

For other dominoes, the figure is appropriately rotated and translated, while the arrows always point from a black cube to a white one. Thus, pipes enter the domino through a neighborhood of the center of a face of the black cube and exit the domino near the center of a face of the white cube.

If we have a tiling $\mathbf{t}$ of a region $\mathcal{R}$, the same construction is performed in each domino. This produces a vector field $\xi_{\mathbf{t}}$ in $\mathcal{R}$. Assume that the flux through each pipe is equal to the same $\varphi > 0$. We may assume that $\xi_{\mathbf{t}}$ is smooth in the interior of $\mathcal{R}$ with the support in the union of the pipes. Notice that the restriction of $\xi_{\mathbf{t}}$ to a thin neighborhood of the boundary $\partial \mathcal{R}$ does not depend on the choice of the tiling $\mathbf{t}$.

5.2. **Proof of theorem on relative flux.** Here we prove Theorem 1.2 that the relative flux $\mathrm{RFlux}(\mathbf{t}) \in H_1(\mathcal{R}, \partial\mathcal{R})$ of a tiling $\mathbf{t}$ coincides modulo a factor with the relative rotation class $[\xi_{\mathbf{t}}]$ the vector field $\xi_{\mathbf{t}}$ obtained via the 5-pipe construction: $[\xi_{\mathbf{t}}] = 6\varphi\, \mathrm{RFlux}(\mathbf{t})$.

Proof of Theorem 1.2. Adjust the field $\xi_{\mathbf{t}}$ as follows. Add to it one more pipe in the shape of a long torus, see Figure 5, which will follow the black-to-white direction as on Figure 4 and then returns after a U-turn, without self-linking or linking with any other pipe.

With this addition the new field $\bar{\xi}_{\mathbf{t}}$ in the tile: (1) is still divergence-free without singular points, (2) has 6 pipes in the black-to-white direction and 1 tube in the
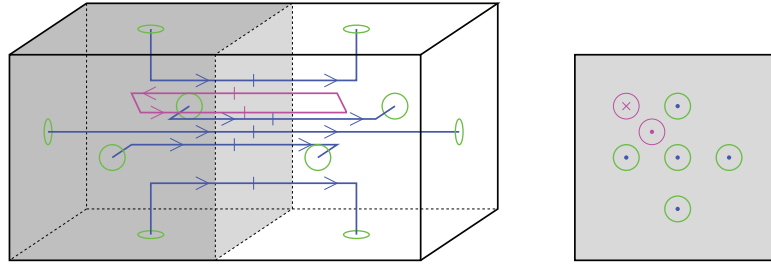
FIGURE 5. An additional narrow toroidal pipe with a field flux φ and directed along the axis of the tile

opposite direction, (3) has one pipe from each face towards the vicinity of the center of the black cube and one pipe from (the vicinity of) the center of the white cube towards its faces, (4) the flux in each narrow pipe is φ.

With this construction, mimicking the definition of the chain $6\mathbf{t}-\mathbf{q}_1$, one achieves that the intersection number of the latter (multiplied by φ) with any closed 2D surface in $\mathcal{R}$ transversal to this chain is equal to the flux of the corresponding vector field $\bar{\xi}_{\mathbf{t}}$ through such a surface. Hence $[\bar{\xi}_{\mathbf{t}}]$ realizes the same homology class in $H_1(\mathcal{R}, \partial\mathcal{R})$ as $\varphi[6\mathbf{t} - \mathbf{q}_1]$.   □

Remark 5.1. While this adjusted "6-pipe" field $\bar{\xi}_{\mathbf{t}}$ more adequately represents the $6\mathbf{t} - \mathbf{q}_1$, the contribution of the extra "long solid torus" pipe vanishes to both the flux and the helicity, discussed below, so one can confine to the 5-pipe construction. However, it explains the origin of the factor 6 in the formulas.

Remark 5.2. The chain $\mathbf{q}_1$, as well as the part of the field $\bar{\xi}_{\mathbf{t}}$ defined by it, is determined solely by the coloring of the region $\mathcal{R}$. It defines a net of pipes with sources at each white center and sinks at each black center, "following" the chain $\mathbf{q}$. Then the tiling $\mathbf{t}$ can be regarded as an additional system of pipes, 6 times stronger, joining the neighboring centers according to the tiling geometry and satisfying the Kirchhoff junction rule.

Note that the same claim about proportionality of the relative flux $\mathrm{RFlux}(\mathbf{t}) \in H_1(\mathcal{R}, \partial\mathcal{R})$ and the relative rotation class $[\xi_{\mathbf{t}}]$ of the corresponding field holds mutatis mutandis for tilings in any dimension.

5.3. **An isolating shell.** Consider first the case where $\mathcal{R} \subset \mathbb{R}^3$ is a well behaved region, perhaps a box. We apply the construction discussed in Section 4.3. Let $M = \mathbb{R}^3$, $A = \mathcal{R}$ (possibly rounded at the corners) and $B = M \smallsetminus A$. Choose a divergence-free vector field ζ in B coinciding with $\xi_{\mathbf{t}}$ near $\partial\mathcal{R}$. For simplicity, we may choose ζ to be also confined to a few tubes. We call ζ or, more precisely, (M, A, B, ζ), an *isolating shell*. Given a tiling $\mathbf{t}$, let $\tilde{\xi}_{\mathbf{t}}$ be the corresponding smooth, divergence-free vector field in M. Since M is contractible, the extended vector field $\tilde{\xi}_{\mathbf{t}}$ is trivially null-homologous.

