BIOGRAPHICAL MEMOIRS

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Vladimir Arnold was a pre-eminent mathematician of the second half of the twentieth and early twenty-first century. Kolmogorov–Arnold–Moser (KAM) theory, Arnold diffusion, Arnold tongues in bifurcation theory, Liouville–Arnold theorem in completely integrable systems, Arnold conjectures in symplectic topology—this is a very incomplete list of notions and results named after him. Arnold was a charismatic leader of a mathematical school, a prolific writer, a flamboyant speaker and a tremendously erudite person. Our biographical sketch describes his extraordinary personality and his major contributions to mathematics.

THE PERSONALITY

Family background

Vladimir Arnold was born on 12 June 1937 in Odessa (now Ukraine, then the Soviet Union) and grew up in Moscow. He had two younger siblings: a brother Dmitry, a physicist, and a sister Katya, an artist.

His father, Igor Vladimirovich Arnold, was a mathematician specializing in algebra. (In particular, he learned modern algebra from Emmy Noether during her stay in Moscow in 1928–29). He was among the first corresponding members of the Soviet Academy of Pedagogical Sciences, recognized for his work in mathematical education. In the preface to (Arnold 1946), Vladimir Arnold recalled:

I turned 11 in the year when he died, but I have not learned anything mathematical from him: instead, he taught me mountaineering and love for long journeys, wood carving and building huts of branches, fishing and skiing.

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Biographical Memoirs

Vladimir's mother, Nina Alexandrovna Arnold, née Isakovich, was an art historian working in a Moscow museum.

The Arnolds were related to several people who made outstanding contributions to Russian science and culture. One of Vladimir's great-uncles was Leonid Mandelstam (1879–1944), the founder of a major school of theoretical physics in the Soviet Union and co-discoverer, jointly with Grigory Landsberg and, independently, with Chandrasekhara Raman and Kariamanickam Krishnan, of the effect of the combinatorial scattering of light. Another of Vladimir's great-uncles was writer and traveller Boris Zhitkov (1882–1938), whose main book, a novel *Viktor Vavich* about the first Russian revolution, was banned in the Soviet Union and became available only in 1999.

Early years

Vladimir was an avid student of mathematics and natural sciences from an early age. His first mathematical book, at the age of 12, was the Russian translation of *Von Zahlen und Figuren* by Hans Rademacher and Otto Toeplitz. He read it slowly, a few pages a day (Khesin & Tabachnikov 2014). A year later, his uncle, the engineer Nikolai Zhitkov, introduced him to mathematical analysis. After that, Vladimir started to devour mathematical books from the library of his father.

The home-based Children Learned Society,¹ organized by a prominent mathematician and computer scientist, Alexey Andreevich Lyapunov, played an important role in Vladimir Arnold's early education. The curriculum included mathematics, natural sciences (physics, chemistry, biology) and, in particular, genetics, that was proclaimed 'bourgeois pseudo-science' in the Soviet Union. To quote from the memoir of Natalia Lyapunova, Alexey Lyapunov's daughter (Lyapunova *et al.* 2011):

One cannot forget the talk 'Waves' by $Dima^2$ Arnold. We had a huge dinner table, extendable to six sections. The table was unfolded, an aquarium with water was put into the hole, and a slide projector was put underneath... The light went through the water whose surface projected on the ceiling. Two corks were floating in the aquarium; one needed to give them a push, and the waves started: circular, counter, interference! Dima is lecturing, and visual demonstrations follow... I was then in the fourth grade.

Many a participant of the Children Learned Society later became an accomplished scientist, including Sergei Novikov, who was awarded the Fields Medal in 1970.

Vladimir Arnold attended one of the best Moscow schools (No 59). He frequented mathematical circles, widespread at the time, and participated in the Moscow mathematical olympiad. He was awarded the second prize in his junior and senior years.

Arnold's spirit of exploration was present from an early age. When Vladimir was seven, he took his four-year-old brother for a cartography expedition along the 'Garden Ring', a circular highway of about 16 kilometres around the central part of Moscow. As a result, he discovered substantial differences with the published maps, which, for fear of spies, were routinely distorted.

¹ Known by its Russian acronym DNO (detskoe nauchnoe obshestvo), which is also a word play: 'dno' means bottom.

² An unusual diminutive of Vladimir; he was 10 years old. Arnold was known under this name to many mathematicians all his life.



Figure 1. Left: on a hike, 1960s. Right: Yakov Sinai and Vladimir Arnold in front of the main building of the Moscow State University, 1963.

Arnold's seminar

Vladimir Arnold belonged to an exceptional cohort of Soviet mathematicians who were born between 1935–38: Dmitri Anosov, Alexandre Kirillov, Yuri Manin, Sergei Novikov, Yakov Sinai (see figure 1). According to Arnold,³ the explanation of this phenomenon was that the preceding generation of Soviet mathematicians largely perished in World War II. As a result, his generation had more freedom to develop independently, growing like trees on a cleared land, rather than in the shade of taller trees in a dense forest.

