



# Pseudo-Riemannian geodesics and billiards

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## Abstract

In pseudo-Riemannian geometry the spaces of space-like and time-like geodesics on a pseudo-Riemannian manifold have natural symplectic structures (just like in the Riemannian case), while the space of light-like geodesics has a natural contact structure. Furthermore, the space of all geodesics has a structure of a Jacobi manifold. We describe the geometry of these structures and their generalizations. We also introduce and study pseudo-Euclidean billiards, emphasizing their distinction from Euclidean ones. We present a pseudo-Euclidean version of the Clairaut theorem on geodesics on surfaces of revolution. We prove pseudo-Euclidean analogs of the Jacobi–Chasles theorems and show the integrability of the billiard in the ellipsoid and the geodesic flow on the ellipsoid in a pseudo-Euclidean space.

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## 1. Introduction

The space of oriented lines in the Euclidean  $n$ -space has a natural symplectic structure. So does the space of geodesics on a Riemannian manifold, at least locally. The structures on the space of geodesics on a *pseudo-Riemannian* manifold are more subtle. It turns out that the spaces of space-like lines and time-like lines in a pseudo-Euclidean space have natural symplectic structures (and so do the corresponding spaces of the geodesics on a pseudo-Riemannian manifold), while the space of light-like (or, null) lines or geodesics has a natural contact structure. Moreover, the corresponding symplectic structures on the manifolds of space- and time-like geodesics blow up as one approaches the border between them, the space of the null geodesics. On the other hand, the space of all (space-like, time-like, and null) geodesics together locally has a structure of a smooth Jacobi manifold. Below we describe these structures in the pseudo-Riemannian setting, emphasizing the differences from the Riemannian case (see Section 2).

Many other familiar facts in Euclidean/Riemannian geometry have their analogs in the pseudo-Riemannian setting, but often with an unexpected twist. For example, assign the oriented normal line to each point of a cooriented hypersurface in pseudo-Euclidean space; this gives a smooth map from the hypersurface to the space of oriented lines whose image is Lagrangian in the space of space-like and time-like lines, and Legendrian in the space of light-like lines, see Section 2.6. Another example: a convex hypersurface in Euclidean space  $\mathbf{R}^n$  has at least  $n$  diameters. It turns out that a convex hypersurface in pseudo-Euclidean space  $V^{k+l}$  with  $k$  space directions and  $l$  time directions has at least  $k$  space-like diameters and at least  $l$  time-like ones, see Section 3.4.

In Section 3 we introduce pseudo-Euclidean billiards. They can be regarded as a particular case of projective billiards introduced in [26]. The corresponding pseudo-Euclidean billiard map has a variational origin and exhibits peculiar properties. For instance, there are special (“singular”) points, where the normal to the reflecting surface is tangent to the surface itself (the phenomenon impossible for Euclidean reflectors), at which the billiard map is not defined. These points can be of two different types, and the reflection near them is somewhat similar to the reflection in the two different wedges, with angles  $\pi/2$  and  $3\pi/2$ , in a Euclidean space. As an illustration, we study in detail the case of a circle on a Lorentz plane, see Section 5.

We prove a Lorentz version of the Clairaut theorem on the complete integrability of the geodesic flow on a surface of revolution. We also prove pseudo-Euclidean analogs of the Jacobi–Chasles theorems and show the integrability of the billiard in the ellipsoid and the geodesic flow on the ellipsoid in a pseudo-Euclidean space. Unlike the Euclidean situation, the number of “pseudo-confocal” conics passing through a point in pseudo-Euclidean space may differ for different points of space, see Section 4.3.

Throughout the paper, we mostly refer to “pseudo-Euclidean spaces” or “pseudo-Riemannian manifolds” to emphasize arbitrariness of the number of space- or time-like directions. “Lorentz” means that the signature is of the form  $(k, 1)$  or  $(1, l)$ . Note that the contact structure on null geodesics was previously known, at least, for the Lorentz case – see [14,21], and it had been important in various causality questions in the physics literature. Apparently, pseudo-Euclidean billiards have not been considered before, nor was the integrability of pseudo-Riemannian geodesic flows on quadratic surfaces different from pseudospheres.

## 2. Symplectic and contact structures on the spaces of oriented geodesics

### 2.1. General construction

Let  $M^n$  be a smooth manifold with a pseudo-Riemannian metric  $\langle \cdot, \cdot \rangle$  of signature  $(k, l)$ ,  $k + l = n$ . Identify the tangent and cotangent spaces via the metric. Let  $H : T^*M \rightarrow \mathbf{R}$  be the Hamiltonian of the metric:  $H(q, p) = \langle p, p \rangle / 2$ . The geodesic flow in  $T^*M$  is the Hamiltonian vector field  $X_H$  of  $H$ .

A geodesic curve in  $M$  is a projection of a trajectory of  $X_H$  to  $M$ . Let  $\mathcal{L}_+$ ,  $\mathcal{L}_-$ ,  $\mathcal{L}_0$  be the spaces of oriented non-parameterized space-, time- and light-like geodesics (that is,  $H = \text{const} > 0$ ,  $< 0$  or  $= 0$ , respectively). Let  $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_- \cup \mathcal{L}_0$  be the space of all oriented geodesic lines. We assume that these spaces are smooth manifolds (locally, this is always the case); then  $\mathcal{L}_0$  is the common boundary of  $\mathcal{L}_\pm$ .

Consider the actions of  $\mathbf{R}^*$  on the tangent and cotangent bundles by rescaling (co)vectors. The Hamiltonian  $H$  is homogeneous of degree 2 in the variable  $p$ . Refer to this action as the dilations. Let  $E$  be the Euler field in  $T^*M$  that generates the dilations.

**Theorem 2.1.** *The manifolds  $\mathcal{L}_\pm$  carry symplectic structures obtained from  $T^*M$  by Hamiltonian reduction on the level hypersurfaces  $H = \pm 1$ . The manifold  $\mathcal{L}_0$  carries a contact structure whose symplectization is the Hamiltonian reduction of the symplectic structure in  $T^*M$  (without the zero section) on the level hypersurface  $H = 0$ .*

**Proof.** Consider three level hypersurfaces:  $N_{-1} = \{H = -1\}$ ,  $N_0 = \{H = 0\}$  and  $N_1 = \{H = 1\}$ . The Hamiltonian reduction on the first and the third yields the symplectic structures in  $\mathcal{L}_\pm$ . This is the same as in the Riemannian case, see, e.g., [2].

Consider the hypersurface  $N_0$  in  $T^*M$  without the zero section. We have two vector fields on it,  $X_H$  and  $E$ , satisfying  $[E, X_H] = X_H$ . Denote the Hamiltonian reduction of  $N_0$  by  $P$ , it is the quotient of  $N_0$  by the  $\mathbf{R}$ -action with the generator  $X_H$  (sometimes,  $P$  is called the space of *scaled* light-like geodesics). Then  $\mathcal{L}_0$  is the quotient of  $P$  by the dilations; denote the projection  $P \rightarrow \mathcal{L}_0$  by  $\pi$ . Note that  $E$  descends on  $P$  as a vector field  $\bar{E}$ . Denote by  $\bar{\omega}$  the symplectic form on  $P$ . Let  $\bar{\lambda} = i_{\bar{E}}\bar{\omega}$ . We have:

$$d\bar{\lambda} = \bar{\omega}, \quad L_{\bar{E}}(\bar{\omega}) = \bar{\omega}, \quad L_{\bar{E}}(\bar{\lambda}) = \bar{\lambda}.$$

Thus  $(P, \bar{\omega})$  is a homogeneous symplectic manifold with respect to the Euler field  $\bar{E}$ . Consider the distribution  $\text{Ker } \bar{\lambda}$  on  $P$ . Since  $\bar{E}$  is tangent to this distribution,  $\text{Ker } \bar{\lambda}$  descends to a distribution on  $\mathcal{L}_0$ . This is a contact structure whose symplectization is  $(P, \bar{\omega})$ .

To prove that the distribution on  $\mathcal{L}_0$  is indeed contact, let  $\eta$  be a local 1-form defining the distribution. Then  $\pi^*(\eta) = \bar{\lambda}$ . Hence

$$\pi^*(\eta \wedge d\eta^{n-2}) = \bar{\lambda} \wedge \bar{\omega}^{n-2} = \frac{1}{n-1} i_{\bar{E}} \bar{\omega}^{n-1}.$$

Since  $\bar{\omega}^{n-1}$  is a volume form, the last form does not vanish.  $\square$

## 2.2. Examples

**Example 2.2.** Let us compute the area form on the space of lines in the Lorentz plane with the metric  $ds^2 = dx dy$ . A vector  $(a, b)$  is orthogonal to  $(a, -b)$ . Let  $D(a, b) = (b, a)$  be the linear operator identifying vectors and covectors via the metric.

The light-like lines are horizontal or vertical, the space-like have positive and the time-like negative slopes. Each space  $\mathcal{L}_+$  and  $\mathcal{L}_-$  has two components. To fix ideas, consider space-like lines having the direction in the first coordinate quadrant. Write the unit directing vector of a line as  $(e^{-u}, e^u)$ ,  $u \in \mathbf{R}$ . Drop the perpendicular  $r(e^{-u}, -e^u)$ ,  $r \in \mathbf{R}$ , to the line from the origin. Then  $(u, r)$  are coordinates in  $\mathcal{L}_+$ . Similarly one introduces coordinates in  $\mathcal{L}_-$ .

**Lemma 2.3.** (Cf. [4, 7].) *The area form  $\omega$  on  $\mathcal{L}_+$  is equal to  $2du \wedge dr$ , and to  $-2du \wedge dr$  on  $\mathcal{L}_-$ .*

**Proof.** Assign to a line with coordinates  $(u, r)$  the covector  $p = D(e^{-u}, e^u) = (e^u, e^{-u})$  and the point  $q = r(e^{-u}, -e^u)$ . This gives a section of the bundle  $N_1 \rightarrow \mathcal{L}_+$ , and the symplectic form  $\omega = dp \wedge dq$  equals, in the  $(u, r)$ -coordinates,  $2du \wedge dr$ . The computation for  $\mathcal{L}_-$  is similar.  $\square$

**Example 2.4.** Consider the Lorentz space with the metric  $dx^2 + dy^2 - dz^2$ ; let  $H^2$  be the upper sheet of the hyperboloid  $x^2 + y^2 - z^2 = -1$  and  $H^{1,1}$  the hyperboloid of one sheet  $x^2 + y^2 - z^2 = 1$ . The restriction of the ambient metric to  $H^2$  gives it a Riemannian metric of constant negative curvature and the restriction to  $H^{1,1}$  a Lorentz metric of constant curvature. The geodesics of these metrics are the intersections of the surfaces with the planes through the origin; the light-like lines of  $H^{1,1}$  are the straight rulings of the hyperboloid. The central projection on a plane induces a (pseudo)-Riemannian metric therein whose geodesics are straight lines (for  $H^2$ , this is the Beltrami–Klein model of the hyperbolic plane).

The scalar product in the ambient space determines duality between lines and points by assigning to a vector the orthogonal plane. In particular, to a point of  $H^2$  there corresponds a space-like line in  $H^{1,1}$  (which is a closed curve). More precisely,  $H^2$  (which is the upper sheet of the hyperboloid) is identified with the space of positively (or “counterclockwise”) oriented space-like lines of  $H^{1,1}$ , while the lower sheet of the same hyperboloid (that is,  $H^2$  with the opposite orientation) is identified with the space of negatively oriented lines. On the other hand,  $H^{1,1}$  is identified with the space of oriented lines in  $H^2$ . The space of oriented time-like lines of  $H^{1,1}$  (which are not closed) is also identified with  $H^{1,1}$  itself. The area forms on the spaces of oriented lines coincide with the area forms on the respective surfaces, induced by the ambient metric.

This construction is analogous to the projective duality between points and oriented great circles of the unit sphere in 3-space.

### 2.3. Pseudo-Euclidean space

Let  $V^{n+1}$  be a vector space with an indefinite non-degenerate quadratic form. Decompose  $V$  into the orthogonal sum of the positive and negative subspaces; denote by  $v_1, v_2$  the positive and negative components of a vector  $v$ , and likewise, for covectors. The scalar product in  $V$  is given by the formula  $\langle u, v \rangle = u_1 \cdot v_1 - u_2 \cdot v_2$  where  $\cdot$  is the Euclidean dot product. Let  $S_{\pm}$  be the unit pseudospheres in  $V$  given by the equations  $|q_1|^2 - |q_2|^2 = \pm 1$ .

The next result and its proof are similar to the familiar Euclidean case.

**Proposition 2.5.**  $\mathcal{L}_{\pm}$  is (anti)symplectomorphic to  $T^*S_{\pm}$ .

**Proof.** Consider the case of  $\mathcal{L}_+$ . Assign to a space-like line  $\ell$  its unit vector  $v$ , so that  $|v_1|^2 - |v_2|^2 = 1$ , and a point  $x \in \ell$  whose position vector is orthogonal to  $v$ , that is,  $\langle x, v \rangle = 0$ . Then  $v \in S_+$  and  $x \in T_v S_+$ . Let  $\xi \in T_v^* S_+$  be the covector corresponding to the vector  $x$  via the metric:  $\xi_1 = x_1, \xi_2 = -x_2$ . Then the canonical symplectic structure in  $T^*S_+$  is  $d\xi \wedge dv = dx_1 \wedge dv_1 - dx_2 \wedge dv_2$ .

The correspondence  $\ell \mapsto (q, p)$ , where  $q = x$  and  $p = (p_1, p_2) = (v_1, -v_2)$  is the covector corresponding to the vector  $v$  via the metric, is a section of the bundle  $N_1 \rightarrow \mathcal{L}_+$ . Thus the symplectic form  $\omega$  on  $\mathcal{L}_+$  is the pull-back of the form  $dp \wedge dq$ , that is,  $\omega = dv_1 \wedge dx_1 - dv_2 \wedge dx_2$ . Up to the sign, this is the symplectic structure in  $T^*S_+$ .  $\square$

A light-like line is characterized by its point  $x$  and a vector  $v$  along the line; one has  $\langle v, v \rangle = 0$ . The same line is determined by the pair  $(x + sv, tv)$ ,  $s \in \mathbf{R}, t \in \mathbf{R}_+^*$ . The respective vector fields  $v\partial x$  and  $v\partial v$  are the Hamiltonian and the Euler fields, in this case.

