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A HIERARCHY OF CENTRALLY EXTENDED ALGEBRAS AND THE LOGARITHM OF THE DERIVATIVE OPERATOR

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The algebra of differential operators on the circle is one of the main objects in conformal field theory and the theory of integrable systems [B], [PRS]. This algebra contains the subalgebra of vector fields on the circle and has a unique central extension as does its subalgebra [G]. Following [KK], we describe in Section 1 an explicit construction of this extension and a new concept of the logarithm of the derivative operator log D. This concept turns out to be very useful for defining the corresponding 2-cocycle; it also gives a central extension of a larger algebra of pseudodifferential operators on the circle. Moreover, the analogous extended algebra exists for differential operators with coefficients in any reductive algebra $\mathscr{G}$. In particular, the Kac-Moody and the Virasoro algebras are the simplest restrictions of the described cocycle on the zeroth and the first-order differential operators. (See also [KP].) We define here a whole hierarchy of subalgebras admitting nontrivial central extensions. (See Section 2.)

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1. The logarithm of the derivative operator. The ring $\mathscr{R}$ of pseudodifferential operators is the ring of formal series $A(x, D) = \sum_{-\infty}^{n} a_i(x)D^i$ with respect to D, where $a_i(x) \in C^\infty(S^1, k)$, $k \in \mathbb{R}, \mathbb{C}$, and the variable D corresponds to d/dx. The multiplication law in $\mathscr{R}$ is given by the product of symbols

$$(1) \qquad A(x, \xi) \circ B(x, \xi) = \sum_{n\geqslant 0} \frac{1}{n!} A_\xi^{(n)}(x, \xi) B_x^{(n)}(x, \xi)$$

(where $A_\xi^{(n)} = d^nA/d\xi^n$ and $B_x^{(n)} = d^nB/dx^n$), and it coincides with the usual multiplication law on the subalgebra $\mathscr{R}_+ \subset \mathscr{R}$ of *differential* operators (i.e., on polynomials with respect to D). This law determines a Lie algebra structure on $\mathscr{R}$:

$$(2) \qquad [A, B] = A \circ B - B \circ A.$$

There is an operator res: $\mathscr{R} \to C^\infty(S^1)$ on the ring $\mathscr{R}$: $res(\sum a_i D^i) = a_{-1}(x)$. The main property of res is $\int res[A, B] = 0$ for arbitrary $A, B \in \mathscr{R}$. (Here and below, we integrate over the circle S^1.)

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Now we consider the formal expression $\log D$. Certainly, for any pseudodifferential operator $A \in \mathcal{R}$, the formal product $A \circ \log D$ (according to (1), where $\log \xi$ is the symbol of $\log D$) does not belong to $\mathcal{R}$.

The crucial point is that the formal commutator $[\log D, A] = \log D \circ A - A \circ \log D$ is an element of $\mathcal{R}$. Thus, $\log D$ acts on $\mathcal{R}$ by commutation $[\log D, *]$. In coordinate form, if $A = \sum_{i=-\infty}^{n} a_i(x) D^i$, then (due to (1–2))

$$(3) \qquad [\log D, A] = \sum_{k \geqslant 1} \frac{(-1)^{k+1}}{k} A_x^{(k)} D^{-k}$$

where $A_x^{(k)}$ is the operator $A_x^{(k)} = \sum_{i=-\infty}^{n} a_i^{(k)} D^i$ with $a_i^{(k)} = d^k a_i / dx^k$. Note that, even for a differential operator $A \in \mathcal{R}_+$, the result $[\log D, A]$ is, generally speaking, a pseudodifferential operator.

THEOREM 1 ([KK]). *A nontrivial central extension of the Lie algebra $\mathcal{R}$ is given by the 2-cocycle*

$$c(L, M) = \int res([\log D, L] \circ M) = \int res\left(\sum_{k \geqslant 1} \frac{(-1)^{k+1}}{k} L_x^{(k)} D^{-k} \circ M\right)$$

where L and M are arbitrary pseudodifferential symbols on S^1. The restriction of this cocycle to $\mathcal{R}_+$ gives the nontrivial central extension of $\mathcal{R}_+$.

Remark 2. The restriction of this cocycle to the subalgebra of vector fields (i.e., to the first-order differential operators) is the Gelfand-Fuchs cocycle of the Virasoro algebra. Indeed,

$$\begin{aligned} c(f(x)D, g(x)D) &= \int res([\log D, f(x)D] \circ g(x)D) \\ &= \int res((f' D^0 - f'' D^{-1}/2 + f''' D^{-2}/3 - \cdots) \circ g D^1) \\ &= \int res(\cdots + f''(x) g'(x) D^{-1}/6 + \cdots) = \frac{1}{6} \int f''(x) g'(x)\, dx. \end{aligned}$$

This observation implies the nontriviality of the cocycle on $\mathcal{R}$ and $\mathcal{R}_+$.