Moscow State University was one of the leading mathematical centres in the world (Zdravkovska 1987), and mathematical life revolved around seminars. The charismatic personalities of their organizers and mathematics of the highest quality made these seminars strong attractors for students and for established researchers alike. The seminars were weekly two-hour events.⁴ This meant that a typical talk included proofs (unlike the one-hour talks common in the West).

Arnold's seminar was one of the most influential seminars in Moscow since the mid 1960s, and it continued until his death in 2010. Several participants left detailed memoirs of this unforgettable mathematical event (Khesin & Tabachnikov 2014).

Giving a talk in Arnold's seminar was a serious challenge (both authors of this article had this experience more than once). Sitting at the front desk, Arnold was always an active listener, regularly asking pointed and sometimes forcefully posed questions. He could say: 'You are writing some formulas on the blackboard that are impossible to penetrate. Erase everything, and explain from the very beginning, so that we would be finally able to understand.'

Arnold's contribution went way beyond mere criticism. He could interrupt the speaker and outline the origin of the problem under discussion, its connections with other results and theories—a typical mistake of an inexperienced speaker is to assume that the audience is familiar with the context—and fantasize how he would prove and generalize the result which, often exceeding the expectations of the speaker, suddenly started to shine. Accompanied by Arnold's characteristic chuckle, this left no hard feelings and the speakers were usually grateful to Arnold for explaining to them their own results.

³ In a conversation with the second author of this article.

⁴ With a notable exception of Gelfand's seminar, that could last twice as long.

Traditionally, the first meeting of every semester was Arnold's talk on open problems. These problems, collected over the years by the participants, were published in a unique book (18)^{*}. The book consists of two parts: the formulations of the problems (from 1956 to 2003, total number 861) and comments on the status of the problems written by various authors, most of them the alumni of the seminar. The problems cover a vast variety of topics across mathematics. Most of them are formulated not in the most general form, but rather as the first non-trivial case of a general phenomenon (Arnold referred to this as the 'Russian style' of posing problems). A typical formulation of a problem is a paragraph long (see also Tabachnikov 2007).

Mathematical philosophy

Arnold viewed mathematics as a part of natural philosophy, in the spirit of Isaac Newton:

Mathematics is a part of physics. Physics is an experimental science, a part of natural sciences. Mathematics is the part of physics where experiments are cheap (15).

He believed that the difference between a pure and an applied mathematician is mostly social—one is paid for making mathematical discoveries, another for solving given problems (Lui 1997)—and he liked to quote Louis Pasteur that there never have been and never will be any 'applied sciences', there are only applications of sciences. In (16), he quipped:

All mathematics is divided into three parts: cryptography (paid for by CIA, KGB and the like), hydrodynamics (supported by manufacturers of atomic submarines) and celestial mechanics (financed by military and other institutions dealing with missiles, such as NASA).

Arnold maintained that mistakes were as important and instructive a part of mathematics as proofs. For example, he said, 'The story of the Poincaré conjecture started from his confusion of homotopy with homology' (16).

In his own research, he was guided by several general principles. One of them was the principle of general position: most interesting are generic, typical phenomena, while degenerations must be considered along with their deformations. Another was a quest for geometric and, more specifically, symplectic and contact origins of facts. Yet another principle concerns the 'mysterious mathematical triads', the intuitive operations of complexification and quaternization (16). Examples abound: quadratic forms—Hermitian forms—Hyperhermitian forms; the Hopf bundles $S^0 \rightarrow S^1 \rightarrow S^1$, $S^1 \rightarrow S^3 \rightarrow S^2$, $S^3 \rightarrow S^7 \rightarrow S^4$; Stiefel–Whitney classes—Chern classes—Pontryagin classes, and so on. Here is but one example of how this philosophy led to a discovery.

Analysing Gudkov's work on Hilbert's sixteenth problem on the mutual position of the ovals of real plane algebraic curves, Arnold was led to the problem: how to complexify the notion of manifold with boundary? His answer was that the complexification of the real surface with boundary f = 0 is the two-fold covering of the complement, in the complex projective plane, of the Riemann surface f = 0, ramified along this Riemann surface. This made it possible for Arnold, and then for Vladimir Rokhlin, to apply powerful methods of 4-dimensional topology and to prove a conjecture of Gudkov, correcting Hilbert's expectations in his sixteenth problem.

^{*} Numbers in this form refer to the bibliography at the end of the text.



Figure 2. Left: at Moscow State University, ca 1985. Right: at a lecture, 2000s.