The more general case is similar, with a few adjustments. If $\partial\mathcal{R} = \emptyset$, nothing needs to be done. Otherwise, recall the assumption that $\partial\mathcal{R}$ is a topological manifold. We can therefore construct an open manifold M_0 with $A = \mathcal{R} \subset M_0$ by taking M_0 to be the union of A with a tubular neighborhood of $\partial\mathcal{R}$. A minor difficulty is that $M_0 \smallsetminus A$ may well be disconnected. If this happens, we construct $M \supseteq M_0$ by adding tubes (disjoint from $\mathcal{R}$) connecting the components of $M_0 \smallsetminus A$. We thus

have an open manifold M, $A = \mathcal{R} \subset M$ and a connected subset $B = M \smallsetminus A$. As above, the vector fields $\xi_{\mathbf{t}}$ for all tilings $\mathbf{t}$ coincide in a neighborhood of $\partial\mathcal{R} \subset M$. Again, choose a divergence-free vector field ζ in B, also confined to a few tubes and coinciding with $\xi_{\mathbf{t}}$ near $\partial\mathcal{R}$. The desired isolating shell is (M, A, B, ζ). We need to check whether the extended vector field $\tilde{\xi}_{\mathbf{t}} = \xi_{\mathbf{t}} + \zeta$ is null-homologous in M.

Lemma 5.3. *Consider a cubiculated region $\mathcal{R}$ and a domino tiling $\mathbf{t}$. For $M \supset \mathcal{R}$ as above, the vector field ζ can be chosen so that $\tilde{\xi}_{\mathbf{t}} = \xi_{\mathbf{t}} + \zeta$ is null-homologous in M if and only if $\mathrm{RFlux}(\mathbf{t}) = 0$.*

Proof. In this proof, we interpret the vector fields as thin tubular neighborhoods of weighted oriented curves. Thus, we can obtain a class in $H_1(M)$ (with real coefficients) from a vector field.

Assume first that ζ has been chosen so that $\tilde{\xi}_{\mathbf{t}}$ is null-homologous in M. Inclusion defines a map $i : H_1(M) \to H_1(M, B) = H_1(\mathcal{R}, \partial\mathcal{R})$. Since $[\tilde{\xi}_{\mathbf{t}}] = 0$ we have $i([\tilde{\xi}_{\mathbf{t}}]) = 0 \in H_1(\mathcal{R}, \partial\mathcal{R})$. But $i([\tilde{\xi}_{\mathbf{t}}]) = [\xi_{\mathbf{t}}] = 6\varphi\,\mathrm{RFlux}(\mathbf{t})$ (from Theorem 1.2), proving one implication.

Conversely, assume that $\mathrm{RFlux}(\mathbf{t}) = 0$. Construct a weighted oriented surface S in $\mathcal{R}$ such that ∂S coincides with $\mathbf{t} - \frac{\mathbf{q}_1}{6}$ in the interior of $\mathcal{R}$. Use the part of ∂S on $\partial\mathcal{R}$ as guides to construct the required pipes in the intersection of B with a thin tubular neighborhood of $\partial\mathcal{R}$, i.e., to construct ζ. The surface S is a witness to the fact that $\tilde{\xi}_{\mathbf{t}}$ is null-homologous in M. $\square$

6. KEY EXAMPLES

In this section we present the first examples of isolating shells and of the five pipes construction. The results of these examples will be used in later proofs.

Example 6.1. Let $\mathcal{R} = [0,2] \times [0,2] \times [0,1]$, the $2 \times 2 \times 1$ box: $\mathcal{R}$ admits exactly two tilings, $\mathbf{t}_0$ and $\mathbf{t}_1$, shown in Figure 6.

FIGURE 6. The only two tilings of the $2 \times 2 \times 1$ box are joined by a flip
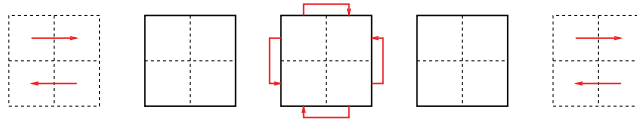

FIGURE 7. A valid isolating shell for the $2 \times 2 \times 1$ box. We show the planes $z = -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}$.

Figure 7 shows a valid isolating shell for $\mathcal{R}$. Figure 8 shows the two vector fields $\tilde{\xi}_{\mathbf{t}_0}$ and $\tilde{\xi}_{\mathbf{t}_1}$: both have zero helicity. Indeed, a curve C_0 is *trivial* (for a family of curves (C_i) including C_0) if there exists a contractible open neighborhood $A \supset C_0$ intersecting no other curve C_i and $slk(C_0) = 0$. In this case, C_0 can be discarded from the family without changing the helicity. In the present example, every curve is trivial.