We shall now describe the contact structure in  $\mathcal{L}_0$  geometrically.

Assign to a line  $\ell \in \mathcal{L}_0$  the set  $\Delta(\ell) \subset \mathcal{L}$  consisting of the oriented lines in the affine hyperplane, orthogonal to  $\ell$ . Then  $\ell \in \Delta(\ell)$  and  $\Delta(\ell)$  is a smooth  $(2n - 2)$ -dimensional manifold, the space of oriented lines in  $n$ -dimensional space. Denote by  $\xi(\ell) \subset T_{\ell}\mathcal{L}$  the tangent hyperplane to  $\Delta(\ell)$  at point  $\ell$ .

Denote by  $S_0$  the spherization of the light cone:  $S_0$  consists of equivalence classes of non-zero vectors  $v \in V$  with  $\langle v, v \rangle = 0$  and  $v \sim tv, t > 0$ . Let  $E$  be the 1-dimensional  $\mathbf{R}_+^*$ -bundle over  $S_0$  whose sections are functions  $f(v)$ , homogeneous of degree 1. Denote by  $J^1 E$  the space of 1-jets of sections of  $E$ ; this is a contact manifold.

**Proposition 2.6.**

- 1)  $\xi(\ell)$  is the contact hyperplane of the contact structure in  $\mathcal{L}_0$ .
- 2)  $\mathcal{L}_0$  is contactomorphic to  $J^1 E$ .<sup>1</sup>

**Proof.** By construction of Theorem 2.1, the contact hyperplane at  $\ell$  is the projection to  $T_{\ell}\mathcal{L}_0$  of the kernel of the Liouville form  $v dx$  (identifying vectors and covectors via the metric). Write an infinitesimal deformation of  $\ell = (x, v)$  as  $(x + \varepsilon y, v + \varepsilon u)$ . This is in  $\text{Ker } v dx$  if and only if  $\langle y, v \rangle = 0$ . The deformed line is light-like, hence  $\langle v + \varepsilon u, v + \varepsilon u \rangle = 0 \bmod \varepsilon^2$ , that is,  $\langle u, v \rangle = 0$ .

<sup>1</sup> In the case of a Lorentz space,  $\mathcal{L}_0$  is also contactomorphic to the space of cooriented contact elements of a Cauchy surface, see [14].

Thus both the foot point and the directional vector of the line  $\ell$  move in the hyperplane, orthogonal to  $\ell$ , and therefore the contact hyperplane at  $\ell$  is contained in  $\xi(\ell)$ . Since the dimensions coincide,  $\xi(\ell)$  is this contact hyperplane. In particular, we see that  $\Delta(\ell)$  is tangent to  $\mathcal{L}_0$  at point  $\ell$ . This proves the first statement.

Assign to  $\ell = (x, v)$  the 1-jet of the function  $\phi(\ell) = \langle x, \cdot \rangle$  on  $S_0$ . This function is homogeneous of degree 1. The function  $\phi(\ell)$  is well defined: since  $v$  is orthogonal to  $v$  and to  $T_v S_0$ , the function  $\phi(\ell)$  does not change if  $x$  is replaced by  $x + sv$ . Thus we obtain a diffeomorphism  $\phi: \mathcal{L}_0 \rightarrow J^1 E$ .

To prove that  $\phi$  preserves the contact structures, let  $f$  be a test section of  $E$ . By definition of the contact structure in  $J^1 E$ , the 1-jet extension of  $f$  is a Legendrian manifold. Set  $x(v) = \nabla f(v)$  (gradient taken with respect to the pseudo-Euclidean structure). We claim that  $\phi(x(v), v) = j^1 f(v)$ . Indeed, by the Euler formula,

$$\langle x(v), v \rangle = \langle \nabla f(v), v \rangle = f(v), \quad (1)$$

that is, the value of the function  $\langle x(v), \cdot \rangle$  at point  $v$  is  $f(v)$ . Likewise, let  $u \in T_v S_0$  be a test vector. Then the value of the differential  $d\langle x(v), \cdot \rangle$  on  $u$  is  $\langle \nabla f(v), u \rangle = df_v(u)$ .

It remains to show that the manifold  $\phi^{-1}(j^1 f) = \{(x(v), v)\}$  is Legendrian in  $\mathcal{L}_0$ . Indeed, the contact form is  $v dx$ . One has:

$$v d\langle x(v), v \rangle = d\langle x(v), v \rangle - x(v) dv = df - \nabla f dv = 0;$$

the second equality is due to (1). Therefore  $\phi^{-1}(j^1 f)$  is a Legendrian submanifold, and the second claim follows.  $\square$

#### 2.4. Symplectic, Poisson and contact structures

The contact manifold  $\mathcal{L}_0$  is the common boundary of the two open symplectic manifolds  $\mathcal{L}_\pm$ . Suppose that  $n \geq 2$ , that is, we consider lines in at least three-dimensional space  $V^{n+1}$ .

**Theorem 2.7.** *Neither the symplectic structures of  $\mathcal{L}_\pm$ , nor their inverse Poisson structures, extend smoothly across the boundary  $\mathcal{L}_0$  to the corresponding structure on the total space  $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_0 \cup \mathcal{L}_-$ .*

**Remark 2.8.** When  $n = 1$  the symplectic structures go to infinity as we approach the one-dimensional manifold  $\mathcal{L}_0$ . The corresponding Poisson structures, which are inverses of the symplectic ones, extend smoothly across  $\mathcal{L}_0$ .

This can be observed already in the explicit computations of Example 2.2. Recall that for the metric  $ds^2 = dx dy$  in  $V^2$  and the lines directed by vectors  $(e^{-u}, e^u)$ ,  $u \in \mathbf{R}$ , the symplectic structure in the corresponding coordinates  $(u, r)$  at  $\mathcal{L}_+$  has the form  $2du \wedge dr$ , see Lemma 2.3.

Now consider a neighborhood of a light-like line among all lines, that is, a neighborhood of a point in  $\mathcal{L}_0$  regarded as a boundary submanifold between  $\mathcal{L}_+$  and  $\mathcal{L}_-$ . Look at the variation  $\xi_\varepsilon = (1, \varepsilon)$  of the horizontal (light-like) direction  $\xi_0 = (1, 0)$ , and regard  $(\varepsilon, r)$  as the coordinates in this neighborhood. For  $\varepsilon > 0$  the corresponding half-neighborhood lies in  $\mathcal{L}_+$ , while the coordinates  $u$  and  $\varepsilon$  in this half-neighborhood are related as follows. Equating the slope of  $(1, \varepsilon)$  to the slope of  $(e^{-u}, e^u)$  we obtain the relation  $\varepsilon = e^{2u}$  or  $u = \frac{1}{2} \ln \varepsilon$ . Then the symplectic structure  $\omega = 2du \wedge dr = d \ln \varepsilon \wedge dr = \frac{1}{\varepsilon} d\varepsilon \wedge dr$ . One sees that  $\omega \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ . The Poisson structure,

inverse to  $\omega$ , is given by the bivector field  $\varepsilon \frac{\partial}{\partial \varepsilon} \wedge \frac{\partial}{\partial r}$  and it extends smoothly across the border  $\varepsilon = 0$ .

**Example 2.9.** Let us compute the symplectic strictures on lines in the 3-dimensional space  $V^3$  with the metric  $dx dy - dz^2$ . We parametrize the space-like directions by  $\xi = (e^{-u} \cosh \phi, e^u \cosh \phi, \sinh \phi)$ , where  $u \in \mathbf{R}, \phi \in \mathbf{R}$ . The operator  $D$  identifying vectors and covectors has the form  $D(a, b, c) = (b/2, a/2, -c)$ . Choose the basis of vectors orthogonal to  $\xi$  as

$$e_1 = (e^{-u} \sinh \phi, e^u \sinh \phi, \cosh \phi) \quad \text{and} \quad e_2 = (e^{-u}, -e^u, 0).$$

The symplectic structure  $\omega = dp \wedge dq$  for  $q = r_1 e_1 + r_2 e_2$  and  $p = D\xi = (e^u \cosh \phi/2, e^{-u} \cosh \phi/2, -\sinh \phi)$  has the following explicit expression in coordinates  $(u, \phi, r_1, r_2)$ :

$$\omega = -d\phi \wedge dr_1 + \cosh \phi du \wedge dr_2 - r_2 \sinh \phi d\phi \wedge du.$$

Now we are in a position to prove Theorem 2.7 on non-extendability.

**Proof.** The impossibility of extensions follows from the fact that the “eigenvalues” of the symplectic structures  $\omega$  of  $\mathcal{L}_\pm$  go to both 0 and  $\infty$ , as we approach  $\mathcal{L}_0$  from either side. (Of course, according to the Darboux theorem, the eigenvalues of the symplectic structures are not well defined, but their zero or infinite limits are.) More precisely, let  $\alpha = \sum a_{ij} dx_i \wedge dx_j$  be a meromorphic 2-form written in local coordinates  $\{x_i\}$  in a neighborhood of a point  $P$ .

**Lemma 2.10.** *The number of eigenvalues of the matrix  $A = (a_{ij})$  which go to 0 or  $\infty$  as  $x \rightarrow P$  does not depend on the choice of coordinates  $\{x_i\}$ .*

**Proof.** Indeed, under a coordinate change  $x = \eta(y)$ , the matrix  $A$  changes to  $(J\eta)^* A (J\eta)$  in coordinates  $\{y_j\}$ , where  $J\eta$  is the Jacobi matrix of the diffeomorphism  $\eta$ . Since  $J\eta$  is bounded and non-degenerate, this change preserves (in)finiteness or vanishing of the limits of the eigenvalues of  $A$ .  $\square$

Now the theorem follows from

**Lemma 2.11.** *The eigenvalues of the 2-form  $\omega$  in coordinates  $(u, \phi, r_1, r_2)$  go to both 0 and  $\infty$  as  $r_2 \rightarrow \infty$  (while keeping other coordinates fixed).*

**Proof.** Indeed, the matrix of  $\omega$  has the following (biquadratic) characteristic equation:  $\lambda^4 + a\lambda^2 + b = 0$ , where  $a = 1 + r_2^2 \sinh^2 \phi + \cosh^2 \phi$ ,  $b = \cosh^2 \phi$ . As  $r_2 \rightarrow \infty$ , so does  $a$ , whereas  $b$  does not change. Thus the sum of the squares of the roots goes to infinity, whereas their product is constant. Hence the equation has one pair of roots going to 0, while the other goes to infinity.  $\square$

The limit  $r_2 \rightarrow \infty$  means that one is approaching the boundary of the space  $\mathcal{L}_+$ . The infinite limit of the eigenvalues means that the symplectic structure  $\omega$  does not extend smoothly across  $\mathcal{L}_0$ , while the zero limit of them means that the Poisson structure inverse to  $\omega$  is non-extendable as well. The case of higher dimensions  $n$  can be treated similarly.  $\square$

**Remark 2.12.** The contact planes in  $\mathcal{L}_0$  can be viewed as the subspaces of directions in the tangent spaces  $T_*\mathcal{L}_0$ , on which the limits of the  $\mathcal{L}_\pm$ -symplectic structures are finite. One can also see that the existence of extensions of the symplectic or Poisson structures would mean the presence of other intrinsic structures, different from the contact one, on the boundary  $\mathcal{L}_0$ . Indeed, the existence of a symplectic structure extension would imply the existence of a presymplectic structure (and hence, generically, a characteristic direction field), rather than of a contact distribution, on  $\mathcal{L}_0$ .

On the other hand, consider the Poisson structures on  $\mathcal{L}_\pm$  which are inverses of the corresponding symplectic structures. The assumption of a smooth extension of such Poisson structures would mean the existence of a Poisson structure on  $\mathcal{L}_0$  as well. The corresponding foliation of  $\mathcal{L}_0$  by symplectic leaves would be integrable, while the contact distribution is not.

### 2.5. Local Lie algebra of geodesics and its Poissonization

It turns out that the space of all pseudo-Riemannian geodesics (i.e., including all three types: space-, time-, and light-like ones) has a structure of a Jacobi manifold, or a local Lie algebra. This structure is not canonical and it depends on the choices described below, but it shows how symplectic ( $\mathcal{L}_\pm$ ) and contact ( $\mathcal{L}_0$ ) manifolds constituting  $\mathcal{L}$  smoothly fit together in the framework of a Jacobi manifold.

Recall that a manifold is said to have a Jacobi structure if the space of functions on it (or, more generally, the space of sections of a line bundle over it) is equipped with a Lie bracket (a bilinear skew-symmetric operation satisfying the Jacobi identity) which is local over the manifold. Locality of the bracket means that it is defined by differential operators on functions or sections [13]. A. Kirillov proved that such a manifold naturally decomposes into a union of presymplectic and contact manifolds with natural “pre-Poisson” or Lagrange brackets on them, respectively [13]. A presymplectic manifold is a manifold equipped with a 2-form which is the product of a symplectic 2-form and a nonvanishing function.

In [6] it was shown that any Jacobi manifold can be obtained from a homogeneous Poisson manifold of dimension one bigger, called its Poissonization, by choosing a hypersurface in the latter. It turns out that the space  $\mathcal{L}$  of all geodesics on a pseudo-Riemannian manifold  $M$  has a natural Poissonization with a simple canonical structure. As in Section 2, we assume that the spaces of geodesics are smooth manifolds (or consider the local situation).

**Theorem 2.13.** *The space  $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_0 \cup \mathcal{L}_-$  of all geodesics on a pseudo-Riemannian manifold  $M$  can be identified with the quotient of a homogeneous Poisson manifold with respect to dilations. The images of the symplectic leaves of this Poisson manifold in the quotient correspond to the spaces  $\mathcal{L}_+$ ,  $\mathcal{L}_0$ , and  $\mathcal{L}_-$ .*

**Proof.** Consider  $T^*M$  (without the zero section) with the standard symplectic structure. Abusing the notation, in the proof below we denote it by the same symbol  $T^*M$ , and use other notations of Theorem 2.1. Let  $H = \langle p, p \rangle / 2$  be the Hamiltonian, and  $X_H$  the corresponding Hamiltonian field.