Pedagogy

Arnold was a truly engaging lecturer (figure 2) and the author of several mathematical textbooks, including the classics *Ordinary Differential Equations* and *Mathematical Methods of Classical Mechanics* that changed the way those subjects are taught (7,9). In later years, Arnold wrote several small books for general mathematical audiences. Some are now available in English (20). Arnold was a prolific writer, proud of being able to produce 30 pages a day (in English; his output in his native Russian was even higher). For example, the small book (12) was written in two days!

Arnold succinctly expressed his teaching philosophy:

The aim of a mathematical lecture should be not the logical derivation of some incomprehensible assertions from others (equally incomprehensible): it is necessary to explain to the audience what the discussion is about and to teach them to use not only the results presented, but—and this is major—the methods and the ideas (13).

Arnold emphasized that he learned a lot from his students, and that he never assigned thesis topics to them ('This is like assigning them a spouse. I merely show them what is known and unknown': Lyapunova *et al.* 2011).

Arnold was heavily involved in various educational projects in Russia. One was the Independent University of Moscow (IUM), created in 1991 as a truly independent counterpart to the Department of Mathematics of Moscow State University (Mekhmat) whose scientific reputation was declining and whose shameless discrimination in the entrance exams⁵ had compromised its integrity. Arnold served as Chairman of the Board of Trustees of IUM.

⁵ The reader interested in this chapter of Soviet mathematical history may consult Shifman (2005).

Another major educational project is the summer school, Modern Mathematics, for students in their last years of high school and first years of university. Started in 2001, this two-week event takes place every year at Dubna (home of the international Joint Institute for Nuclear Research), near Moscow. Arnold was one of the founders and a permanent lecturer at this school.

Some of the mathematical events of the distant past were as lively to Arnold as if they were happening now. In particular, this concerns the second half of the seventeenth century, the time of Huygens, Hooke and Newton. Arnold read Newton's *Principia* as if it were written by his contemporary and a kindred spirit. For him, the book was full of surprises and fresh ideas.

For example, Arnold connected Proposition VII, Problem II, in *Principia* ('If a body revolves in the circumference of a circle; it is proposed to find the law of centripetal force directed to any given point') with the Bohlin theorem about the duality of laws of central attraction whose strength is proportional to a power of distance to the centre. One of the two self-dual laws occurs when the force is inversely proportional to the fifth power of the distance from the centre, the case studied by Newton. The discussion of this topic in (11) reads like a detective story!

Arnold used to say (exaggerating, as usual) that much of what he knew in mathematics he learned from Felix Klein's book *Development of Mathematics in the 19th Century*.

Arnold, a polemicist

Arnold was known for his 'in your face, take no prisoners' polemical style.

One of his permanent targets was the formal style of presenting and teaching mathematics that he called 'Bourbakism' (sometimes with the adjective 'criminal'). The following quotation from (17) is characteristic:

Unfortunately, the simple texts of Poincaré are difficult for mathematicians raised upon set theory. Poincaré would have said 'Pete washed his hands,' where a contemporary mathematician would simply write instead 'There exists a $t_1 < 0$ such that the image of the point t_1 under the natural mapping $t \mapsto \text{Pete}(t)$ belongs to the set of people with dirty hands, and a $t_2 \in (t_1, 0]$ such that $\text{Pete}(t_2)$ belongs to the complement of the set mentioned above.'

Arnold valiantly fought against the decline of the standards of mathematical education, be this in Russia or in the West. In an unpublished draft of a preface to (Fuchs & Tabachnikov 2007), he remarked that the book might be too difficult for the readers, including most modern students in the USA and France, who do not know that $1/2 + 1/3 \neq 2/5$.

His view of the future of mathematics and, in general, of our culture, was pessimistic: he warned that the united bureaucrats of all countries may be able to stop all kinds of creative activity.

Soviet mathematicians were isolated from their colleagues in the West and, until the Iron Curtain fell, personal contacts were severely limited. (The main Russian mathematical journals were translated into English, but the quality of translation was sometimes substandard.) Arnold felt very strongly that the contributions of Soviet mathematicians were not properly acknowledged and their results were not properly attributed.

This fits well with what Sir Michael Berry calls the first (out of three) law of discovery (Berry 1997):

Arnold's law (implied by statements in his many letters disputing priority, usually in response to what he sees as neglect of Russian mathematicians): 'Discoveries are rarely attributed to the correct person.' (Of course Arnold's law is self-referential.)

Arnold was sceptical about fads and fashions in mathematics (although many times he himself was a trendsetter):

Development of mathematics resembles a fast revolution of a wheel: sprinkles of water are flying in all directions. Fashion – it is the stream that leaves the main trajectory in the tangential direction. These streams of epigone works attract most attention, and they constitute the main mass, but they inevitably disappear after a while because they parted with the wheel. To remain on the wheel, one must apply the effort in the direction perpendicular to the main stream. (Khesin & Tabachnikov 2014)

Pushing the envelope

From an early age onwards, Arnold enjoyed outdoor sports and was an intrepid adventurer. For example, while visiting MSRI in Berkeley in 1988, he tried to swim across the Golden Gate strait during ebb tide. Fortunately, he realized the futility of this attempt and turned back; otherwise he would have been swept at least a mile out to sea.