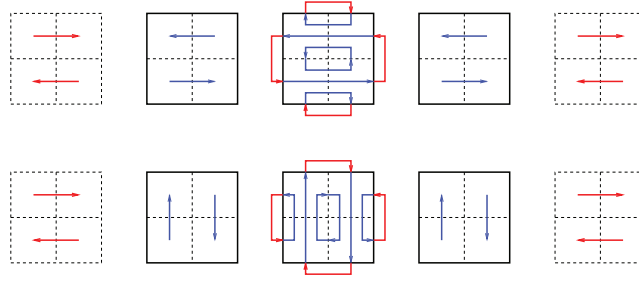FIGURE 8. The vector fields $\tilde{\xi}_{\mathbf{t}_i}$ for the only two tilings of the $2 \times 2 \times 1$ box (with the same planes as Figure 7)

Example 6.2. Let $\mathcal{R}$ be the region shown in Figure 9, i.e.,

$$\mathcal{R} = [0,2]^3 \smallsetminus (((1,2] \times (1,2] \times [0,1)) \cup ([0,1) \times [0,1) \times (1,2])).$$

It is a $2 \times 2 \times 2$ cube without two small, $1 \times 1 \times 1$ cubes on a diagonal. The region $\mathcal{R}$ admits precisely two tilings $\mathbf{t}_0$ and $\mathbf{t}_1$, also shown in Figure 9. Notice that the two tilings differ by a positive trit from $\mathbf{t}_0$ to $\mathbf{t}_1$. They also differ by reflection in the plane $x = y$. In this case there is no natural definition of $\mathrm{Tw}(\mathbf{t})$ as an integer, that is, we need to choose a base tiling in a more or less arbitrary manner. If we choose $\mathbf{t}_0$ as a base tiling we then have $\mathrm{Tw}(\mathbf{t}_0) = 0$ and $\mathrm{Tw}(\mathbf{t}_1) = 1$.
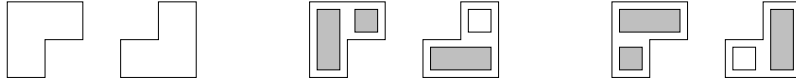

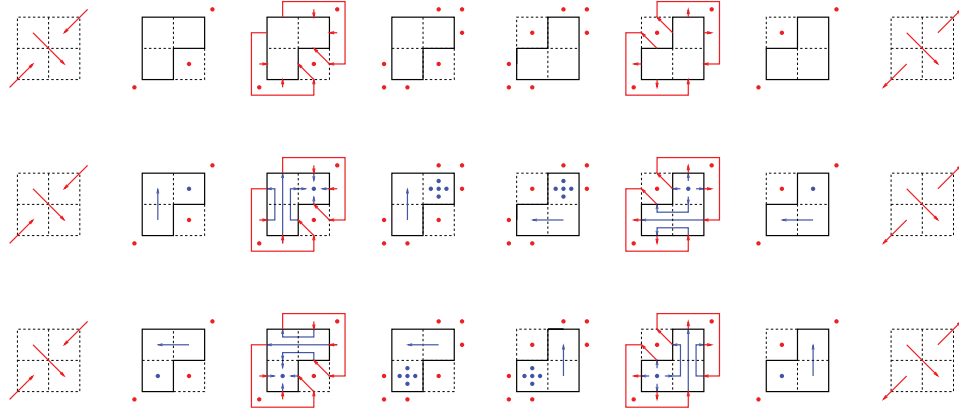FIGURE 9. The region $\mathcal{R}$ and its two tilings $\mathbf{t}_0$ and $\mathbf{t}_1$

FIGURE 10. A valid isolating shell and the vector fields $\tilde{\xi}_{\mathbf{t}_0}$ and $\tilde{\xi}_{\mathbf{t}_1}$. We show the following planes: $-\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \frac{9}{4}$.

Figure 10 shows a valid isolating shell for $\mathcal{R}$ and the two vector fields $\tilde{\xi}_{\mathbf{t}_0}$ and $\tilde{\xi}_{\mathbf{t}_1}$. Notice that $\tilde{\xi}_{\mathbf{t}_1}$ is the mirror image of $\tilde{\xi}_{\mathbf{t}_0}$ under reflection in the plane $x = y$. This implies $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_1}) = -\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_0})$.
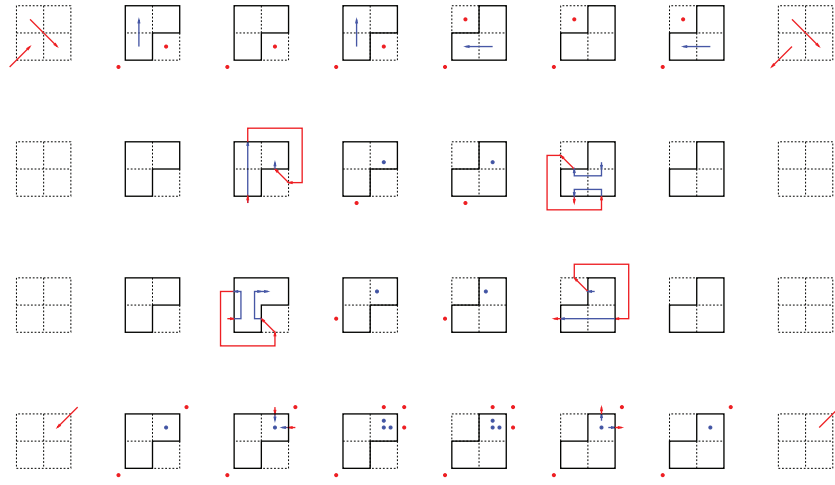
FIGURE 11. Curves for the vector fields $\tilde{\xi}_{\mathbf{t}_0}$ from Example 6.2. The first three rows show three nontrivial curves; the last row shows several trivial curves.
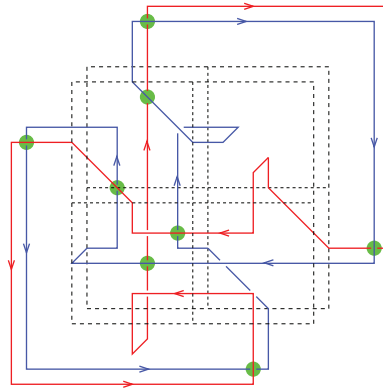