Now we consider the manifold  $PM := T^*M / X_H$ , that is, instead of first confining ourselves to the levels of  $H$  as in Theorem 2.1, we take the quotient with respect to the  $\mathbf{R}$ -action of the Hamiltonian field  $X_H$  right away. Then  $PM$  is a Poisson manifold as a quotient of the Poisson manifold  $T^*M$  along the action of the field  $X_H$ , which respects the Poisson structure. Furthermore, the symplectic leaves of  $PM$  have codimension 1, since  $PM$  was obtained as a quotient



of a symplectic manifold (i.e., a non-degenerate Poisson manifold) by a one-dimensional group. These leaves are exactly the levels of  $H$  in  $PM$ .

We claim that the manifold  $PM$  can be regarded as the Poissonization of the space  $\mathcal{L}$  of all geodesics: it has a natural Poisson structure, homogeneous with respect to the action of dilations, and such that the quotient space coincides with  $\mathcal{L}$ :  $\mathcal{L} = PM/\mathbf{R}^*$ . Indeed, consider the action of  $\mathbf{R}^*$ -dilations  $E$  on  $PM$ . It is well defined on  $PM$  due to the relation  $[E, X_H] = X_H$ . Note that the symplectic leaves, i.e.  $H$ -levels, are transversal to  $E$  wherever  $H \neq 0$ , and are tangent to  $E$  when  $H = 0$ .

For  $H \neq 0$ , the quotient space with respect to the  $\mathbf{R}^*$ -action  $E$  can be described by the levels  $H = \pm 1$ , and the latter correspond to the spaces of space-like or time-like geodesics. (Note that here we have made the same Hamiltonian reduction as in Theorem 2.1, but in the opposite order: first taking the quotient, and then passing to the restriction.)

For  $H = 0$ , we have one leaf with the field  $E$  in it, which exactly constitutes the setting for defining the space  $\mathcal{L}_0$  of light-like geodesics in the proof of Theorem 2.1, where this leaf is called  $P$ , and the field is  $\bar{E}$ . This leads to the contact structure on  $\mathcal{L}_0$  after taking the quotient.  $\square$

**Corollary 2.14.** *The quotient space  $\mathcal{L} = PM/\mathbf{R}^*$  can be (locally) endowed with a smooth Jacobi structure upon choosing any section of this  $\mathbf{R}^*$ -bundle over  $\mathcal{L}$ .*

**Remark 2.15.** The formulas relating the homogeneous Poisson structures and the Jacobi structures on the sections in the general setting can be found in [6]. The smooth Jacobi structure on the manifold  $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_0 \cup \mathcal{L}_-$  depends on the choice of the hypersurface in  $PM$  realizing the section. To describe the structure “independent of this choice,” one can consider the Poisson structure on  $\mathcal{L}_\pm$ , which, up to conformal changes, gives rise to a “conformal cosymplectic structure” and captures many features of the neighboring symplectic structures. This approach is developed in [12].

On the other hand, as we have seen in the preceding section, the symplectic structures of the two open submanifolds  $\mathcal{L}_\pm$  blow up as one approaches their common contact boundary  $\mathcal{L}_0$ . To see how the coexistence of a smooth Jacobi structure and the blowing up symplectic structures fit together, we consider the corresponding homogeneous cone  $P = \{H = 0\}$  (consisting of scaled null geodesics) over the space of null geodesics. Then the space  $\mathcal{L}_+$  of space-like geodesics “approaches the cone  $P$  at infinity” (same for  $\mathcal{L}_-$ ). Now consider the family of spaces  $\{H = \lambda\}$ , all isomorphic to the set of space-like geodesics. The picture is similar to a family of hyperboloids approaching the quadratic cone. At any given point of the cone the convergence will be smooth, since we are taking different (closer and closer) hyperboloids, while the corresponding structures on the quotients, when we fix one hyperboloid, and where the structure “comes from infinitely remote points” will not necessarily have a nice convergence.

One can show that this type of blow-up of the symplectic structures in Jacobi manifolds is typical: there is a version of the Darboux theorem showing that locally all such degenerations look alike, cf. [18].

## 2.6. Hypersurfaces and submanifolds

Let  $M \subset V$  be an oriented smooth hypersurface. Assign to a point  $x \in M$  its oriented normal line  $\ell(x)$ . We obtain a Gauss map  $\psi : M \rightarrow \mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_- \cup \mathcal{L}_0$ . Denote by  $\psi_+$ ,  $\psi_-$  and  $\psi_0$  its space-, time- and light-like components, i.e. the restriction of the map  $\psi$  to those parts of  $M$ ,

where the normal  $\ell(x)$  is respectively space-, time- or light-like. Note that if the normal line is light-like then it is tangent to the hypersurface.

**Proposition 2.16.** *The images of  $\psi_{\pm}$  are Lagrangian and the image of  $\psi_0$  is Legendrian.*

**Proof.** Consider the case of  $\psi_+$  (the time-like case being similar). Denote by  $\nu(x)$  the “unit” normal vector to  $M$  at point  $x$  satisfying  $\langle \nu(x), \nu(x) \rangle = 1$ . The line  $\ell(x)$  is characterized by its vector  $\nu(x)$  and its point  $x$ ; the correspondence  $\ell \mapsto (\nu(x), x)$  is a section of the bundle  $N_1 \rightarrow \mathcal{L}_+$  over the image of  $\psi_+$ . We need to prove that the form  $dx \wedge d\nu(x)$  vanishes on this image. Indeed, the 1-form  $\nu(x) dx$  vanishes on  $M$  since  $\nu(x)$  is a normal vector, hence its differential is zero as well.

In the case of  $\psi_0$ , we do not normalize  $\nu(x)$ . The 1-form  $\nu(x) dx$  is still zero on  $M$ , and this implies that the image of the Gauss map in  $\mathcal{L}_0$  is Legendrian, as in the proof of Proposition 2.6.  $\square$

**Remark 2.17.** The maps  $\psi_{\pm}$  are immersions but  $\psi_0$  does not have to be one. For example, let  $M$  be a hyperplane such that the restriction of the metric to  $M$  has a 1-dimensional kernel. This kernel is the normal direction to  $M$  at each point. These normal lines foliate  $M$ , the leaves of this foliation are the fibers of the Gauss map  $\psi_0$ , and its image is an  $(n - 1)$ -dimensional space.

**Remark 2.18.** More generally, let  $M \subset V$  be a smooth submanifold of any codimension. Assign to a point  $x \in M$  the set of all oriented normal lines to  $M$  at  $x$ . This also gives us a Gauss map  $\psi : \mathcal{N}_M \rightarrow \mathcal{L}$  of the normal bundle  $\mathcal{N}_M$  of the submanifold  $M \subset V$  into  $\mathcal{L}$  with space-, time- and light-like components  $\psi_+$ ,  $\psi_-$  and  $\psi_0$ , respectively. In this setting, Proposition 2.16 still holds, while the proof requires only cosmetic changes.

Note that the Jacobi structure approach, discussed in the last section, explains why one obtains Lagrangian/Legendrian submanifolds by considering spaces of normals to various varieties in a pseudo-Riemannian space: they are always Lagrangian submanifolds in the Poissonization, before taking the quotient.

**Example 2.19.** The set of oriented lines through a point provides an example of a submanifold in  $\mathcal{L}$  whose intersection with  $\mathcal{L}_+ \cup \mathcal{L}_-$  is Lagrangian and with  $\mathcal{L}_0$  Legendrian.

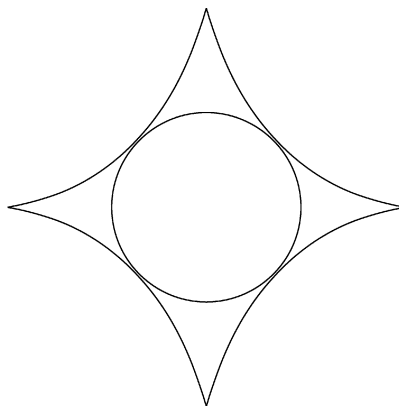


Fig. 1. The caustic of a circle in the Lorentz plane.

**Example 2.20.** Consider the circle  $x^2 + y^2 = 1$  on the Lorentz plane with the metric  $dx^2 - dy^2$ . Then the caustic, that is, the envelope of the normal lines to the circle, is the astroid  $x^{2/3} + y^{2/3} = 2^{2/3}$ , see Fig. 1 (note that the caustic of an ellipse in the Euclidean plane is an astroid too). The role of Euclidean circles is played by the pseudocircles, the hyperbolas  $(x - a)^2 - (y - b)^2 = c$ : their caustics degenerate to points.

### 3. Billiard flow and billiard transformation

#### 3.1. Definition of the billiard map

Let  $M$  be a pseudo-Riemannian manifold with a smooth boundary  $S = \partial M$ . The billiard flow in  $M$  is a continuous time dynamical system in  $TM$ . The motion of tangent vectors in the interior of  $M$  is free, that is, coincides with the geodesic flow. Suppose that a vector hits the boundary at point  $x$ . Let  $\nu(x)$  be the normal to  $T_x S$ . If  $x$  is a singular point, that is, the restriction of the metric on  $S$  is singular or, equivalently,  $\langle \nu(x), \nu(x) \rangle = 0$ , then the billiard trajectory stops there. Otherwise the billiard reflection occurs.

Since  $x$  is a non-singular point,  $\nu(x)$  is transverse to  $T_x S$ . Let  $w$  be the velocity of the incoming point. Decompose it into the tangential and normal components,  $w = t + n$ . Define the billiard reflection by setting  $w_1 = t - n$  to be the velocity of the outgoing point. Clearly  $|w|^2 = |w_1|^2$ . In particular, the billiard reflection does not change the type of a geodesic: time-, space- or light-like.

We view the billiard map  $T$  as acting on oriented geodesics and sending an incoming ray to the outgoing one.

**Example 3.1.** Let the pseudocircle  $x^2 - y^2 = c$  be a billiard curve (or an ideally reflecting mirror) in the Lorentz plane with the metric  $dx^2 - dy^2$ . Then all normals to this curve pass through the origin, and so every billiard trajectory from the origin reflects back to the origin. The same holds in multi-dimensional pseudo-Euclidean spaces.

**Example 3.2.** In the framework of Example 2.4, consider two billiards, inner and outer, in the hyperbolic plane  $H^2$ . (The latter is an area preserving mapping of the exterior of a strictly convex curve  $\gamma$  defined as follows: given a point  $x$  outside of  $\gamma$ , draw a support line to  $\gamma$  and reflect  $x$  in the support point; see [29,30].) The duality between  $H^2$  and  $H^{1,1}$  transforms the inner and outer billiard systems in  $H^2$  to the outer and inner billiard systems in  $H^{1,1}$ . Given a convex closed curve in  $H^2$ , the dual curve in  $H^{1,1}$  (consisting of the points, dual to the tangent lines of the original curve) is space-like. Thus any outer billiard in  $H^2$  provides an example of a billiard in  $H^{1,1}$  whose boundary is a space-like curve.

**Remark 3.3.** Similarly to the Riemannian case, the origin of the billiard reflection law is variational. One can show that a billiard trajectory from a fixed point  $A$  to a fixed point  $B$  in  $M$  with reflection at point  $X \in S$  is an extremal of the following variational problem:

$$I_\tau(\gamma_1, \gamma_2) = \int_0^\tau \langle \gamma_1'(t), \gamma_1'(t) \rangle dt + \int_\tau^1 \langle \gamma_2'(t), \gamma_2'(t) \rangle dt,$$

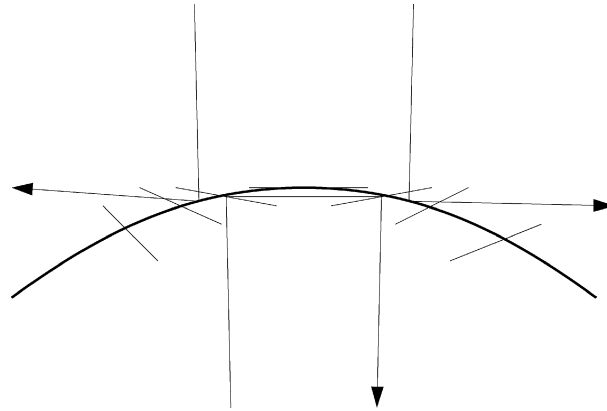


Fig. 2. Two types of the billiard reflection near a singular point: rays coming from above get scattered, while rays coming from below have the second reflection.

where  $\gamma_1(t)$ ,  $0 \leq t \leq \tau$ , is a path from  $A$  to a point  $X$  of  $S$  and  $\gamma_2(t)$ ,  $\tau \leq t \leq 1$ , is a path from the point  $X$  to  $B$ , and where  $\tau \in [0, 1]$  is also a variable.

### 3.2. Reflection near a singular point

Let us look more carefully at the billiard reflection in a neighborhood of a singular point of a curve in the Lorentz plane. First of all we note that typical singular points can be of two types, according to whether the inner normal is oriented toward or from the singular point as we approach it along the curve. These two types are shown in the same Fig. 2, where we regard the curve as the billiard boundary either for rays coming from above or from below. In both cases after the reflection the rays are “squeezed” between the tangent and the normal to the curve. In the first case, when rays come from above and the normals to the curve are oriented *from* the singular point, this implies that the reflected rays scatter away from the point. In the second case, when rays come from below and the normals point *toward* the singular point, the reflected ray hits the boundary again.

One can see that the smooth boundary of a strictly convex domain in the Lorentz plane has singular points of the former type only. Indeed, up to a diffeomorphism, there exists a unique germ of normal line field at a singular point of a quadratically non-degenerate curve in the Lorentz plane – the one shown in Fig. 2. The billiard table may lie either on the convex (lower) or the concave (upper) side of the curve, whence the distinction between the two cases.

Note also that, at a singular point, the caustic of the curve always touches the curve (cf. Example 2.20). The above two cases differ by the location of the caustic: it can touch the curve from (a) the exterior or (b) the interior of the billiard domain. The billiard inside a circle in the Lorentz plane has singular points of the former type only, cf. Fig. 1.