In an interview, reprinted in Khesin & Tabachnikov (2014), he describes his way of overcoming mathematical difficulties:

When a problem resists a solution, I jump on my cross country skis. Forty kilometers later a solution (or at least an idea for a solution) always comes. Under scrutiny, an error is often found. But this is a new difficulty that is overcome in the same way.

A weekend cross country ski trip was a tradition of Arnold's seminar. The distance was about 50 km; Arnold, dressed only in swimming trunks, led the group. The exhausted participants who could not keep up with Arnold would leave by bus at occasional crossroads (15–20 km apart). This ski trip included bathing in small rivers and streams that did not freeze in winter.

One of Arnold's favourite summer sports was bicycling (see figure 3). In 1999, he had a serious bicycle accident in Paris, resulting in traumatic brain injury. He wrote the book (19) during his slow recovery in a hospital.

Awards

Arnold was a recipient of many awards, among them the Lenin Prize (1965), the Crafoord Prize (1982), the Lobachevsky Prize of the Russian Academy of Sciences (1992), the Harvey Prize (1994), the Dannie Heineman Prize for Mathematical Physics (2001), the Wolf Prize (2001), the State Prize of the Russian Federation (2007) and the Shaw Prize in Mathematical Sciences (2008). Asteroid Vladarnolda, discovered in 1981 and registered under No. 10031, is named after him.

One notable omission in this list is the Fields Medal. Arnold was nominated in 1974, but the award was blocked by Lev Pontryagin, who served as the Soviet representative in the International Mathematical Union (IMU). A probable reason for Arnold's falling into disfavour with the Soviet authorities was that, in 1968, he signed a letter, along with 98 other mathematicians, in defence of Alexander Esenin-Volpin, a mathematician and dissident who was forcibly confined to a psychiatric hospital (see Zdravkovska & Duren 1993).



Figure 3. Arnold with his wife Elya, discussing his ski and bike trips.

A Renaissance man

Arnold had very broad interests and was a captivating storyteller. His friends remember his 'universal knowledge of everything' (Dmitry Fuchs: Khesin & Tabachnikov 2014). He knew his two favorite cities, Moscow and Paris, very well. He had grown up in Moscow and spent most of his life there. He had been allowed to visit Paris in the mid 1960s and, since 1993, he spent one semester a year there as a professor of Université Paris–Dauphine. The sight-seeing tours of these cities that he gave to his friends, colleagues and students far exceeded what one could expect from a professional guide.

Perhaps the most important work of Russian literature is Pushkin's novel in verse *Eugene Onegin*. The novel has an epigraph, in French, attributed by Pushkin to a private letter:

Pétri de vanitéil avait encore plus de cette espèce d'orgueil qui fait avouer avec la même indifférence les bonnes comme les mauvaises actions, suite d'un sentiment de supériorité, peut-être imaginaire.⁶

It is believed that this 'private letter' is Pushkin's hoax, and he wrote the epigraph himself. Arnold published a short article (14) where he argued that the source of the epigraph, with all adjectives inverted, was *Dangerous Liaisons* by Choderlos de Laclos, an observation that was new to the experts. In a somewhat tongue-in-cheek manner, and in the style that, to an extent, imitates Pushkin, Arnold writes:

Not being a literary scholar by profession (and even less a Pushkinist), but a mathematician, in my work I must constantly depend not on proofs, but on sensations, guesses and hypotheses, moving from one fact to another by means of the kind of insight that lets one see commonalities in things that an observer may think completely unrelated.

⁶ In A. S. Kline's translation: 'Formed by vanity, he possessed still more of that species of pride that leads one to confess to good and evil actions with a like indifference, due to a sense of superiority which is perhaps merely imagined.'



Figure 4. Arnold's tombstone at the Novodevichy Cemetery in Moscow.

These words, along with mathematical formulas representing his discoveries, are engraved on Arnold's tombstone (figure 4).

THE MATHEMATICIAN

Hilbert's thirteenth problem

In 1900 David Hilbert presented the mathematical community with 23 problems for the new century. It is one of the highest honours for a mathematician to contribute to a solution of Hilbert's problems. Arnold did this more than once.

Being just a third year undergraduate student of Moscow State University, Arnold became famous in the mathematical world by completing the solution (started by Kolmogorov) of Hilbert's thirteenth problem.