FIGURE 12. Example of computation of the linking number, between the curves in the second and third rows of Figure 11; see Example 6.2

Figure 11 shows the several curves describing $\tilde{\xi}_{\mathbf{t}_0}$. There are three nontrivial curves C_i, $1 \leq i \leq 3$ (one per row in Figure 11) and several trivial curves. An explicit computation shows that $lk(C_i, C_j) = -2$ for all $i \neq j$ and $slk(C_i) = -2$ for all i. Figure 12 shows a sample computation that $lk(C_2, C_3) = -2$. Indeed, we draw projections of these two curves onto (a small perturbation of) the horizontal xy plane (red for C_2, blue for C_3). This perturbed plane is chosen so that the curves' projections on it intersect transversally. There are eight intersection points between the projections of different curves (indicated in green). Computing the signs, we verify that six of them are negative and two are positive, yielding $lk(C_2, C_3) = -2$. Similarly, Figure 13 shows that $slk(C_2) = -2$. Here we project C_2 (red) and a satellite curve C_2^{sat} (blue). There are six intersection points between
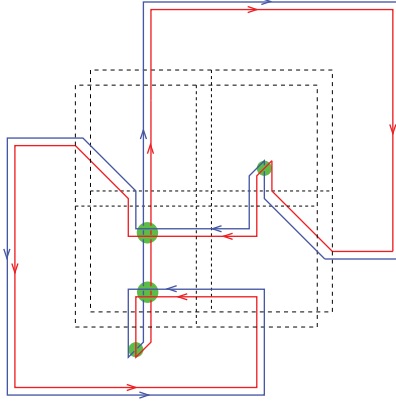
FIGURE 13. Example of computation of the self-linking number
for the curve in the second row of Figure 11; see Example 6.2

different curves (indicated in green, with larger disks indication pairs of twin inter-
section points). There are five negative and one positive intersection and therefore
$slk(C_2) = lk(C_2, C_2^{\mathrm{sat}}) = -2$. Other cases are treated similarly.

It is convenient to keep this data in the 3×3 tabulation matrix L whose entries
are $l_{ij} = lk(C_i, C_j)$ for $i \neq j$ and $l_{ij} = slk(C_i)$ for $i = j$:

$$L = \begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{pmatrix}.$$

Proposition 4.3 implies $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_0}) = -18\varphi^2$ and $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_1}) = 18\varphi^2$. We then have (for
all $\mathbf{t} \in \mathcal{T}(\mathcal{R})$)

(3) $$\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}}) = 36\varphi^2 \mathrm{Tw}(\mathbf{t}) + C, \qquad C = -18\varphi^2.$$

The value of the constant C depends on two arbitrary choices made above: the
choice of base tiling and the choice of isolating shell.

Remark 6.3. Strictly speaking, the region $\mathcal{R}$ in Example 6.2 must be massaged to
make its boundary $\partial\mathcal{R}$ a manifold, since the point $(1, 1, 1)$ is singular. This can be
easily achieved by considering the boundary $\partial\mathcal{R}_\epsilon$ of a small ϵ-neighborhood $\mathcal{R}_\epsilon$ of
$\mathcal{R}$, as this will not affect the computations of Hel and Tw. Another solution is to
include in $\mathcal{R}$ a domino covering either of the missing cubes, such as $[1, 3] \times [1, 2] \times
[0, 1]$. The isolating shell needs to be slightly modified but we arrive at the same
conclusions.

7. Proof of the main theorem on twist and helicity

Our aim is to prove Theorem 1.1, i.e., to relate $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}})$ with the twist $\mathrm{Tw}(\mathbf{t})$ for
a tiling of zero relative flux and satisfying the properties of Theorem 3.2. We prove
two lemmas in this direction.

Lemma 7.1. *Let $\mathcal{R}$ be a cubiculated region with a fixed isolating shell. Let $\mathbf{t}_0, \mathbf{t}_1$ be domino tilings of $\mathcal{R}$ of zero relative flux and let $\xi_{\mathbf{t}_i}$ be the corresponding divergence-free vector fields.*

(1) *If $\mathbf{t}_0$ and $\mathbf{t}_1$ differ by a flip then $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_1}) = \mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_0})$.*
(2) *If there is a positive trit from $\mathbf{t}_0$ to $\mathbf{t}_1$ then $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_1}) = \mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_0}) + 36\varphi^2$.*

Proof. To see the invariance under the flip (item (1)) we start with A being the union of the four unit cubes involved in the flip (as in Example 6.1) and $B = M \smallsetminus A$ its any isolating shell. The fields $\xi_{\mathbf{t}_i}$ are defined and different in A, but continued into B as the same field ζ. One can see that the relative helicities of $\tilde{\xi}_{\mathbf{t}_i} = \xi_{\mathbf{t}_i} + \zeta$ coincide, $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_1}) = \mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_0})$. (For instance, for the choice of continuation ζ in Figure 7 both helicities vanish.) For a general region $\mathcal{R}$ and a fixed isolating shell B with $M := \mathcal{R} \cup B$ we still define A as the four cubes "participating in the flip" from $\xi_{\mathbf{t}_0}$ to $\xi_{\mathbf{t}_1}$, while regard its complement $B' := M \smallsetminus A$ as a new isolating shell. The same example shows the invariance of relative helicity, $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_1}) = \mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_0})$.