The billiard reflections are drastically different in these two cases. In case (b), a generic family of rays gets dispersed in opposite directions on different sides from the singular point. In case (a), the situation is quite different: the scattered trajectories are reflected toward the singular point and hit the curve one more time in its vicinity. Thus one considers the square of the billiard map  $T^2$ .

**Proposition 3.4.** *Assume that a smooth billiard curve  $\gamma$  in the Lorentz plane is quadratically non-degenerate at a singular point  $O$ . Consider a parallel beam of lines  $\{\ell\}$  reflecting in an arc*

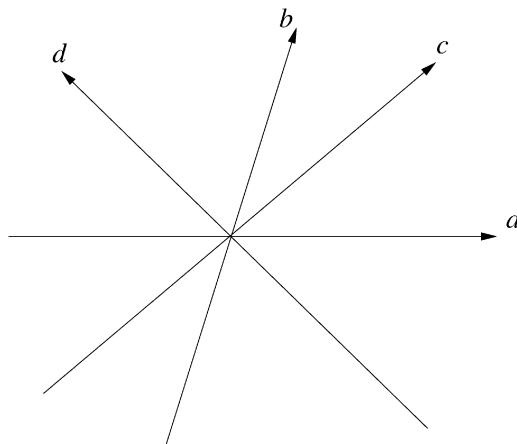


Fig. 3. Harmonic quadruple of lines given by four vectors.

of  $\gamma$  near point  $O$ , on the convex side. Then, as the reflection points tend to  $O$ , the lines  $T^2(\ell)$  have a limiting direction, and this direction is parallel to  $\ell$ .

**Proof.** Let the metric be  $dx dy$ . In this metric, a vector  $(a, b)$  is orthogonal to  $(-a, b)$ . We may assume that the singular point is the origin, and that  $\gamma$  is the graph  $y = f(x)$  where  $f(0) = f'(0) = 0$  and  $f''(0) > 0$ . Consider a downward incoming ray with slope  $u$  reflecting in  $\gamma$  at point  $(s, f(s))$ , then at point  $(t, f(t))$ , and escaping upward with slope  $v$ .

The law of billiard reflection can be formulated as follows: the incoming billiard ray, the outgoing one, the tangent line and the normal to the boundary of the billiard table at the impact point constitute a harmonic quadruple of lines. The following criterion is convenient, see [26] for this and for a study of more general projective billiards. Let the lines be given by vectors  $a, b, c, d$ , see Fig. 3. Then the lines constitute a harmonic quadruple if and only if

$$[a, c][b, d] + [a, d][b, c] = 0, \quad (2)$$

where  $[,]$  is the cross product of two vectors.

For the first reflection, we have

$$a = (1, f'(s)), \quad b = (-1, f'(s)), \quad c = (t - s, f(t) - f(s)), \quad d = (u, 1).$$

Substitute to (2) and compute the determinants to obtain

$$u(f'(s))^2(t - s) = f(t) - f(s).$$

Similarly, for the second reflection, we have

$$v(f'(t))^2(t - s) = f(t) - f(s),$$

and hence

$$u(f'(s))^2 = v(f'(t))^2. \quad (3)$$

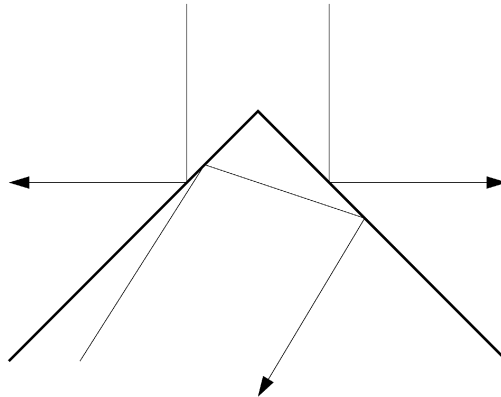


Fig. 4. Euclidean billiard reflection in a right angle.

Write  $f(x) = ax^2 + O(x^3)$ , then  $f'(x) = 2ax + O(x^2)$ , and

$$\frac{f(t) - f(s)}{t - s} = a(s + t) + O(s^2, st, t^2).$$

The above quantity equals  $u(f'(s))^2$ , which is  $O(s^2)$ , hence  $t = -s + O(s^2)$ . It follows that  $(f'(t))^2 = (f'(s))^2 = 4a^2s^2 + O(s^3)$  and, by (3), that  $v = u$ .  $\square$

Thus a ray meeting a curve near a singular point emerges, after two reflections, in the opposite direction. This resembles the billiard reflection in a right angle in the Euclidean plane, see Fig. 4. In contrast, the reflection of a parallel beam on the concave side of a Lorentz billiard near a singular point resembles the Euclidean billiard reflection from the angle  $3\pi/2$  (cf. Figs. 2 and 4). Of course, this behavior excludes the existence of smooth caustics in Lorentz billiards, cf., e.g., [29,30] for the Euclidean case.

### 3.3. Symplectic and contact properties of the billiard map

Now we discuss symplectic properties of the billiard transformation. To fix ideas, let the billiard table be geodesically convex.<sup>2</sup> Denote by  $\mathcal{L}^0$  the set of oriented lines that meet  $S$  at non-singular points. The billiard map  $T$  preserves the space-, time-, and light-like parts of  $\mathcal{L}^0$ , so we have billiard transformations  $T_+$ ,  $T_-$ , and  $T_0$  acting on  $\mathcal{L}_+^0$ ,  $\mathcal{L}_-^0$ , and  $\mathcal{L}_0^0$ , respectively. The (open dense) subsets  $\mathcal{L}_\pm^0 \subset \mathcal{L}_\pm$  and  $\mathcal{L}_0^0 \subset \mathcal{L}_0$  carry the same symplectic or contact structures as the ambient spaces.

**Theorem 3.5.** *The transformations  $T_+$  and  $T_-$  are symplectic and  $T_0$  is a contact transformation.*

**Proof.** We adopt the approach of R. Melrose [16,17]; see also [2]. Identify tangent vectors and covectors via the metric. We denote vectors by  $v$  and covectors by  $p$ .

Consider first the case of space-like geodesics (the case of time-like ones is similar). Let  $\Sigma \subset T^*M$  be the hypersurface consisting of vectors with foot-point on  $S$ . Let  $Z = N_1 \cap \Sigma$  and let  $\Delta \subset Z$  consists of the vectors tangent to  $S$ .

<sup>2</sup> Alternatively, one may consider the situation locally, in a neighborhood of an oriented line transversally intersecting  $S$  at a non-singular point.

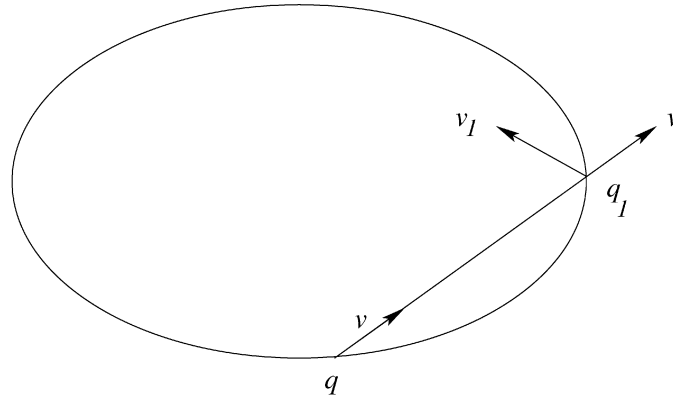


Fig. 5. The billiard map  $F = \sigma \circ \tau$  is a composition of two involutions:  $\tau(q, v) = (q_1, v)$ ,  $\sigma(q_1, v) = (q_1, v_1)$ .

Denote by  $v(q) \in T_q^*M$  a conormal vector to  $S$  at point  $q \in S$ . Consider the characteristics of the canonical symplectic form  $\omega$  in  $T^*M$  restricted to  $\Sigma$ . We claim that these are the lines  $(q, p + tv(q))$ ,  $t \in \mathbf{R}$ .

Indeed, in local Darboux coordinates,  $\omega = dp \wedge dq$ . The line  $(q, p + tv(q))$  lies in the fiber of the cotangent bundle  $T^*M$  over the point  $q$  and the vector  $\xi = v(q)\partial/\partial p$  is tangent to this line. Then  $i_\xi \omega = v(q) dq$ . This 1-form vanishes on  $\Sigma$  since  $v(q)$  is a conormal vector to  $S$  at  $q$ . Thus  $\xi$  has the characteristic direction. Note that the quotient space by the characteristic foliation is  $T^*S$ .

Next we claim that the restriction of  $\omega$  to  $Z - \Delta$  is a symplectic form. Indeed,  $Z - \Delta \subset N_1$  is transverse to the trajectories of the geodesic flow, that is, the leaves of the characteristic foliation of  $N_1 \subset T^*M$ .

The intersections of  $Z$  with the leaves of the characteristic foliation on  $N_1$  determine an involution,  $\tau$ , which is free on  $Z - \Delta \subset N_1$ . If  $(q, v) \in Z$  is a vector, let  $q_1 \in S$  be the other intersection point of the geodesic generated by  $(q, v)$  with  $S$  and  $v_1$  the vector translated to point  $q_1$  along the geodesic. Then  $\tau(q, v) = (q_1, v_1)$ .

Consider the intersections of  $Z$  with the leaves of the characteristic foliation on  $\Sigma$ . We claim that this also determines an involution,  $\sigma$ , which is free on  $Z - \Delta \subset \Sigma$ . Indeed, let  $(q, v) \in Z$ , i.e.,  $q \in S$ ,  $\langle v, v \rangle = 1$ . The characteristic line is  $(q, v + tv(q))$ , where  $v(q)$  is a normal vector, and its intersection with  $Z$  is given by the equation  $\langle p + tv(q), p + tv(q) \rangle = 1$ . Since  $\langle v(q), v(q) \rangle \neq 0$ , this equation has two roots and we have an involution. One root is  $t = 0$ , the other is different from 0 if  $\langle v, v(q) \rangle \neq 0$ , that is,  $v$  is not tangent to  $S$ .

Let  $F = \sigma \circ \tau$ ; this is the billiard map on  $Z$ , see Fig. 5. Since both involutions are defined by intersections with the leaves of the characteristic foliations, they preserve the symplectic structure  $\omega|_Z$ . Thus  $F$  is a symplectic transformation of  $Z - \Delta$ . Let  $P : Z - \Delta \rightarrow \mathcal{L}_+^0$  be the projection. Then  $P$  is a symplectic 2-to-1 map and  $P \circ F = T_+ \circ P$ . It follows that  $T_+$  preserves the symplectic structure in  $\mathcal{L}_+^0$ .

In the case of  $T_0$ , we have the same picture with  $N_0$  replacing  $N_1$  and its symplectic reduction  $P$  in place of  $\mathcal{L}_+^0$ . We obtain a symplectic transformation of  $P$  that commutes with the action of  $\mathbf{R}_+^*$  by dilations. Therefore the map  $T_0$  preserves the contact structure of  $\mathcal{L}_0^0$ .  $\square$

**Remark 3.6.** Consider a convex domain  $D$  in the Lorentz plane with the metric  $dx dy$ . The light-like lines are either horizontal or vertical. The billiard system in  $D$ , restricted to light-like lines, coincides with the system described in [1] in the context of Hilbert's 13th problem (namely, see

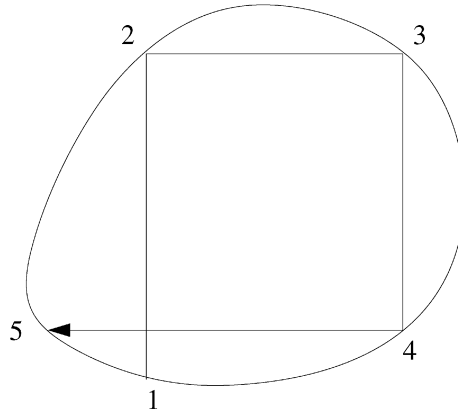


Fig. 6. A dynamical system on an oval.

Fig. 6 copied from Fig. 3 in [1, p. 8]). The map that moves a point of the curve first along a vertical and then along a horizontal chord is a circle map that, in case  $D$  is an ellipse, is conjugated to a rotation. The same map is discussed in [25] in the context of the Sobolev equation, approximately describing fluid oscillations in a fast rotating tank.

### 3.4. Diameters

A convex hypersurface in  $\mathbf{R}^n$  has at least  $n$  diameters, which are 2-periodic billiard trajectories in this hypersurface. In a pseudo-Euclidean space with signature  $(k, l)$  the result is as follows.

**Theorem 3.7.** *A smooth strictly convex closed hypersurface has at least  $k$  space-like and  $l$  time-like diameters.*

**Proof.** Denote the hypersurface by  $Q$ . Consider the space of chords  $Q \times Q$  and set  $f(x, y) = \langle x - y, x - y \rangle / 2$ . Then  $f$  is a smooth function on  $Q \times Q$ . The group  $\mathbf{Z}_2$  acts on  $Q \times Q$  by interchanging points, and this action is free off the diagonal  $x = y$ . The function  $f$  is  $\mathbf{Z}_2$ -equivariant.

First we claim that a critical point of  $f$  with non-zero critical value corresponds to a diameter (just as in the Euclidean case). Indeed, let  $u \in T_x Q$ ,  $v \in T_y Q$  be test vectors. Then  $d_x f(u) = \langle x - y, u \rangle$  and  $d_y f(v) = \langle x - y, v \rangle$ . Since these are zeros for all  $u, v$ , the (non-degenerate) chord  $x - y$  is orthogonal to  $Q$  at both end-points. Note that such a critical chord is not light-like, due to convexity of  $Q$ .

Fix a sufficiently small generic  $\varepsilon > 0$ . Let  $M \subset Q \times Q$  be a submanifold with boundary given by  $f(x, y) \geq \varepsilon$ . Since the boundary of  $M$  is a level hypersurface of  $f$ , the gradient of  $f$  (with respect to an auxiliary metric) has inward direction along the boundary, and the inequalities of Morse theory apply to  $M$ . Since  $\mathbf{Z}_2$  acts freely on  $M$  and  $f$  is  $\mathbf{Z}_2$ -equivariant, the number of critical  $\mathbf{Z}_2$ -orbits of  $f$  in  $M$  is not less than the sum of  $\mathbf{Z}_2$  Betti numbers of  $M/\mathbf{Z}_2$ .