A general equation of degree 7 can be reduced to the form $x^7 + x^3 + ax^2 + bx + c = 0$ and its roots can be regarded as functions of the coefficients, x = x(a, b, c).

Hilbert's thirteenth problem: Show that the function x(a, b, c), defined by the equation

$$x^7 + x^3 + ax^2 + bx + c = 0,$$

cannot be represented by superpositions of continuous functions in two variables.

Andrei Kolmogorov and Vladimir Arnold proved that in fact such a representation does exist (1), thus solving the problem negatively. Namely, their theorem states that every continuous function of n > 2 variables can be represented by a superposition of continuous functions of two variables (and moreover, every continuous function of two variables can be expressed via sums and compositions of functions of one variable only).

The first approach to this problem (based on the notion of the Kronrod tree of connected components of level sets) was found by Kolmogorov in 1956, who proved the reductions of many to three variables, as well as two to one. The final step of overcoming low-dimensional difficulties and reducing three-variable functions to those of two variables was made by Arnold, who was then 19 years of age.

Apparently, the solution for continuous functions (the most direct understanding of Hilbert's question) was the beginning of Arnold's interest in the 'genuine' (and still open) Hilbert problem on whether the root of the general degree 7 polynomial considered as *an algebraic function* of its coefficients can be written as a superposition of *algebraic* functions of two variables. In 1970 Arnold proved the impossibility of such a representation for polynomials of degree $n = 2^r$.

Arnold's work on and around Hilbert's thirteenth problem was very influential and led to numerous new developments. To mention a few: the computation of the cohomology of braid group by Dmitry Fuchs; the extension, by Egbert Brieskorn, of Arnold's approach to braid groups via topology of configuration spaces to braid groups associated with reflection groups; the Orlik–Solomon theory of hyperplane arrangements; Smale's concept of topological complexity of algorithms and Victor Vassiliev's results on this subject, etc.

KAM theory

One of the most important chapters of modern mathematics associated with Arnold is the Kolmogorov–Arnold–Moser theory, the theory of quasiperiodic motions in nonintegrable dynamical systems. It is widely regarded as one of the major discoveries of twentieth-century mathematics and mathematical physics.

For a long time it was believed that a typical dynamical system exhibits chaotic behaviour and does not admit any invariant submanifolds, but, in 1954, Kolmogorov made an astonishing discovery.

The phase space of a completely integrable Hamiltonian system with *n* degrees of freedom carries action-angle coordinates (I, φ) . The phase space is foliated by invariant *n*-tori, the level surfaces of the action variables, and they carry quasiperiodic motions $\dot{\varphi} = \omega(I)$.

Kolmogorov showed that if the frequencies do change over the set of tori, i.e. if $\det(\partial \omega/\partial I) \neq 0$, then most of these tori are not destroyed by a small Hamiltonian perturbation, but are only slightly deformed in the phase space. To be more precise, a torus $\{I = I^0\}$ persists under a perturbation whenever the frequencies $\omega_1(I^0), \ldots, \omega_n(I^0)$ are Diophantine (strongly incommensurable). In other words, nonintegrable systems in the vicinity of an integrable one still exhibit a rather regular behaviour of trajectories on a large set, retaining many of their invariant tori and same-frequency quasiperiodic motion on them.



Figure 5. Left: with former students, Yuri Chekanov, Vladimir Zakalyukin and Askold Khovanskii, Fields Institute, Toronto, 1997. Right: Vladimir Arnold and Jürgen Moser, Euler Institute, St Petersburg, 1991.

This was contrary to the general opinion of the physical community of that time. To prove this fundamental theorem, Kolmogorov proposed a new powerful method of constructing an infinite sequence of canonical coordinate transformations with fast convergence, a variation on Newton's method of finding roots of equations.

In the early 1960s, Arnold published a series of papers on 'small denominators' in which he generalized Kolmogorov's theorem to various systems with degeneracies. His 1961 paper (2) also contained the first detailed exposition of Kolmogorov's method (Kolmogorov did not publish detailed proof of his result). These studies culminated in Arnold's celebrated result (3) on stability in planetary-like systems of celestial mechanics.

While Kolmogorov and Arnold dealt with analytic Hamiltonian systems, Jürgen Moser, trying to reconstruct Kolmogorov's proof from his address at the 1954 International Congress of Mathematicians, succeeded in proving a similar theorem in the finitely smooth case by ingenious fusion of regularity theory of elliptic PDEs and Nash's smoothing method (see figure 5). The acronym 'KAM' was coined by physicists Izrailev and Chirikov in 1968.

In the case of two degrees of freedom, the invariant two-dimensional tori divide a threedimensional energy level, and the solution is trapped between tori. Arnold discovered the universal mechanism of instability of the action variables in nearly integrable Hamiltonian systems with more than two degrees of freedom and constructed an explicit example where such instability occurs (4). This chaotic evolution along resonances between the Kolmogorov tori is called Arnold diffusion. The study of Arnold diffusion remains an active field of contemporary research (e.g. John Mather, Vadim Kaloshin and others).