The change of relative helicity under the trit (item (2)) we define A to be the union of the six unit cubes involved in the trit. The rest of the proof is similar, except that we now use Example 6.2. $\qquad\square$

Our next lemma considers refinements. We must first define how to refine an isolating shell. Consider a cubiculated region $\mathcal{R}$, a tiling $\mathbf{t}$ and a valid isolating shell (M, A, B, ζ). The boundary $\partial\mathcal{R}$ is quadriculated: in the middle of each square, ζ draws a pipe with flux φ, pointing in or out according to color. That pipe is connected to somewhere in $\partial\mathcal{R}$ to another square of the opposite color.

When we refine $\mathcal{R}$ to obtain $\mathcal{R}'$, each old large square of $\partial\mathcal{R}$ is decomposed into 25 new small squares of $\partial\mathcal{R}'$. Assume without loss of generality that the old pipe in ζ is now in the central small square. Construct 12 new short pipes matching the other 24 small squares as in Figure 14. (Note that different choices of the horizontal direction on a square face of $\partial\mathcal{R}$ define equivalent isolating shells.) This defines ζ' and the desired isolating shell for $\mathcal{R}'$.
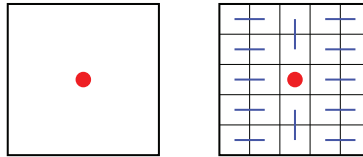

FIGURE 14. A square in $\partial\mathcal{R}$ is decomposed into 25 small squares in $\partial\mathcal{R}'$. We refine an isolating shell by adding short pipes joining the centers of the unmatched new squares.

Lemma 7.2. *Let $\mathcal{R}$ be a cubiculated region with a fixed isolating shell. Refine $\mathcal{R}$ to obtain $\mathcal{R}'$ and a corresponding isolating shell. Let $\mathbf{t}$ be a domino tiling of $\mathcal{R}$ and $\mathbf{t}'$ be its refinement, a tiling of $\mathcal{R}'$. Let $\xi_{\mathbf{t}}$ and $\xi_{\mathbf{t}'}$ be the corresponding divergence-free vector fields. We have $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}}) = \mathrm{Hel}(\tilde{\xi}_{\mathbf{t}'})$.*
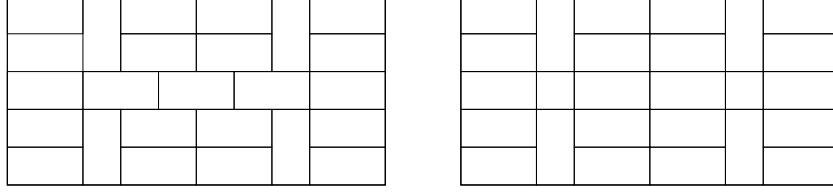
FIGURE 15. The tiling $\mathbf{t}^{\star}$, a minor modification of the refinement $\mathbf{t}'$. We show here two floors of a domino of $\mathbf{t}$. The large domino is drawn horizontally. At the left, we draw the central floor. At the right, any of the other 4 floors.

Proof. We first perform a few flips on $\mathbf{t}'$ to obtain a more drawable tiling $\mathbf{t}^{\star}$ of $\mathcal{R}'$: by Lemma 7.1, $\mathrm{Hel}(\xi_{\mathbf{t}^{\star}}) = \mathrm{Hel}(\xi_{\mathbf{t}'})$. The tiling $\mathbf{t}^{\star}$ is drawn in Figure 15. Notice that each domino of $\mathbf{t}$ is tiled by $125 = 5^3$ small dominoes of $\mathbf{t}^{\star}$.

We now construct the vector field $\xi_{\mathbf{t}^{\star}}$, that is, the pipe system corresponding to the tiling $\mathbf{t}^{\star}$ as in Figure 16.
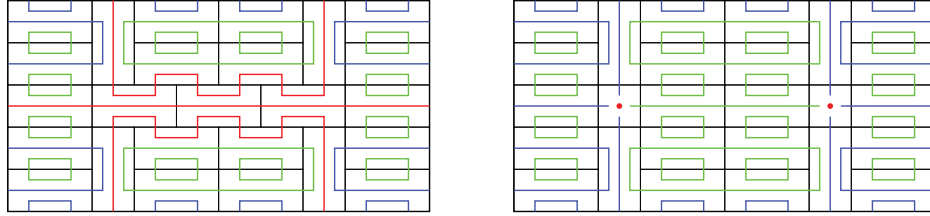

FIGURE 16. The vector field $\xi_{\mathbf{t}^{\star}}$. The position is the same as in Figure 15. The pipes which were already present in $\xi_{\mathbf{t}}$ are drawn in red and remain essentially unchanged. The remaining pipes correspond to new curves, all trivial.