We claim that  $M$  is homotopically equivalent to  $S^{k-1}$  and  $M/\mathbf{Z}_2$  to  $\mathbf{RP}^{k-1}$ . Indeed,  $M$  is homotopically equivalent to the set of space-like oriented lines intersecting  $Q$ . Retract this set to the set of space-like oriented lines through the origin. The latter is the spherization of the cone  $|q_1|^2 > |q_2|^2$ , and the projection  $(q_1, q_2) \mapsto q_1$  retracts it to the sphere  $S^{k-1}$ .

Since the sum of  $\mathbf{Z}_2$  Betti numbers of  $\mathbf{RP}^{k-1}$  is  $k$ , we obtain at least  $k$  space-like diameters. Replacing  $M$  by the manifold  $\{f(x, y) \leq -\varepsilon\}$  yields  $l$  time-like diameters.  $\square$



**Problem 3.8.** In Euclidean geometry, the fact that a smooth closed convex hypersurface in  $\mathbf{R}^n$  has at least  $n$  diameters has a far-reaching generalization due to Pushkar' [23]: a generic immersed closed manifold  $M^k \rightarrow \mathbf{R}^n$  has at least  $(B^2 - B + kB)/2$  diameters, that is, chords that are perpendicular to  $M$  at both end-points; here  $B$  is the sum of the  $\mathbf{Z}_2$ -Betti numbers of  $M$ . It is interesting to find a pseudo-Euclidean analog of this result.

**Problem 3.9.** Another generalization, in Euclidean geometry, concerns the least number of periodic billiard trajectories inside a closed smooth strictly convex hypersurface. In dimension 2, the classical Birkhoff theorem asserts that, for every  $n$  and every rotation number  $k$ , coprime with  $n$ , there exist at least two  $n$ -periodic billiard trajectories with rotation number  $k$ , see, e.g. [29,30]. In higher dimensions, a similar result was obtained recently [8,9]. It is interesting to find analogs for billiards in pseudo-Euclidean space. A possible difficulty is that variational problems in this set-up may have no solutions: for example, not every two points on the hyperboloid of one sheet in Example 2.4 are connected by a geodesic [3]!

#### 4. The geodesic flow on a quadric and the billiard inside a quadric

##### 4.1. Geodesics and characteristics

Let us start with a general description of the geodesics on a hypersurface in a pseudo-Riemannian manifold.

Let  $M^n$  be a pseudo-Riemannian manifold and  $S^{n-1} \subset M$  a smooth hypersurface. The geodesic flow on  $S$  is a limiting case of the billiard flow inside  $S$  when the billiard trajectories become tangent to the reflecting hypersurface. Assume that  $S$  is free of singular points, that is,  $S$  is a pseudo-Riemannian submanifold: the restriction of the metric in  $M$  to  $S$  is non-degenerate. The infinitesimal version of the billiard reflection law gives the following characterization of geodesics: a geodesic on  $S$  is a curve  $\gamma(t)$  such that  $\langle \gamma'(t), \gamma'(t) \rangle = \text{const}$  and the acceleration  $\gamma''(t)$  is orthogonal to  $S$  at point  $\gamma(t)$  (the acceleration is understood in terms of the covariant derivative) – see [22]. Note that the type (space-, time-, or light-like) of a geodesic curve remains the same for all  $t$ .

Let  $Q \subset \mathcal{L}$  be the set of oriented geodesics tangent to  $S$ . Write  $Q = Q_+ \cup Q_- \cup Q_0$  according to the type of the geodesics. Then  $Q_{\pm}$  are hypersurfaces in the symplectic manifolds  $\mathcal{L}_{\pm}$  and  $Q_0$  in the contact manifold  $\mathcal{L}_0$ .

Recall the definition of characteristics on a hypersurface  $X$  in a contact manifold  $Y$ , see [2]. Assume that the contact hyperplane  $C$  at point  $x \in X$  is not tangent to  $X$ ; we say that  $x$  is a non-singular point. Then  $C \cap T_x X$  is a hyperplane in  $C$ . Let  $\lambda$  be a contact form. Then  $\omega = d\lambda$  is a symplectic form on  $C$ ; a different choice of the contact form,  $f\lambda$ , gives a proportional symplectic form  $f(x)\omega$  on  $C$ . The characteristic line at  $x$  is the skew-orthogonal complement of the hyperplane  $C \cap T_x X$  in  $C$ .

##### Theorem 4.1.

- 1) The characteristics of the hypersurfaces  $Q_{\pm} \subset \mathcal{L}_{\pm}$  consist of oriented geodesics in  $M$  tangent to a fixed space- or time-like geodesic on  $S$ .
- 2) The hypersurface  $Q_0 \subset \mathcal{L}_0$  consists of non-singular points and its characteristics consist of oriented geodesics in  $M$  tangent to a fixed light-like geodesic on  $S$ .

**Proof.** The argument is a variation on that given in the proof of Theorem 3.5 (cf. [2]), and we use the notation from this proof. In particular, we identify vectors and covectors using the metric.

Consider  $Q_+$  (the case of  $Q_-$  being similar). We have the submanifold  $\Delta \subset N_1$  consisting of the unit space-like vectors tangent to  $S$ ; the projection  $N_1 \rightarrow \mathcal{L}_+$  identifies  $\Delta$  with  $Q_+$ . Likewise, the projection  $\Sigma \rightarrow T^*S$  makes it possible to consider  $\Delta$  as a hypersurface in  $T^*S$ . The characteristics of  $\Delta \subset T^*S$  are the geodesics on  $S$ .

We need to show that the two characteristic directions on  $\Delta$ , induced by its inclusions into  $\mathcal{L}_+$  and into  $T^*S$ , coincide. We claim that the restriction of the canonical symplectic structure  $\omega$  in  $T^*M$  on its codimension 3 submanifold  $\Delta$  has 1-dimensional kernel at every point. If this holds then both characteristic directions on  $\Delta$  coincide with these kernels and therefore with each other.

The kernel of the restriction of  $\omega$  on  $\Delta$  is odd-dimensional. Assume its dimension is 3; then  $\Delta \subset T^*M$  is a coisotropic submanifold. We will show that this is not the case.

Let  $v(q)$  be the normal vector to  $S$  at point  $q \in S$ . Since  $q$  is not singular,  $v(q)$  is transverse to  $T_q S$ . Thus the vector  $u = v(q) \partial/\partial q$  is transverse to  $\Delta \subset T^*M$ . So is the vector  $v = v(q) \partial/\partial p$ . Let  $w$  be another transverse vector such that  $u, v, w$  span a transverse space to  $\Delta$ . Note that

$$\omega(v, u) = (dp \wedge dq)(v(q) \partial/\partial p, v(q) \partial/\partial q) = \langle v(q), v(q) \rangle \neq 0.$$

Since  $\omega$  is a symplectic form,  $0 \neq i_{u \wedge v} \omega^n = C \omega^{n-1}$  with  $C \neq 0$ , and the  $2n-3$  form  $i_w \omega^{n-1}$  is a volume form on  $\Delta$ . This contradicts to the fact that  $T_{(q,p)} \Delta$  contains a 3-dimensional subspace skew-orthogonal to  $T_{(q,p)} \Delta$ , and the first statement of the theorem follows.

For the second statement, we replace  $N_1$  by  $N_0$  and  $\mathcal{L}_+$  by the space of scaled light-like geodesics  $P$ . Then  $P$  and  $\Delta$  are acted upon by the dilations. Using the notation from Theorem 2.1,  $\pi(\Delta) = Q_0$ . The characteristics of  $\Delta \subset P$  consist of scaled oriented geodesics tangent to a fixed light-like geodesic on  $S$ .

To show that the points of  $Q_0 \subset \mathcal{L}_0$  are non-singular, it suffices to prove that the hypersurface  $\Delta \subset P$  is not tangent to the kernel of the 1-form  $\bar{\lambda}$ . We claim that this kernel contains the vector  $v$  transverse to  $\Delta$ . Indeed,

$$\bar{\lambda}(v) = \bar{\omega}(\bar{E}, v) = (dp \wedge dq)(p \partial/\partial p, v(q) \partial/\partial p) = 0,$$

and hence  $\ker \bar{\lambda} \neq T_{(q,p)} \Delta$ .

Finally, the characteristics of the conical hypersurface  $\Delta = \pi^{-1}(Q_0)$  in the symplectization  $P$  of the contact manifold  $\mathcal{L}_0$  project to the characteristics of  $Q_0 \subset \mathcal{L}_0$ , see [2], and the last claim of the theorem follows.  $\square$

#### 4.2. Geodesics on a Lorentz surface of revolution

Geodesics on a surface of revolution in the Euclidean space have the following Clairaut first integral:  $r \sin \alpha = \text{const}$ , where  $r$  is the distance from a given point on the surface to the axis of revolution, and  $\alpha$  is the angle of the geodesic at this point with the projection of the axis to the surface.

Here we describe an analog of the Clairaut integral for Lorentz surfaces of revolution. Let  $S$  be a surface in the Lorentz space  $V^3$  with the metric  $ds^2 = dx^2 + dy^2 - dz^2$  obtained by a revolution of the graph of a function  $f(z)$  about the  $z$ -axis: it is given by the equation  $r = f(z)$

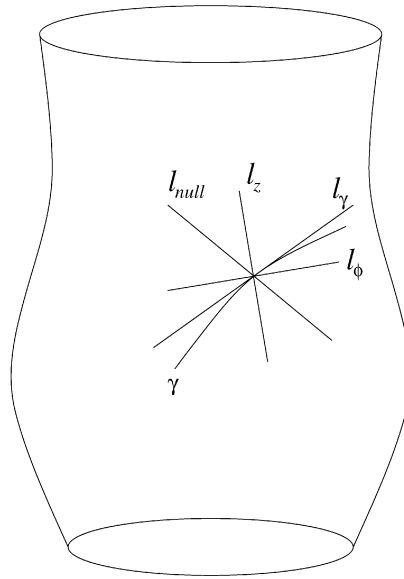


Fig. 7. A quadruple of tangent lines on a surface of revolution.

for  $r^2 = x^2 + y^2$ . We assume that the restriction of the ambient metric to the surface is pseudo-Riemannian.

Consider the tangent plane  $T_P S$  to the surface  $S$  at a point  $P$  on a given geodesic  $\gamma$ . Define the following 4 lines in this tangent plane: the axis projection  $l_z$  (meridian), the revolution direction  $l_\phi$  (parallel), the tangent to the geodesic  $l_\gamma$ , and one of the two null directions  $l_{\text{null}}$  on the surface at the point  $P$ , see Fig. 7. We denote the corresponding cross-ratio of this quadruple of lines as  $\text{cr} = \text{cr}(l_z, l_\phi, l_\gamma, l_{\text{null}})$ .

**Theorem 4.2.** *The function  $(1 - \text{cr}^2)/r^2$  is constant along any geodesic  $\gamma$  on the Lorentz surface of revolution.*

**Proof.** The Clairaut integral in either Euclidean or Lorentz setting is a specification of the Noether theorem, which gives the invariance of the angular momenta  $m = r \cdot v_\phi$  with respect to the axis of revolution. (Here  $(r, \phi)$  are the polar coordinates in the  $(x, y)$ -plane.)

In the Euclidean case, we have  $r\phi' = |v| \sin \alpha$  and combining the invariance of the magnitude of  $v$  along the geodesics with the preservation of  $m = r^2\phi'$  we immediately obtain the Clairaut integral  $r \sin \alpha = \text{const}$ .

In the Lorentz setting, we first find the cross-ratio discussed above. Let  $v$  be the velocity vector along a geodesic, and  $v_r, v_\phi, v_z$  be its radial, angle and axis projections, respectively. Suppose the point  $P$  has coordinates  $(0, y_0, z_0)$ ; choose  $(x, z)$  as the coordinates in the corresponding tangent plane  $T_P S$ . The lines  $l_z, l_\phi$ , and  $l_\gamma$  have the directions  $(1, 0)$ ,  $(0, 1)$ , and  $(v_\phi, v_z)$ , respectively. The direction of null vectors in this tangent plane is the intersection of the cone of null vectors  $x^2 + y^2 - z^2 = 0$  (in the coordinates centered at  $P$ ) with the plane  $y = f'(z_0)z$  tangent to  $S$  at  $P$ . Thus the corresponding null directions are  $x = \pm \sqrt{1 - (f'(z_0))^2}z$ . We choose the “plus” direction for  $l_{\text{null}}$  and find the cross-ratio

$$\text{cr} = \text{cr}(l_z, l_\phi, l_\gamma, l_{\text{null}}) = \frac{[l_z, l_{\text{null}}][l_\gamma, l_\phi]}{[l_z, l_\gamma][l_{\text{null}}, l_\phi]} = \frac{v_z \sqrt{1 - (f'(z_0))^2}}{v_\phi},$$

that is  $v_z^2(1 - (f'(z_0))^2) = v_\phi^2 \cdot cr^2$ . (By choosing the other sign for the null direction  $l_{\text{null}}$  we obtain the same relation.)

Now recall that the Lorentz length of  $v$  is preserved:  $v_r^2 + v_\phi^2 - v_z^2 = 1$ . Taking into account that  $v_r = f'(z_0)v_z$  in the tangent plane  $T_P S$ , we exclude from this relation both  $v_r$  and  $v_z$  and express  $v_\phi$  via the cross-ratio:  $v_\phi^2 = 1/(1 - cr^2)$ . Thus the preservation of the angular momenta  $m = r \cdot v_\phi$  yields the Lorentz analog of the Clairaut theorem:  $1/m^2 = (1 - cr^2)/r^2 = \text{const}$  along any given geodesic on  $S$ .  $\square$

We consider the conservation of the quantity  $(1 - cr^2)/r^2$ , inverse to  $m^2$ , since  $r \neq 0$  and this quantity makes sense even when  $cr = \pm 1$ . Note that this invariant immediately implies that a geodesic with a light-like initial condition stays light-like forever: along such geodesics,  $cr^2 = 1$ . If the geodesic is time-like ( $ds^2 < 0$ ), it can be continued until it hits the “tropic” consisting of singular points.