Topological fluid dynamics

Arnold's interest in hydrodynamics is rooted in Kolmogorov's study of turbulence, and started with the programme outlined by Kolmogorov for his seminar in 1958–59. Kolmogorov conjectured an increase of randomness in dynamical systems related to hydrodynamical partial differential equations as viscosity of the fluid tends to zero, which would imply the practical impossibility of long-term weather forecasts. Arnold's take on hydrodynamics was, however, completely different from Kolmogorov's and involved Lie groups and topology.

Arnold's 1966 paper on the hydrodynamical Euler equation as a geodesic flow (6) had the effect of a bombshell. In the next several years Arnold laid out the foundations for the study of hydrodynamical stability, the use of Hamiltonian methods and the study of the topology of steady flows.

The Euler equation of an ideal incompressible fluid filling a domain is the evolution equation

$$\partial_t v + (v, \nabla v) = -\nabla p$$

on the fluid velocity field v; this field is assumed to be divergence-free and tangent to the domain boundary (and the pressure p is defined uniquely, up to an additive constant, by these conditions on v). Arnold showed that this Euler equation can be regarded as the equation of the geodesic flow on the group of volume-preserving transformations of the domain. The corresponding metric on this infinite-dimensional group is the right-invariant metric defined by the kinetic energy of the fluid.

Arnold proved that the sectional curvature of this metric is negative in most directions. The geodesics on negatively curved manifolds diverge exponentially fast. Applied to the group of volume-preserving transformations, this implies instability of the corresponding fluid flows.

In particular, this gives a qualitative explanation of unreliability of long-term weather forecasts (thus solving, in a way, the problem posed by Kolmogorov in the 1950s). Arnold estimated that a prediction of the weather in two months requires knowledge of the state of the Earth's atmosphere with five more orders of accuracy, compared with the forecast. (Curiously, an assumption in this work was that the Earth had the shape of a torus. Later calculations showed that this 'geological' assumption was purely technical and that the same estimates work for a sphere as well.)

Arnold's geometric view on hydrodynamics opened a multitude of research directions. Many other evolution equations were found to describe geodesic flows on appropriate Lie groups with respect to one-sided invariant metrics. These included the Euler top and its higherdimensional analogues, the Kirchhoff equations for rigid body dynamics in a fluid, the inviscid Burgers equation, the Korteweg–de Vries, Camassa–Holm and Hunter–Saxton equations, and the self-consistent magnetohydrodynamics equations describing plasma motion.

Teasing the physicists, Arnold used to say that their gauge groups are too simple to serve as a model for hydrodynamics.

Two more gems of topological hydrodynamics are Arnold's description of topology of steady fluid flows and the development of the theory of an asymptotic Hopf invariant, generalizing the work of Keith Moffatt and others. Arnold's stability criterion in hydrodynamics became yet another cornerstone of the subject. Note that Arnold's interest in magnetohydrodynamics was, to a large extent, stimulated by his acquaintance with the prominent physicists Yakov Zeldovich and Andrei Sakharov. One of the results of their interaction was a model of the fast dynamo on a three-dimensional Riemannian manifold, derived from Arnold's cat map on a torus, see figure 6.

Singularity theory

For many mathematicians, physicists, and engineers, singularity theory is closely associated with Arnold's name. Yulij Ilyashenko recalls (Khesin & Tabachnikov 2014):



Figure 6. Arnold's cat map.

In 1965 Arnold came back from France where he spent almost a year. From there he brought a keen interest in the newborn singularity theory, of which he became one of the founding fathers. He also brought the philosophy of general position invented by René Thom, which became sort of a compass in Arnold's investigations in differential equations and bifurcation theory. In the form that Arnold gave to it, this philosophy claimed that one should first investigate objects in general position, then the simplest degenerations, together with their unfoldings. It makes no sense to study degenerations of higher codimension until those of smaller codimension have been investigated.

Singularities and critical points of functions became one of the main topics of Arnold's seminar. Arnold classified simple singularities of critical points in 1972, unimodal ones in 1973 and bimodal ones two years later.

Simple singularities form two series A_n , D_n , and three exceptional cases E_6 , E_7 , E_8 , and they are closely related with the simple Lie algebra of the like-named types at several levels.⁷ (For example, the Dynkin diagram of the intersection form on vanishing cohomology at a simple singularity of an odd number of variables coincides with the Dynkin diagram of the corresponding Lie algebra, the monodromy group of the simple singularity is related to the Coxeter number of the corresponding Lie algebra.)