The old pipes (that is, the pipes which were already in $\xi_{\mathbf{t}}$) remain essentially as they were (in $\xi_{\mathbf{t}}$), with no change of linking or self-linking numbers. The new pipes define only trivial links. This implies $\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}}) = \mathrm{Hel}(\tilde{\xi}_{\mathbf{t}^{\star}})$, completing the proof. □

We are ready to state and prove our main result, Theorem 1.1. We restate it in a slightly different way.

Theorem 7.3 (=Theorem 1.1′). *Let $\mathcal{R}$ be a cubiculated region. Let $\mathbf{t}_0$ be a tiling of $\mathcal{R}$ such that $\mathrm{RFlux}(\mathbf{t}_0) = 0 \in H_1(\mathcal{R}, \partial\mathcal{R})$. Construct an isolating shell for $\mathbf{t}_0$. There exists a constant $C \in \mathbb{R}$ such that*

$$\mathrm{Tw}(\mathbf{t}) = \frac{1}{36\varphi^2}\,\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}}) + C$$

for all $\mathbf{t} \in \mathcal{T}_0(\mathcal{R})$.

Proof. Recall that given an initial tiling $\mathbf{t}_0$, the set of all refinements of tilings $\mathcal{T}_0(\mathcal{R})$ with the same flux is denoted by $\mathcal{T}_0(\mathcal{R}^{(*)})$. Define $\widetilde{\mathrm{Tw}} : \mathcal{T}_0(\mathcal{R}^{(*)}) \to \mathbb{R}$ by

$$\widetilde{\mathrm{Tw}}(\mathbf{t}) = \frac{1}{36\varphi^2}\left(\mathrm{Hel}(\tilde{\xi}_{\mathbf{t}}) - \mathrm{Hel}(\tilde{\xi}_{\mathbf{t}_0})\right).$$

We claim that $\widetilde{\mathrm{Tw}}$ assumes integer values and is the twist function, as in Theorem-Definition 3.2. Indeed, properties (1) and (2) follow from items (1) and (2) of Lemma 7.1, while its property (3) follows from Lemma 7.2. The fact that $\widetilde{\mathrm{Tw}}$ assumes integer values now follows from the connectivity of graph $\mathcal{T}_0(\mathcal{R}^{(*)})$ (Theorem 3.1). Uniqueness of Tw up to an additive constant (which also follows from Theorem 3.1) completes the proof. □

Remark 7.4. Theorem 7.3 also gives an alternative proof of Theorem-Definition 3.2. Indeed, we constructed a twist function Tw assuming values in $\mathbb{Z}$. In [8] also the setting with $m \neq 0$ is discussed from a combinatorial viewpoint, where m is related to the relative flux of the initial tiling. One can prove that $m = 0$ if and only if $\mathrm{RFlux}(\mathbf{t}_0) = 0$, but we do not discuss this here as the proof uses different (combinatorial) tools.

8. A 3D 'CROSSES-AND-TOES' EXAMPLE

In this section we consider yet one more, larger example of tiling, which is of interest by itself. One can use this example and a slightly different construction of the set A_{new} for the proof of the second part of Lemma 7.1.

Example 8.1. Let $\mathcal{R} = [0, 3] \times [0, 3] \times [0, 2]$, the $3 \times 3 \times 2$ box, the same shown in Figure 2. A valid isolating shell is shown in Figure 17. Figure 18 shows $\tilde{\xi}$ (i.e., the tubes) for the fifth tiling in Figure 2, the one with nine vertical dominoes. Even though there are many curves, they are all trivial and helicity is again zero.
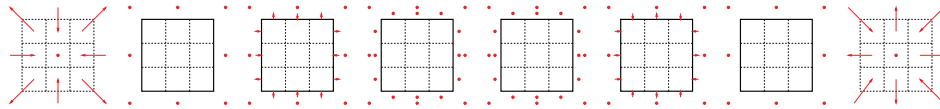

FIGURE 17. A valid isolating shell for the $3 \times 3 \times 2$ box. We show the following planes: $-\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{3}{2}$, $\frac{7}{4}$ and $\frac{9}{4}$. The red dots are vertical tubes. There is yet another central vertical tube, not entirely shown.
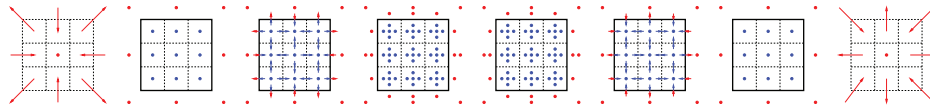

FIGURE 18. The five-tube version of the vertical tiling of the $3 \times 3 \times 2$ box (Example 8.1)