**Corollary 4.3.** *At the singular points of a surface of revolution all geodesics become tangent to the direction of the axis projection  $l_z$ .*

**Proof.** If the geodesic is “vertical,” i.e.,  $l_\gamma = l_z$ , it stays vertical forever, and the statement is evident. If its initial velocity is non-vertical,  $l_\gamma \neq l_z$ , then the cross-ratio is initially finite. Therefore, when the geodesic hits a singular point, the cross-ratio still has a finite limit which is obtained from the Clairaut invariant along the geodesic. On the other hand, when approaching the “tropic,” the null direction  $l_{\text{null}}$  tends to the axis projection  $l_z$ . This forces the tangent element  $l_\gamma$  to approach  $l_z$  as well.  $\square$

Finally, consider space-like geodesics. Depending on the initial velocity, they can either hit the tropic or stay away from it. It is interesting to compare the latter with geodesics in the Riemannian case. The Riemannian Clairaut theorem implies that a “non-vertical” geodesic does not enter the regions with a too narrow neck on the surface of revolution. Indeed, the invariance of  $r \sin \alpha$  implies, for  $\alpha \neq 0$ , that  $r$  cannot be too small, since  $|\sin \alpha|$  is bounded above. It turns out that, on a Lorentz surface of revolution, the phenomenon is exactly the opposite: such geodesics do not enter the regions where the neck is too wide:

**Corollary 4.4.** *For any space-like geodesic, there is an upper bound  $K$  such that the geodesic stays in the region  $|r| \leq K$ .*

**Proof.** For space-like geodesics,  $|cr| < 1$ . Then  $|1 - cr^2| \leq 1$  and the conservation of  $(1 - cr^2)/r^2$  implies that  $|r^2|$  must be bounded above for any such geodesic.  $\square$

A surface of revolution is an example of a warped product. A nice alternative description of the geodesics via Maupertuis’ principle in the general context of warped products is given in [31]. The dichotomy of the Riemannian and Lorentzian cases, shown in Corollary 4.4, also follows from this description of geodesics as particles moving in the potentials differed by sign.

### 4.3. Analogs of the Jacobi and the Chasles theorems

An ellipsoid with distinct axes in Euclidean space

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1 \quad (4)$$

gives rise to the confocal family of quadrics

$$\frac{x_1^2}{a_1^2 + \lambda} + \frac{x_2^2}{a_2^2 + \lambda} + \cdots + \frac{x_n^2}{a_n^2 + \lambda} = 1.$$

The Euclidean theory of confocal quadrics comprises the following theorems: through a generic point in space there pass  $n$  confocal quadrics, and they are pairwise orthogonal at this point (Jacobi);<sup>3</sup> a generic line is tangent to  $n - 1$  confocal quadrics whose tangent hyperplanes at the points of tangency with the line are pairwise orthogonal (Chasles); and the tangent lines to a geodesic on an ellipsoid are tangent to fixed  $n - 2$  confocal quadrics (Jacobi–Chasles) – see [2,19,20].

We shall construct a pseudo-Euclidean analog of this theory and adjust the proofs accordingly.

Consider pseudo-Euclidean space  $V^n$  with signature  $(k, l)$ ,  $k + l = n$ , and let  $E : V \rightarrow V^*$  be the self-adjoint operator such that  $\langle x, x \rangle = E(x) \cdot x$  where dot denotes the pairing between vectors and covectors. Let  $A : V \rightarrow V^*$  be a positive-definite self-adjoint operator defining an ellipsoid  $A(x) \cdot x = 1$ . Since  $A$  is positive-definite, both forms can be simultaneously reduced to principle axes.<sup>4</sup> We assume that  $A = \text{diag}(a_1^2, \dots, a_n^2)$  and  $E = \text{diag}(1, \dots, 1, -1, \dots, -1)$ . An analog of the confocal family is the following “pseudo-confocal” family of quadrics  $Q_\lambda$

$$\frac{x_1^2}{a_1^2 + \lambda} + \frac{x_2^2}{a_2^2 + \lambda} + \cdots + \frac{x_k^2}{a_k^2 + \lambda} + \frac{x_{k+1}^2}{a_{k+1}^2 - \lambda} + \cdots + \frac{x_n^2}{a_n^2 - \lambda} = 1 \quad (5)$$

where  $\lambda$  is a real parameter or, in short,  $(A + \lambda E)^{-1}(x) \cdot x = 1$ .

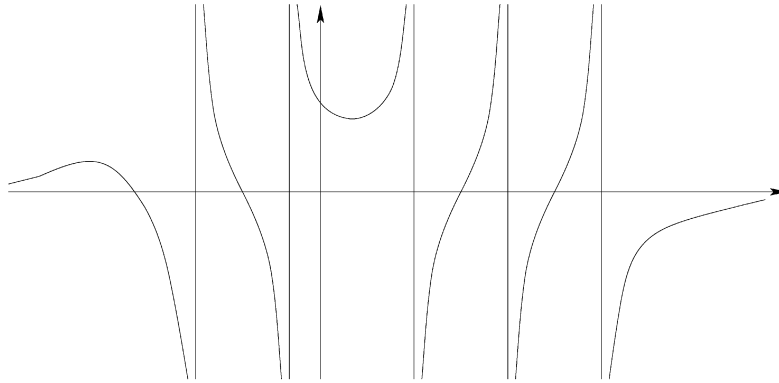
The following result is a pseudo-Euclidean version of the Jacobi theorem.

**Theorem 4.5.** *Through every generic point  $x \in V$  there pass either  $n$  or  $n - 2$  quadrics from the pseudo-confocal family (5). In the latter case, all quadrics have different topological types and in the former two of them have the same type. The quadrics are pairwise orthogonal at point  $x$ .*

**Proof.** Given a point  $x$ , we want to find  $\lambda$  satisfying Eq. (5), which reduces to a polynomial in  $\lambda$  of degree  $n$ . Denote by  $f(\lambda)$  the function on the left-hand side of (5). This function has poles at  $\lambda = -a_1^2, \dots, -a_k^2, a_{k+1}^2, \dots, a_n^2$ . At every negative pole  $f(\lambda)$  changes sign from negative to

<sup>3</sup> The respective values of  $\lambda$  are called the elliptic coordinates of the point.

<sup>4</sup> In general, it is not true that a pair of quadratic forms can be simultaneously reduced to principle axes. The simplest example in the plane is  $x^2 - y^2$  and  $xy$ .

Fig. 8. The graph of the function  $f(\lambda)$  for a time-like point  $x$ .

positive, and at every positive pole from positive to negative. Let us analyze the behavior of  $f(\lambda)$  as  $\lambda \rightarrow \pm\infty$ . One has:

$$f(\lambda) = \frac{1}{\lambda} \langle x, x \rangle - \frac{1}{\lambda^2} \sum_{i=1}^n a_i^2 x_i^2 + O\left(\frac{1}{\lambda^3}\right),$$

hence if  $x$  is not light-like then the sign of  $f(\lambda)$  at  $+\infty$  is equal, and at  $-\infty$  opposite, to that of  $\langle x, x \rangle$ , whereas if  $x$  is light-like then  $f(\lambda)$  at  $\pm\infty$  is negative. The graph of the function  $f(\lambda)$  in the case  $\langle x, x \rangle < 0$  is shown in Fig. 8. Thus  $f(\lambda)$  assumes value 1 at least  $k - 1$  times for negative  $\lambda$  and at least  $l - 1$  times for positive ones. Being a polynomial of degree  $n$ , the number of roots is not greater than  $n$ .

Note that the topological type of the quadric changes each time that  $\lambda$  passes through a pole of  $f(\lambda)$ . It follows that if there are  $n - 2$  quadrics passing through  $x$  then they all have different topological types, and the ellipsoid (corresponding to  $\lambda = 0$ ) is missing. On the other hand, if there are  $n$  quadrics passing through  $x$  then two of them have the same topological type and there are  $n - 1$  different types altogether. Note, in particular, that if  $x$  lies on the original ellipsoid then there are  $n$  quadrics passing through it.

To prove that  $Q_\lambda$  and  $Q_\mu$  are orthogonal to each other at  $x$ , consider their normal vectors (half the gradients of the left-hand sides of (5) with respect to the pseudo-Euclidean metric)

$$N_\lambda = \left( \frac{x_1}{a_1^2 + \lambda}, \frac{x_2}{a_2^2 + \lambda}, \dots, \frac{x_k}{a_k^2 + \lambda}, -\frac{x_{k+1}}{a_{k+1}^2 - \lambda}, \dots, -\frac{x_n}{a_n^2 - \lambda} \right),$$

and likewise for  $N_\mu$ . Then

$$\langle N_\lambda, N_\mu \rangle = \sum_{i=1}^k \frac{x_i^2}{(a_i^2 + \lambda)(a_i^2 + \mu)} - \sum_{i=k+1}^n \frac{x_i^2}{(a_i^2 - \lambda)(a_i^2 - \mu)}. \quad (6)$$

The difference of the left-hand sides of Eqs. (5), taken for  $\lambda$  and  $\mu$ , is equal to the right-hand side of (6) times  $(\mu - \lambda)$ , whereas the right-hand side is zero. Thus  $\langle N_\lambda, N_\mu \rangle = 0$ .  $\square$

**Example 4.6.** Consider the simplest example, which we will study in detail in Section 5:  $A = \text{diag}(1, 1)$ ,  $E = \text{diag}(1, -1)$ . Fig. 9 depicts the partition of the Lorentz plane according to the

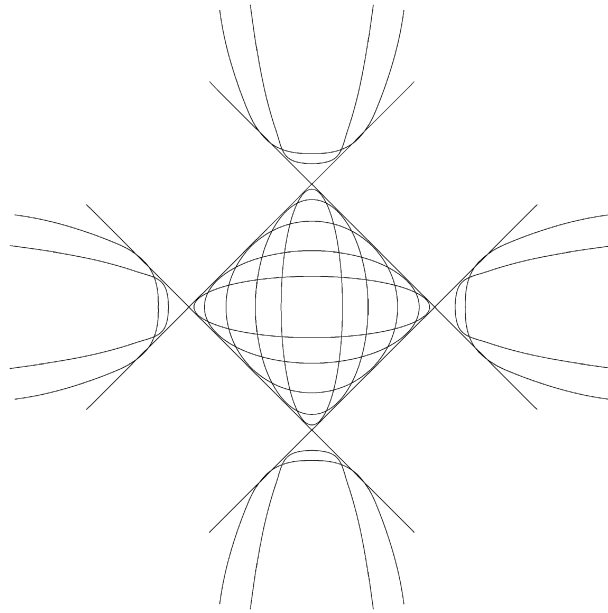


Fig. 9. A pseudo-confocal family of conics.

number of conics from a pseudo-confocal family passing through a point: the boundary consists of the lines  $|x \pm y| = \sqrt{2}$ .

**Problem 4.7.** It is interesting to describe the topology of the partition of  $V$  according to the number of quadrics from the family (5) passing through a point. In particular, how many connected components are there?

Next, consider a pseudo-Euclidean version of the Chasles theorem.

**Theorem 4.8.** *A generic space- or time-like line  $\ell$  is tangent to either  $n - 1$  or  $n - 3$ , and a generic light-like line to either  $n - 2$  or  $n - 4$ , quadrics from the family (5). The tangent hyperplanes to these quadrics at the tangency points with  $\ell$  are pairwise orthogonal.*

**Proof.** Let  $v$  be a vector spanning  $\ell$ . Suppose first that  $v$  is space- or time-like. Project  $V$  along  $\ell$  on the orthogonal complement  $U$  to  $v$ . A quadric determines a hypersurface in this  $(n - 1)$ -dimensional space, the set of critical values of its projection (the apparent contour). If one knows that these hypersurfaces also constitute a family (5) of quadrics, the statement will follow from Theorem 4.5.

Let  $Q \subset V$  be a smooth star-shaped hypersurface and let  $W \subset V^*$  be the annihilator of  $v$ . Suppose that a line parallel to  $v$  is tangent to  $Q$  at point  $x$ . Then the tangent hyperplane  $T_x Q$  contains  $v$ . Hence the respective covector  $y \in V^*$  from the polar dual hypersurface  $Q^*$  lies in  $W$ . Thus polar duality takes the points of tangency of  $Q$  with the lines parallel to  $v$  to the intersection of the dual hypersurface  $Q^*$  with the hyperplane  $W$ .

On the other hand,  $U = V/(v)$  and  $W = (V/(v))^*$ . Therefore the apparent contour of  $Q$  in  $U$  is polar dual to  $Q^* \cap W$ . If  $Q$  belongs to the family of quadrics (5) then  $Q^*$  belongs to the pencil  $(A + \lambda E)y \cdot y = 1$ . The intersection of a pencil with a hyperplane is a pencil of the same type (with the new  $A$  positive definite and the new  $E$  having signature  $(k - 1, l)$  or  $(k, l - 1)$ ),

depending on whether  $v$  is space- or time-like). It follows that the polar dual family of quadrics, consisting of the apparent contours, is of the type (5) again, as needed.

Note that, similarly to the proof of Theorem 4.5, if  $\ell$  is tangent to the original ellipsoid then it is tangent to  $n - 1$  quadrics from the family (5).

If  $v$  is light-like then we argue similarly. We choose as the “screen”  $U = V/(v)$  any hyperplane transverse to  $v$ . The restriction of  $E$  to  $W$  is degenerate: it has 1-dimensional kernel and its signature is  $(k - 1, l - 1, 1)$ . The family of quadrics, dual to the restriction of the pencil to  $W$  is given by the formula

$$\frac{x_1^2}{b_1^2 + \lambda} + \frac{x_2^2}{b_2^2 + \lambda} + \cdots + \frac{x_{k-1}^2}{b_{k-1}^2 + \lambda} + \frac{x_{k+1}^2}{b_{k+1}^2 - \lambda} + \cdots + \frac{x_{k+l-1}^2}{b_{k+l-1}^2 - \lambda} = 1 - \frac{x_k^2}{b_k^2}$$

which is now covered by the  $(n - 1)$ -dimensional case of Theorem 4.5.  $\square$

Note that in Example 4.6 a generic light-like line is tangent to no conic, whereas the four exceptional light-like lines  $|x \pm y| = \sqrt{2}$  are tangent to infinitely many ones.