Soon thereafter powerful methods were developed by participants of Arnold's seminar and by others for the study of various characteristics of critical points, such as monodromy groups and intersection forms, asymptotics of oscillatory integrals, the mixed Hodge structure on vanishing cohomology and computation of the corresponding Hodge numbers in terms of Newton polygons. Numerous works of Arnold concerned singularities of caustics and wave

⁷ The ADE classification is ubiquitous in mathematics. For example, it arises in classification of the quivers of finite type and it made a recent appearance in the theory of cluster algebras of Fomin–Zelevinsky.



Figure 7. An ellipse, its wave fronts (equidistant curves) and their singularity locus, the caustic (or evolute) of the ellipse.

fronts, Lagrangian and Legendrian singularities, singularities of functions on manifolds with boundary, equivariant singularities, etc. (10); see figure 7.

The so-called 'Strange Duality' is Arnold's observation on the relation between triples of numbers, computed by Igor Dolgachev and Andrei Gabrielov and characterizing, respectively, uniformization and monodromy of 14 exceptional unimodal singularities of surfaces. We now recognize this as the first manifestation of Mirror Symmetry, a profound conjecture, discovered by string theorists, and an equivalence between symplectic topology and complex geometry, an active area of contemporary research in mathematics and mathematical physics.

Let us also mention Vassiliev's theory of knot invariants of finite type that revolutionized knot theory in the early 1990s: he derived knot invariants from the study of the discriminant variety in the space of curves immersed in three-dimensional space, in the spirit of Arnold's approach to singularity theory (see Vassiliev 1992).

Bifurcation theory

Arnold often made fun of papers titled something like 'On some property of one solution of a certain differential equation'. As a matter of fact, in bifurcation theory, Arnold always applied the same philosophy of general position that we mentioned earlier, and this approach to the local bifurcation theory had revolutionized the field.

Let us mention but one result that is now considered classic: Arnold's description of normal forms of families of matrices depending on parameters, which is now called the Jordan–Arnold canonical form.

One spectacular application of the bifurcation philosophy was Arnold's reformulation of the sixteenth Hilbert problem. The second part of this problem is concerned with the maximal number of limit cycles that a plane polynomial vector field of a given degree can have.

Arnold proposed an infinitesimal version of this problem, now called the Hilbert-Arnold problem: what is the maximal number of limit cycles, born under a non-conservative

perturbation of a Hamiltonian system, whose Hamiltonian is a polynomial of a given degree? Arnold's reformulation of this problem as an estimate of the number of zeros of Abelian integrals over a family of real algebraic ovals, and his conjecture about non-oscillatory behaviour of the latter, drew new interest to the notoriously difficult second part of the sixteenth Hilbert problem and led to deep and difficult results by a number of mathematicians, mostly from Arnold's school.

Real algebraic geometry

By the time Arnold got interested in real algebraic geometry, it was widely considered to be a theory of the past. Largely due to Arnold's efforts, the situation has changed dramatically.

We already mentioned Arnold's results on the possible mutual positions of ovals of algebraic curves—the first part of Hilbert's sixteenth problem—where four-dimensional topology played an unexpected and critical role.

Another work of Arnold (8) unified the Petrovsky–Oleinik inequalities, providing estimates for the Euler characteristics of real algebraic sets and bringing mixed Hodge structures into real algebraic geometry. He found novel proofs and unexpected generalizations of these inequalities (which later were proved to be exact) based on singularity theory.

Still another contribution of Arnold to the revival of real algebraic geometry was his elegant generalization of the famous Newtonian 'no gravity in the cavity' result (Proposition 70, Theorem 30, in *Principia*). This theorem states that a homogeneous sphere exerts zero gravitational force at every interior point. Arnold extended this result, along with the Laplace–Ivory theorem on the gravitational attraction of ellipsoids, to the level surfaces of hyperbolic polynomials of degree higher than two. This work was continued by Arnold's students, including Alexander Givental and Victor Vassiliev (see Vassiliev 1995).

Symplectic geometry

In the last third of the twentieth century, symplectic geometry became a universal language of a number of disciplines, including Hamiltonian mechanics, quantization, microlocal analysis of differential equations and calculus of variations. Citing Givental (Khesin & Tabachnikov 2014),

One of the first mathematicians who understood the unifying nature of symplectic geometry was Vladimir Arnold, and his work played a key role in establishing this status of symplectic geometry. In particular, his monograph *Mathematical Methods of Classical Mechanics* has become a standard textbook, but 30 years ago it indicated a paradigm shift in a favorite subject of physicists and engineers.

Arnold's important achievements in this domain include Lagrangian Sturm theory, introduction of the Maslov class, initiation of the theory of Lagrange and Legendre cobordisms, the study of Lagrangian and Legendrian singularities, etc.