Let $\mathbf{t}_0$ be the first tiling in Figure 2, the one with nonzero twist, so that $\mathrm{Tw}(\mathbf{t}_0) = -1$. Figure 19 shows $\mathbf{t}_0$ again; on the second row, the tubes for $\xi_{\mathbf{t}_0}$. The other rows show one curve at a time: call them C_1 to C_6, from top to bottom. There is a central vertical curve C_7, not shown.
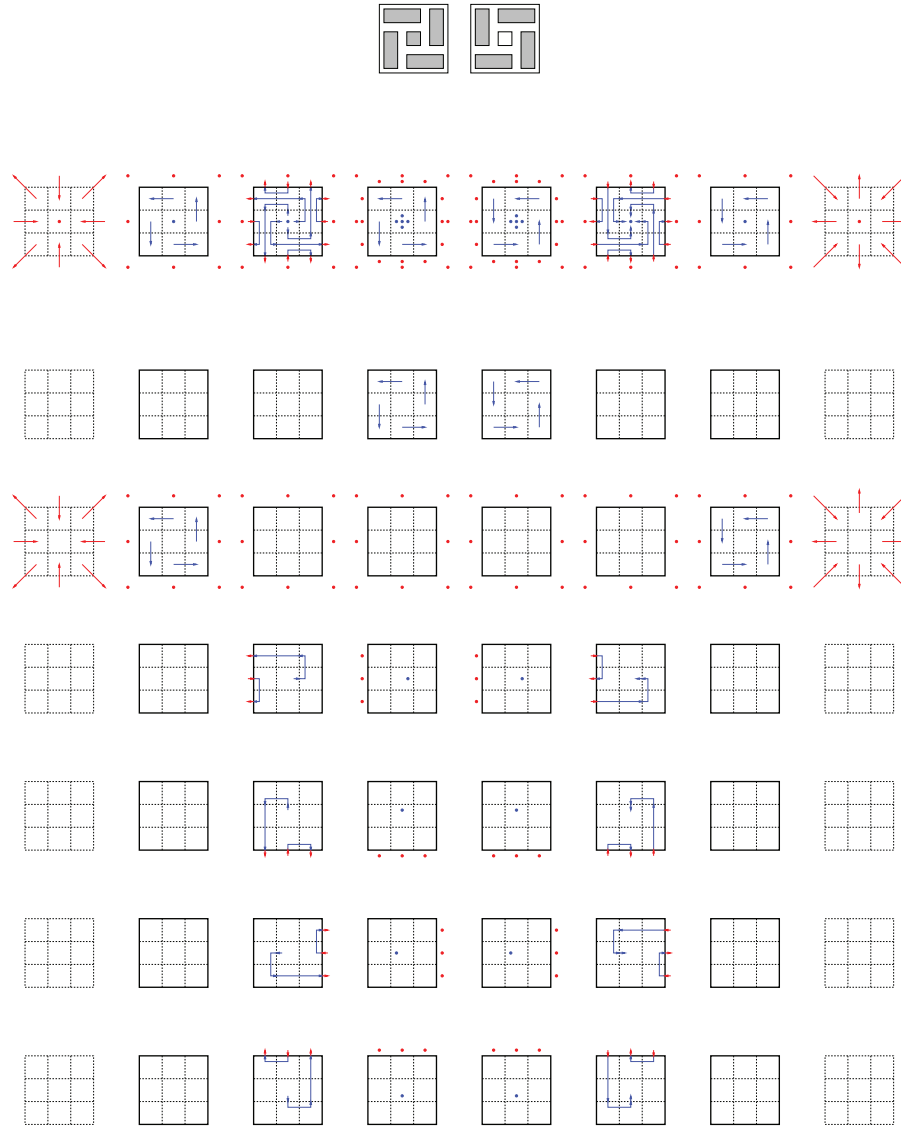
FIGURE 19. On the first row, the tiling $\mathbf{t}_0$ from Example 8.1. On the second, the vector field $\xi_{\mathbf{t}_0}$, i.e., the five-tube version of $\mathbf{t}_0$. Curves are shown separately on the other rows.

Computing linking and self-linking numbers by using signed intersections of curve projections is not hard, but laborious, and we omit the details for now. The results are shown in a 7×7 tabulation matrix L with entries $l_{ij} = lk(C_i, C_j)$ for $i \neq j$ and

$l_{ij} = slk(C_i)$ for $i = j$:

$$L = \begin{pmatrix} 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix};$$

Equation (2) then implies $\mathrm{Hel}(\xi_{t_0}) = -36\varphi^2 = 36\varphi^2 \, \mathrm{Tw}(\mathbf{t}_0)$.

Note that our box $\mathcal{R}$ has precisely 229 tilings. These are $\mathbf{t}_0$ (with twist -1), the mirror image of $\mathbf{t}_0$ (in the plane $z = 1$, for instance; this tiling has twist $+1$) and another 227 tilings of twist 0. In this example, any two tilings of twist 0 can be joined by a finite sequence of flips. It can be verified (and it follows from Theorem 1.1) that

$$(4) \qquad\qquad \mathrm{Hel}(\xi_{\mathbf{t}}) = 36\varphi^2 \, \mathrm{Tw}(\mathbf{t})$$

for all $\mathbf{t} \in \mathcal{T}(\mathcal{R})$. If $\mathbf{t}_1$ is the last tiling in Figure 2 it is easy to verify that $\mathrm{Hel}(\xi_{\mathbf{t}_1}) = 0$ (all linkings are trivial; see Figure 18).

9. Appendix: Helicity via linking and self-linking

Here we recall the derivation of the explicit formula (2):

Proposition 9.1. *For the helicity of a divergence-free field ξ confined to several narrow nonintersecting linked flux pipes $\cup T_i$ in a simply-connected manifold M:*

$$\mathrm{Hel}(\xi) = 2 \sum_{i<j} lk(C_i, C_j) \cdot \mathrm{Flux}_i \cdot \mathrm{Flux}_j + \sum_i slk(C_i) \cdot (\mathrm{Flux}_i)^2.$$

Here closed curves C_i are cores of the pipes T_i and Flux_i are the fluxes in those pipes, while $lk(C_i, C_j)$ is the Gauss linking number of the curves. The self-linking number $slk(C_i)$ is the "average self-linking" of trajectories of ξ in the given pipe. Namely, if all trajectories of the field ξ in a pipe are closed and have the same pairwise linking (for instance, the Poincaré map in the pipe cross-section is a solid rotation by a rational angle), then slk is equal to that pairwise linking number. For an arbitrary field, the self-linking number slk must be replaced by the integral of the asymptotic winding number of the field's trajectories introduced by Fathi [6] and discussed below.