#### 4.4. Complete integrability

The following theorem is a pseudo-Euclidean analog of the Jacobi–Chasles theorem.<sup>5</sup>

##### Theorem 4.9.

- 1) *The tangent lines to a fixed space- or time-like (respectively, light-like) geodesic on a quadric in pseudo-Euclidean space  $V^n$  are tangent to  $n - 2$  (respectively,  $n - 3$ ) other fixed quadrics from the pseudo-confocal family (5).*
- 2) *A space- or time-like (respectively, light-like) billiard trajectory in a quadric in pseudo-Euclidean space  $V^n$  remains tangent to  $n - 1$  (respectively,  $n - 2$ ) fixed quadrics from the family (5).*
- 3) *The sets of space- or time-like oriented lines in pseudo-Euclidean space  $V^n$ , tangent to  $n - 1$  fixed quadrics from the family (5), are Lagrangian submanifolds in the spaces  $\mathcal{L}_\pm$ . The set of light-like oriented lines, tangent to  $n - 2$  fixed quadrics from the family (5), is a codimension  $n - 2$  submanifold in  $\mathcal{L}_0$  foliated by codimension one Legendrian submanifolds.*

**Proof.** Let  $\ell$  be a tangent line at point  $x$  to a geodesic on the quadric  $Q_0$  from the pseudo-confocal family (5). By Theorem 4.8,  $\ell$  is tangent to  $n - 2$  (or  $n - 3$ , in the light-like case) quadrics from this family. Denote these quadrics by  $Q_{\lambda_j}$ ,  $j = 1, \dots, n - 2$ .

Let  $N$  be a normal vector to  $Q_0$  at point  $x$ . Consider an infinitesimal rotation of the tangent line  $\ell$  along the geodesic. Modulo infinitesimals of the second order, this line rotates in the 2-plane generated by  $\ell$  and  $N$ . By Theorem 4.8, the tangent hyperplane to each  $Q_{\lambda_j}$  at its tangency point with  $\ell$  contains the vector  $N$ . Hence, modulo infinitesimals of the second order, the line  $\ell$  remains tangent to every  $Q_{\lambda_j}$ , and therefore remains tangent to each one of them.

<sup>5</sup> See also [10], a follow-up to the present paper, devoted to the case study of the geodesic flow on the ellipsoid in 3-dimensional pseudo-Euclidean space.



The billiard flow inside an ellipsoid in  $n$ -dimensional space is the limit case of the geodesic flow on an ellipsoid in  $(n + 1)$ -dimensional space, whose minor axis goes to zero. Thus the second statement follows from the first one.

Now we prove the third statement. Consider first the case of space- or time-like lines  $\mathcal{L}_{\pm}$ . Let  $\ell$  be a generic oriented line tangent to quadrics  $Q_{\lambda_j}$ ,  $j = 1, \dots, n - 1$ , from the family (5). Choose smooth functions  $f_j$  defined in neighborhoods of the tangency points of  $\ell$  with  $Q_{\lambda_j}$  in  $V^n$  whose level hypersurfaces are the quadrics from the family (5). Any line  $\ell'$  close to  $\ell$  is tangent to a close quadric  $Q_{\lambda'_j}$ . Define the function  $F_j$  on the space of oriented lines whose value at  $\ell'$  is the (constant) value of  $f_j$  on  $Q_{\lambda'_j}$ .

We want to show that  $\{F_j, F_k\} = 0$  where the Poisson bracket is taken with respect to the symplectic structure defined in Section 2. Consider the value  $dF_k(\text{sgrad } F_j)$  at  $\ell$ . The vector field  $\text{sgrad } F_j$  is tangent to the characteristics of the hypersurface  $F_j = \text{const}$ , that is, the hypersurface consisting of the lines, tangent to  $Q_{\lambda_j}$ . According to Theorem 4.1, these characteristics consist of the lines, tangent to a fixed geodesic on  $Q_{\lambda_j}$ . According to statement 1) of the present theorem, these lines are tangent to  $Q_{\lambda_k}$ , hence  $F_k$  does not change along the flow of  $\text{sgrad } F_j$ . Thus  $dF_k(\text{sgrad } F_j) = 0$ , as claimed.

Finally, in the light-like case, consider the homogeneous symplectic manifold  $P^{2n-2}$  of scaled light-like lines, whose quotient is  $\mathcal{L}_0$ . Then, as before, we have homogeneous of degree zero, Poisson-commuting functions  $F_j$ ,  $j = 1, \dots, n - 2$ , on  $P$ . Therefore a level submanifold  $M^n = \{F_1 = c_1, \dots, F_{n-2} = c_{n-2}\}$  is coisotropic: the symplectic orthogonal complement to  $TM$  in  $TP$  is contained in  $TM$ . The commuting vector fields  $\text{sgrad } F_j$  define an action of the Abelian group  $\mathbf{R}^{n-2}$  on  $M$  whose orbits are isotropic submanifolds. Furthermore,  $M$  is invariant under the Euler vector field  $E$  that preserves the foliation on isotropic submanifolds. Hence the quotient by  $E$  is a codimension  $n - 2$  submanifold in  $\mathcal{L}_0$  foliated by codimension one Legendrian submanifolds.  $\square$

**Example 4.10.** Let  $\gamma$  be a geodesic on a generic ellipsoid  $Q_0$  in 3-dimensional Lorentz space and let  $x$  be a point of  $\gamma$ . Then, upon each return to point  $x$ , the curve  $\gamma$  has one of at most two possible directions (a well-known property in the Euclidean case).

Indeed, if  $\gamma$  is not light-like then the tangent lines to  $\gamma$  are tangent to a fixed pseudo-confocal quadratic surface, say  $Q_1$ . The intersection of the tangent plane  $T_x Q_0$  with  $Q_1$  is a conic, and there are at most two tangent lines from  $x$  to this conic. If  $\gamma$  is light-like then its direction at point  $x$  is in the kernel of the restriction of the metric to  $T_x Q_0$  which consists of at most two lines.

**Remark 4.11.** The functions  $F_1, \dots, F_{n-1}$  from the proof of Theorem 4.9 can be considered as functions on the tangent bundle  $TM$ . The proof of Theorem 4.9 implies that these functions and the energy function  $F_0(x, v) = \langle v, v \rangle$  pairwise Poisson commute with respect to the canonical symplectic structure on  $T^*M$  (as usual, identified with  $TM$  via the metric).

We now give explicit formulas for the integrals; these formulas are modifications of the ones given in [19,20].

Write the pseudo-Euclidean metric as

$$\sum_{i=1}^n \tau_i dx_i^2 \quad \text{with } \tau_1 = \dots = \tau_k = 1, \quad \tau_{k+1} = \dots = \tau_n = -1.$$

Let  $v = (v_1, \dots, v_n)$  denote tangent vectors to the ellipsoid. Then the integrals are given by the formulas

$$F_k = \frac{v_k^2}{\tau_k} + \sum_{i \neq k} \frac{(x_i v_k - x_k v_i)^2}{\tau_i a_k^2 - \tau_k a_i^2}, \quad k = 1, \dots, n.$$

These integrals satisfy the relation  $\sum F_k = \langle v, v \rangle$ . One also has a modification of the Joachimsthal integral (functionally dependent on the previous ones):

$$J = \left( \sum_i \frac{x_i^2}{\tau_i a_i^4} \right) \left( \sum_j \frac{v_j^2}{a_j^2} \right).$$

**Remark 4.12.** Another approach to complete integrability of the billiard in the ellipsoid and the geodesic flow on the ellipsoid in Euclidean space is described in [27,28]. In a nutshell, in the case of billiards, one constructs another symplectic form on the space of oriented lines invariant under the billiard map, and for the geodesic flow one constructs another metric on the ellipsoid, projectively equivalent to the Euclidean one: this means that their non-parameterized geodesics coincide. For geodesic flows, this integrability mechanism was independently and simultaneously discovered by Matveev and Topalov [15].

In the present situation, this approach leads to the following result. We do not dwell on details.

As before,  $Q$  is an ellipsoid in pseudo-Euclidean space  $V^n$  given by the equation  $A(x) \cdot x = 1$  and the scalar product is  $\langle u, v \rangle = E(u) \cdot v$  where  $A$  and  $E$  are self-adjoint operators  $V \rightarrow V^*$ . We assume that  $A = \text{diag}(a_1^2, \dots, a_n^2)$  and  $E = \text{diag}(1, \dots, 1, -1, \dots, -1)$ , and denote the billiard map in  $Q$  by  $T$ . Consider the interior of  $Q$  as the projective, or Cayley–Klein, model of hyperbolic geometry and let  $\Omega$  be the respective symplectic structure on the space of oriented lines (obtained by the standard symplectic reduction).

**Theorem 4.13.**

- 1) *The symplectic structure  $\Omega$  is invariant under  $T$ .*
- 2) *The restrictions of the metrics*

$$\langle dx, dx \rangle \quad \text{and} \quad \frac{A(dx) \cdot dx}{\langle A(x), A(x) \rangle} \quad (7)$$

*on the ellipsoid  $Q$  are projectively equivalent.*

## 5. The geometry of the circle billiard in the Lorentz plane

Below we show how the theorems of Section 4 work for a circle billiard in dimension 2. Although its integrability follows from the results above, we made this section self-contained to emphasize the simplicity of the corresponding formulae.

Consider the plane with the metric  $ds^2 = dx dy$ . Then a vector  $(a, b)$  is orthogonal to  $(a, -b)$ . Let  $D(a, b) = (b, a)$  be the linear operator identifying vectors and covectors via the metric.

Consider the circle  $x^2 + y^2 = 1$  and the billiard system inside it. There are four singular points:  $(\pm 1, 0)$ ,  $(0, \pm 1)$ . The phase space consists of the oriented lines intersecting the circle and such

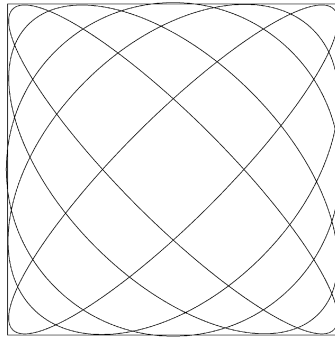


Fig. 10. Ellipses inscribed into a square.

that the impact point is not singular. The billiard map on light-like lines is 4-periodic. One also has two 2-periodic orbits, the diameters having slopes  $\pm 1$ .

Let  $t$  be the cyclic coordinate on the circle. Let us characterize a line by the coordinates of its first and second intersection points with the circle,  $(t_1, t_2)$ . The billiard map  $T$  sends  $(t_1, t_2)$  to  $(t_2, t_3)$ .

### Theorem 5.1.

1) The map  $T$  is given by the equation

$$\cot\left(\frac{t_2 - t_1}{2}\right) + \cot\left(\frac{t_2 - t_3}{2}\right) = 2 \cot 2t_2. \quad (8)$$

2) The area form is given by the formula

$$\omega = \frac{\sin((t_2 - t_1)/2)}{|\sin(t_1 + t_2)|^{3/2}} dt_1 \wedge dt_2. \quad (9)$$

3) The map is integrable: it has an invariant function

$$I = \frac{\sin((t_2 - t_1)/2)}{|\sin(t_1 + t_2)|^{1/2}}. \quad (10)$$

4) The lines containing the billiard segments, corresponding to a fixed value  $\lambda$  of the (squared) integral  $I^2$ , are tangent to the conic

$$x^2 + y^2 + 2\lambda xy = 1 - \lambda^2, \quad \lambda \in \mathbf{R}. \quad (11)$$

These conics for different  $\lambda$  are all tangent to the four lines – two horizontal and two vertical – tangent to the unit circle.

In principal axes (rotated  $45^\circ$ ), the family (11) writes as

$$\frac{x^2}{1 - \lambda} + \frac{y^2}{1 + \lambda} = 1.$$

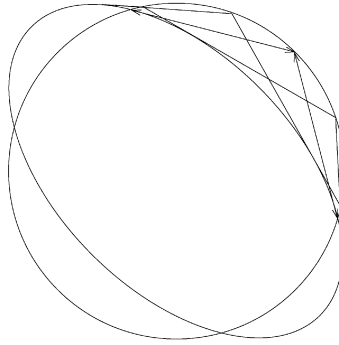


Fig. 11. Two 4-periodic trajectories on the invariant curve with the rotation number  $1/4$ : one orbit is a self-intersecting quadrilateral, and the other one consists of two segments, traversed back and forth.

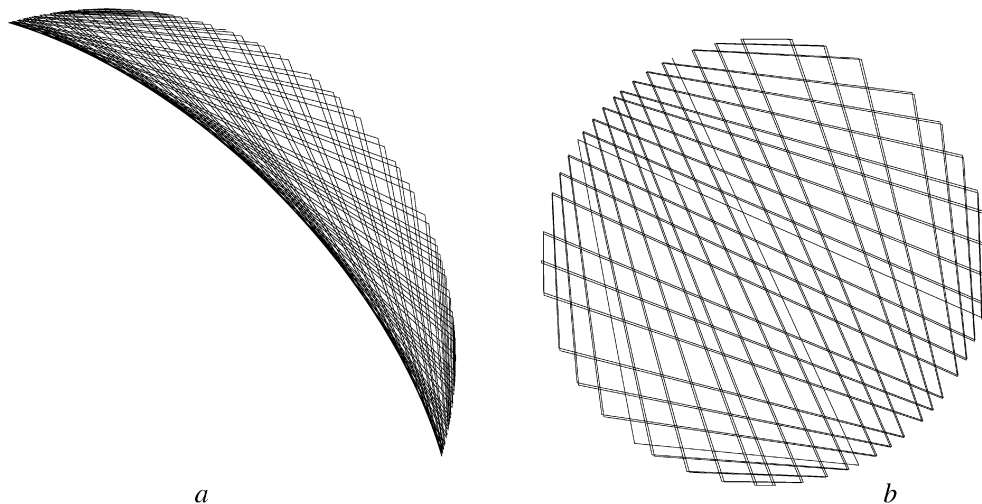


Fig. 12. Two orbits consisting of 100 billiard segments: the lines containing the billiard segments on the left are tangent to an ellipse, and those on the right to a hyperbola.

Before going into the proof of Theorem 5.1 let us make some comments and illustrate the theorem by figures.