Let us comment on the classical result known as the Liouville–Arnold theorem. Liouville's theorem states that a Hamiltonian system with n degrees of freedom having n independent integrals in involution can be integrated in quadratures. Arnold's symplectic geometric point of view substantially clarified the picture: the common level surfaces of the integrals form a Lagrangian foliation of the phase space whose leaves carry n commuting vector fields, and hence they are quotients of \mathbb{R}^n . In particular, they are tori if they are compact.

Symplectic and contact topology

The first result of what later became symplectic topology is the Poincaré last theorem, published by him as conjecture shortly before his untimely death and later proved by Birkhoff. This result claims that an area-preserving diffeomorphism of an annulus, rotating two boundary circles in the opposite directions, has at least two fixed points. By gluing two annuli, one can obtain a similar statement for a torus: a generic area-preserving diffeomorphism of a two-dimensional torus, 'preserving its center of mass', has at least four fixed points.

It was Arnold who recognized the Hamiltonian nature of the statement and formulated it in remarkable generality in (5) as what is now known as *Arnold's Conjecture*: a non-degenerate Hamiltonian diffeomorphism of a symplectic manifold has at least as many fixed points as a Morse function has critical points on that manifold.

Several perturbative versions of this statement were proved by Arnold and Weinstein for Hamiltonian diffeomorphisms sufficiently close to the identity. In Arnold's textbook on *Classical Mechanics* (9) one finds the following formulation:

Theorem: Every symplectic diffeomorphism of a compact symplectic manifold, homologous to the identity, has at least as many fixed points as a smooth function on this manifold has critical points (at least if this diffeomorphism is not too far from the identity).

As one of the main players in the field, Helmut Hofer, puts it (Khesin & Tabachnikov 2014):

The symplectic community is now trying since 1965 to remove the bracketed part of the statement. After tough times from 1965 to 1982 an enormously fruitful period started with the Conley–Zehnder theorem in 1982/83, proving the Arnold conjecture for the standard torus in any (even) dimension using Conley's index theory (a powerful version of variational methods). This was followed by Gromov's pseudoholomorphic curve theory coming from a quite different direction. At this point the highly flexible symplectic language becomes a real asset in the field. Finally Floer combines the Conley–Zehnder view point with that of Gromov, which is the starting point of Floer theory in 1987. As far as the Arnold conjecture is concerned, we understand so far a homological version of the non-degenerate case. A Lusternik–Schnirelmann case (also conjectured by Arnold) is still wide open, though some partial results are known.

Thus it will not be an exaggeration to say that the rich and fast-growing field of symplectic and contact topology has roots in Arnold's conjectures about Hamiltonian fixed points and Lagrangian intersections.

It is not possible to cover all the areas where Arnold's contributions were pivotal: they also include Newton polyhedra, finite type topological invariants of curves and wave fronts, combinatorics and number theory related to statistics of the Euler function, large scale structure of the universe, and many others.

A new *Arnold Mathematical Journal*, published by Springer since 2015, 'presents interdisciplinary results in mathematics in a style that is understandable and always interesting'. It strives to 'maintain and promote the scientific style characteristic of Arnold's best mathematical work'.

One cannot help but ending this biographical sketch by citing Sir Michael Berry again (Khesin & Tabachnikov 2014):

It suddenly occurs to me that in at least four respects Arnold was the mathematical counterpart of Richard Feynman. Like Feynman, Arnold made massive original contributions in his field, with

enormous influence outside it; he was a master expositor, an inspiring teacher bringing new ideas to new and wide audiences; he was uncompromisingly direct and utterly honest; and he was a colorful character, bubbling with mischief, endlessly surprising.

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Boris Khesin studied mathematics at the Moscow State University, Russia. After obtaining his PhD in 1990 under the guidance of Vladimir Arnold, he spent several years at UC Berkeley and Yale University, USA, before moving to Toronto, Canada. Currently he is a Professor of Mathematics at the University of Toronto. His research interests include infinite-dimensional groups and Hamiltonian dynamics. Arnold and Khesin authored the book *Topological Methods in Hydrodynamics*, which appears to be accepted as one of the main references in the field. Together with the co-author of this memoir, Sergei Tabachnikov, Khesin edited the book *ARNOLD: Swimming Against the Tide* (2014), devoted to the memory of his teacher.

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Sergei Tabachnikov obtained his PhD from Moscow State University in 1987; his co-advisors were Dmitry Fuchs and Anatoly Fomenko. He is a Professor of Mathematics at Pennsylvania State University, and he served as Deputy Director of ICERM (Institute for Computational and Experimental Research in Mathematics) at Brown University. His areas of research include geometry, topology and dynamical systems. Tabachnikov (co)-authored a number of books, in particular, *Mathematical Omnibus* (with Dmitry Fuchs, 2007). He is Editor-in-Chief of *Experimental Mathematics*, and Associate Editor of *Mathematical Intelligencer* and *The American Mathematical Monthly*.

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