This formula was a folklore statement since Moffatt's discovery in [12] of the formula

$$\mathrm{Hel}(\xi) = 2 lk(C_1, C_2) \cdot \mathrm{Flux}_1 \cdot \mathrm{Flux}_2$$

for helicity $\mathrm{Hel}(\xi) = \int_M (\xi, \mathrm{curl}^{-1}\xi) \, d^3x$ of the field ξ supported in two tubes $T_1 \cup T_2 \subset \mathbb{R}^3$ with all orbits closed, with equal periods, and having no internal twist inside the tubes, and since Arnold's generalization of it to arbitrary divergence-free fields as the asymptotic linking number of its trajectories. It explicitly appeared e.g. in [16] with slightly stronger assumptions and a special interpretation of slk via the Calugareanu invariant, cf. [13].

The derivation of the formula above can be obtained as a following sequence of statements.

Proof. (1) According to Arnold's theorem [2] the total helicity integral $\mathrm{Hel}(\xi) = \int_M (\xi, \mathrm{curl}^{-1}\xi)\, d^3x$ is equal to the average linking number of trajectories,

$$\mathrm{Hel}(\xi) = \iint_{M \times M} lk(x,y)\mu_x \mu_y,$$

where $lk(x,y)$ is the asymptotic linking of the trajectories $g^s(x)$ and $g^t(y)$ of the field ξ starting at the points $x, y \in M$, and μ is the volume form on M. For a field supported on the union of pipes $\cup T_i$ it splits into pairwise integrals.

(2) For two tubes T_i and T_j carrying a field ξ, all whose orbits are closed with equal periods and with no internal twist, their cross-linking is given by Moffatt's formula. Note that this also holds in the more general setting of not necessarily closed trajectories and for arbitrary internal twist inside the tubes. Indeed, by taking the average linking of one closed curve with trajectories in the other tube (the case studied in [2]), the corresponding linking is proportional to the flux through a Seifert surface for that curve and it does not depend in the internal twist in the tube. Hence in the formula for cross-linking two tubes T_i and T_j, only the mutual linking of their cores C_i with C_j and the product of their fluxes $\mathrm{Flux}_i \cdot \mathrm{Flux}_j$ appears.

(3) What remains to prove is that the helicity for a field inside the ith tube is given by $slk(C_i) \cdot (\mathrm{Flux}_i)^2$. This is essentially the statement from [9], see also [17].

In more detail, use some Riemannian metric on M to introduce trivialization along the tube, which now can be regarded as a solid torus $D^2 \times S^1$ with globally defined cylindrical coordinates (r, ϕ, z), where $r \geq 0$ and z mod 1. (The invariants introduced will not depend on that choice.) Following [6] for any pair of points $x, y \in D$ one considers the asymptotic winding number

$$W_\phi(x,y) = \lim_{T,S \to \infty} \frac{\phi(g^t(y)) - \phi(g^s(x))}{2\pi \cdot T \cdot S}$$

for a pair of trajectories $g^t(y)$, $t \in [0,T]$ and $g^s(x)$, $s \in [0,S]$. It is defined for almost all pairs $x, y \in D$ and it is an integrable function on $D \times D$. It was proved in [9, 17] that its integral over $D \times D$ is equal to the helicity invariant of the field. It is also the Calabi invariant of the corresponding Poincaré map $\Psi : D \to D$ for the flow of ξ in the tube:

$$\mathrm{Hel}(\xi) = \mathrm{Cal}(\Psi) = \iint_{D \times D} W_\phi(x,y)\, \omega_x \wedge \omega_y,$$

where ω is the invariant area form induced by the divergence-free vector field ξ in the pipe cross-section, $\omega_x = i_\xi \mu_x$. Note that for a field ξ in a tube along C with closed trajectories of period 1 the value of $W_\phi(x,y)$ is constant and $W_\phi(x,y) = slk\,(C)$. On the other hand, the integral of the induced area form on the cross-section is equal to Flux_ξ of the corresponding vector field ξ through that cross-section. $\square$

Remark 9.2. Note that the above general formula is, in particular, valid for the case of closed curves with *different periods*. For instance, rescale a divergence-free field ξ as $f\xi$ for a function f on a tube T. The divergence-free constraint on $f\xi$ implies that f is a first integral of the field ξ, while the flux $\mathrm{Flus}_{f\xi}$ of the new field must be the same in all cross-sections. Then $\mathrm{Hel}(f\xi) = (\int_D f(x)\,\omega_x)^2 \cdot \mathrm{Hel}(\xi)$, see

e.g. [9], while the latter expression transforms as follows:

$$\text{Hel}(f\xi) = (\int_D f(x)\,\omega_x)^2 \cdot slk(C) \cdot (\text{Flux}_\xi)^2 = slk(C) \cdot (\text{Flux}_{f\xi})^2\,,$$

which emphasizes universality of the formula (2) via fluxes.

Acknowledgments

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