In the familiar case of the billiard inside an ellipse in the Euclidean plane, the billiard trajectories are tangent to the family of conics, confocal with the given ellipse, see, e.g. [29,30]. These conics are either the confocal ellipses inside the elliptic billiard table or the confocal hyperbolas. The billiard map, restricted to an invariant curve of the integral, is described as follows: take a point  $A$  on the boundary of an elliptic billiard table, draw a tangent line to the fixed confocal ellipse (or hyperbola – for other values of the integral) until the intersection with the boundary ellipse at point  $A_1$ ; take  $A_1$  as the next point of the billiard orbit, etc.

In our case, for a fixed billiard trajectory inside the unit circle, there is a quadric inscribed into a  $2 \times 2$  square to which it is tangent, see the family of ellipses in Fig. 10. For instance, two 4-periodic trajectories in Fig. 11 are tangent to one and the same inscribed ellipse.

Figs. 12a and 12b depict two billiard orbits in the configuration space consisting of 100 time-like billiard segments. It is easy to recognize the inscribed ellipse as an envelope of the segments of the billiard orbit in Fig. 12a: all the reflections occur on one of the two arcs of the circle outside of the ellipse, and the tangent line to the ellipse at its intersection point with the circle is the Lorentz normal to the circle at this point.

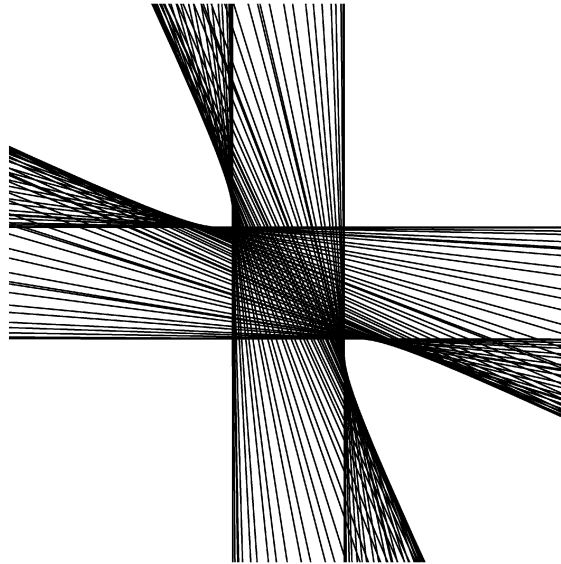


Fig. 13. Extensions of billiard chords tangent to a hyperbola.

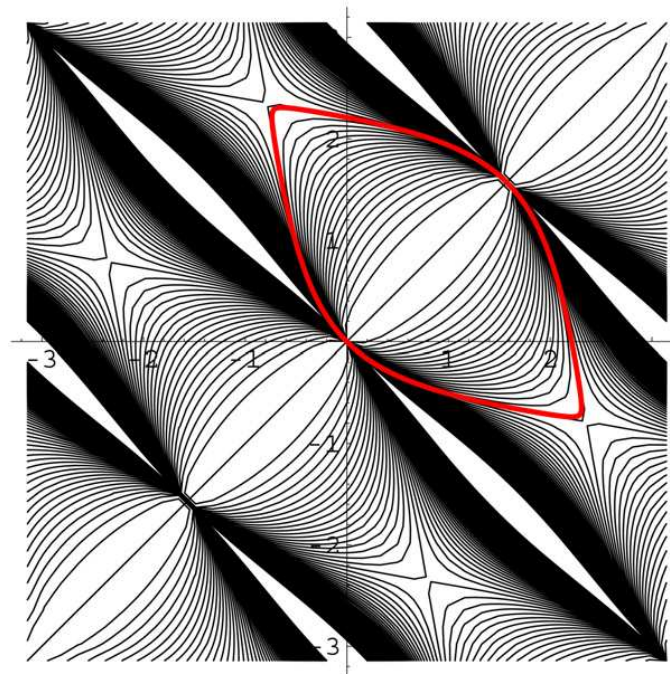


Fig. 14. Level curves of the integral  $I$  in the  $(t_1, t_2)$  coordinates.

The corresponding envelope is less evident for the orbit in Fig. 12b: extensions of the billiard chords have a hyperbola as their envelope, see Fig. 13.

The level curves of the integral  $I$  are shown in Fig. 14 depicting a  $[-\pi, \pi] \times [-\pi, \pi]$  torus with coordinates  $(t_1, t_2)$ . The four hyperbolic singularities of the foliation  $I = \text{const}$  at points

$$(3\pi/4, -\pi/4), (\pi/4, -3\pi/4), (-\pi/4, 3\pi/4), (-3\pi/4, \pi/4)$$

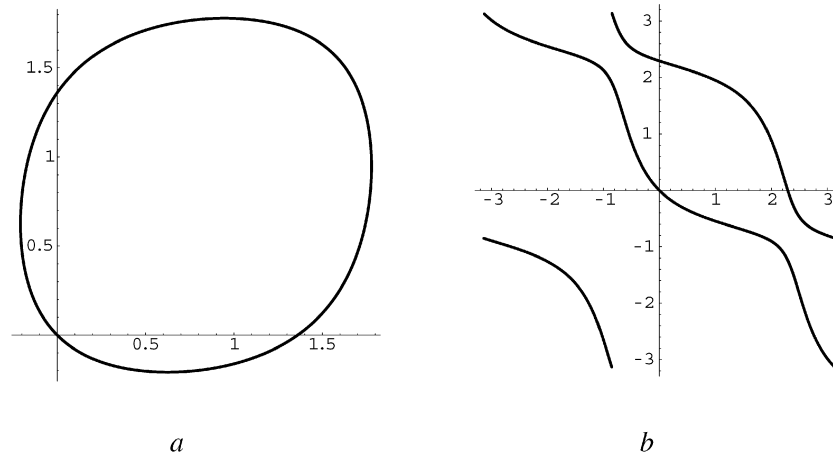


Fig. 15. Two invariant curves (the right one consists of two components, the phase points “jump” from one component to the other).

correspond to two 2-periodic orbits of the billiard map; these orbits are hyperbolically unstable<sup>6</sup> (unlike the case of an ellipse in the Euclidean plane where the minor axis is a stable 2-periodic orbit). The white spindle-like regions surround the lines  $t_1 + t_2 = \pi n$ ,  $n \in \mathbf{Z}$ ; these lines correspond to the light-like rays. The four points  $(0, 0)$ ,  $(\pi/2, \pi/2)$ ,  $(-\pi/2, -\pi/2)$ ,  $(\pi, \pi)$  are singular: every level curve of the integral  $I$  pass through them.

Figs. 15a and 15b show two topologically different invariant curves, and Figs. 12a and 12b discussed above depict two billiard orbits in the configuration space corresponding respectively to those two invariant curves.

Note also that it follows from the Poncelet porism (see, e.g. [5]) that if some point of an invariant circle is periodic then all points of this invariant circle are periodic with the same period, cf. Fig. 11.

Now we shall prove Theorem 5.1.

**Proof.** We use another criterion for harmonicity of a quadruple of lines, similar to (2). Consider four concurrent lines, and let  $\alpha, \phi, \beta$  be the angles made by three of them with the fourth, see Fig. 16. Then the lines are harmonic if and only if

$$\cot \alpha + \cot \beta = 2 \cot \phi. \quad (12)$$

In our situation, the billiard curve is  $\gamma(t) = (\cos t, \sin t)$ . The tangent vector is  $\gamma'(t) = (-\sin t, \cos t)$  and the normal is  $(\sin t, \cos t)$ . Consider the impact point  $t_2$ . By elementary geometry, the rays  $(t_2, t_1)$  and  $(t_2, t_3)$  make the angles  $(t_1 - t_2)/2$  and  $(t_3 - t_2)/2$  with the tangent line at  $\gamma(t_2)$ , and the normal makes the angle  $\pi - 2t_2$  with this tangent line. Then Eq. (12) becomes (8).

It is straightforward to compute the area form from Lemma 2.3 in the  $(t_1, t_2)$ -coordinates; the result (up to a constant factor) is (9).

<sup>6</sup> As indicated by the hyperbolic crosses made by the level curves at these points.

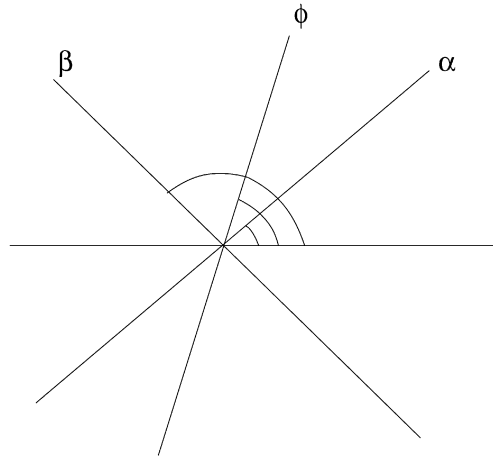


Fig. 16. Harmonic quadruple of lines given by angles.

We shall give two proofs that  $I$  is an integral. First, our Lorentz billiard is a particular case of a projective billiard in a circle. It is proved in [26] that every such billiard map has an invariant area form

$$\Omega = \frac{1}{\sin^2((t_2 - t_1)/2)} dt_1 \wedge dt_2. \quad (13)$$

(This form is the symplectic structure on the space of oriented lines for the projective – or Klein–Beltrami – model of hyperbolic geometry inside the unit disc, see Remark 4.12.) Thus  $T$  has two invariant area forms, and (the cube root of) their ratio is an invariant function.

The second proof imitates a proof that the billiard inside an ellipse in the Euclidean plane is integrable, see [30]. Let us restrict attention to space-like lines. Assign to a line its first intersection point with the circle,  $q$ , and the unit vector along the line,  $v$ . Then  $\langle D(q), q \rangle = 1$  and  $\langle v, v \rangle = 1$ . We claim that  $I = \langle D(q), v \rangle$  is invariant under the billiard map.

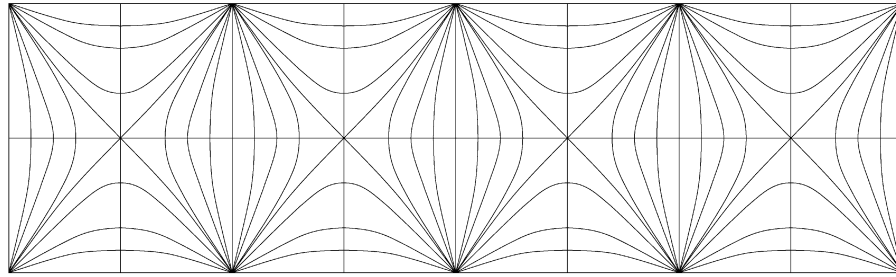
The billiard map is the composition of the involutions  $\tau$  and  $\sigma$ , see proof of Theorem 3.5. It turns out that each involution changes the sign of  $I$ .

Indeed,  $\langle D(q) + D(q_1), q_1 - q \rangle = 0$  since  $D$  is self-adjoint. Since  $v$  is collinear with  $q_1 - q$ , we have:  $\langle D(q) + D(q_1), v \rangle = 0$ , and hence  $I$  is odd with respect to  $\tau$ .

Since the circle is given by the equation  $\langle D(q), q \rangle = 1$ , the normal at point  $q_1$  is  $D(q_1)$ . By definition of the billiard reflection, the vector  $v + v_1$  is collinear with the normal at  $q_1$ , hence  $\langle D(q_1), v \rangle = -\langle D(q_1), v_1 \rangle$ . Thus  $I$  is odd with respect to  $\sigma$  as well. The invariance of  $I = \langle D(q), v \rangle$  follows, and it is straightforward to check that, in the  $(t_1, t_2)$ -coordinates, this integral equals (10).

To find the equation of the envelopes and prove 4) we first rewrite the integral  $I$  in the standard, Euclidean, coordinates  $(p, \alpha)$  in the space of lines:  $p$  is the signed length of the perpendicular from the origin to the line and  $\alpha$  the direction of this perpendicular, see [24]. One has

$$\alpha = \frac{t_1 + t_2}{2}, \quad p = \cos\left(\frac{t_2 - t_1}{2}\right). \quad (14)$$

Fig. 17. Level curves of the integral  $I$  in the  $(\alpha, p)$ -coordinates.

Fix a value of the integral  $I$  by setting

$$\frac{\sin^2((t_2 - t_1)/2)}{\sin(t_1 + t_2)} = \lambda.$$

It follows from (14) that  $1 - p^2 = \lambda \sin 2\alpha$ , and hence  $p = \sqrt{1 - \lambda \sin 2\alpha}$ . (See Fig. 17 which shows the level curves of the (squared) integral  $I^2 = (1 - p^2)/\sin 2\alpha$  in the  $(\alpha, p)$ -coordinates.) We use  $\alpha$  as a coordinate on the level curve corresponding to a fixed value of  $\lambda$ , and  $p$  as a function of  $\alpha$  (this function depends on  $\lambda$  as a parameter).

The envelope of a 1-parameter family of lines given by a function  $p(\alpha)$  is the curve

$$(x(\alpha), y(\alpha)) = p(\alpha)(\cos \alpha, \sin \alpha) + p'(\alpha)(-\sin \alpha, \cos \alpha),$$

see [24]. In our case, we obtain the curve

$$(x(\alpha), y(\alpha)) = (1 - \lambda \sin 2\alpha)^{-1/2}(\cos \alpha - \lambda \sin \alpha, \sin \alpha - \lambda \cos \alpha).$$

It is straightforward to check that this curve satisfies Eq. (11).

It is also clear that the conics (11) are tangent to the lines  $x = \pm 1$  and  $y = \pm 1$ . Indeed, if, for example,  $y = 1$  then the left-hand side of (11) becomes  $(x + \lambda)^2 + 1 - \lambda^2$ , and Eq. (11) has a multiple root  $x_{1,2} = -\lambda$ .  $\square$

**Remark 5.2.** Yet another proof of the integrability of the Lorentz billiard inside a circle can be deduced from the duality (the skew hodograph transformation) between Minkowski billiards discovered in [11]. This duality trades the shape of the billiard table for that of the unit (co)sphere of the metric. In our case, the billiard curve is a circle and the unit sphere of the metric is a hyperbola; the dual system is the usual, Euclidean billiard “inside” a hyperbola.

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