Proof of the theorem. Skew symmetry of $c(L, M)$ is a consequence of the identities

$$[\log D, L \circ M] = [\log D, L] \circ M + L \circ [\log D, M] \quad \text{and} \quad \int res[\log D, A] = 0$$

for any $L, M, A \in \mathcal{R}$. These identities themselves follow immediately from (1–3).

The same identities together with the Jacobi identity on $\mathscr{R}$ allows one to verify the cocycle property

$$\circlearrowleft_{L,M,N} c(L, [M, N])$$

$$= \int res([\log D, L][M, N] + [\log D, N][L, M] + [\log D, M][N, L]) = 0.$$

Remark 3. Assume for a moment that $\log D$ is an element of the algebra $\mathscr{R}$. Then we can define not only the commutator $[\log D, A]$ but also a product $\log D \circ A$, and rewrite the cocycle $c(A, B) = \int res([\log D, A] \circ B)$ as $c(A, B) = \int res(\log D \circ [A, B])$. The last formula means that the cocycle $c(A, B)$ is a 2-coboundary (and hence, trivial) because it is a linear function of the commutator: $c(A, B) = \langle \log D, [A, B] \rangle$. Recalling that $\log D \notin \mathscr{R}$, we get a heuristic proof of the nontriviality of the cocycle.

The same point of view on the Kac-Moody algebras gives a description of the Maurer-Cartan cocycle as a "coboundary": $\int \mathrm{tr}([d/dx, A(x)] \circ B(x)) = \int \mathrm{tr}(d/dx \circ [A(x), B(x)])$. The "logarithmic" theory turns out to be very close to the corresponding theory of the Kac-Moody algebras [RS]. In further publications we will touch on the "logarithmic" analogs of flat connections, gauge transformations, the Floquet theorem, and the Drinfeld-Sokolov hamiltonian reduction.

Remark 4. The value of $c(f(x)D^m, g(x)D^n)$ on the homogeneous generators of $\mathscr{R}$ vanishes for $n + m + 1 < 0$ and, generally speaking, does not vanish for $m + n + 1 \geqslant 0$. The restriction of this cocycle on the differential operators $(n, m \geqslant 0)$ coincides with the formula from [KP]:

$$c(f(x)D^m, g(x)D^n) = \frac{m!n!}{(m+n+1)!} \int f^{(n+1)} g^{(m)} \, dx.$$

(See, for example, [R].)

2. The cocycle on the subalgebras. Let $\mathscr{G}$ be a reductive matrix Lie algebra. We consider pseudodifferential operators on the circle with matrix coefficients $\{A(x, D) = \sum_{-\infty}^{n} a_i(x) D^i$, where $a_i \in C^\infty(S^1, \mathscr{G})\}$. These operators form a ring (and Lie algebra) $\mathscr{R}_\mathscr{G}$ with respect to the same multiplication law (1), where the product $\circ$ includes the matrix product of the corresponding coefficients.

Now $res: \mathscr{R}_\mathscr{G} \to C^\infty(S^1, k)$ means taking the trace of the matrix coefficient in front of D^{-1}: $res(\sum a_i(x) D^i) = \mathrm{tr}\, a_{-1}(x)$. Analogously to Section 1, we define the action of $\log D$ on $\mathscr{R}_\mathscr{G}$:

$$[\log D, A] = \sum_{k \geqslant 1} \frac{(-1)^{k+1}}{k} A_x^{(k)} D^{-k}.$$

The following statement is completely parallel to Theorem 1.

THEOREM 5. *For any reductive Lie algebra $\mathscr{G}$ the algebra of pseudodifferential operators $\mathscr{R}_{\mathscr{G}}$ has a nontrivial central extension given by the cocycle* $c(A, B) = \int res([\log D, A] \circ B)$.

Remark 6. The Lie algebra $\mathscr{R}_{\mathscr{G}}$ has several interesting subalgebras. First of all, analogously to the scalar case, we define the subalgebra $\mathscr{R}_{+,\mathscr{G}}$ of all matrix *differential* operators. Further, for any positive integer $m \in \mathbb{N}$, let $\mathscr{R}_{m,\mathscr{G}}$ be the subalgebra consisting of differential operators $\{A_m(x, D) = \sum_{m \leqslant i} a_i(x)D^i,\ a_i \in C^\infty(S^1, \mathscr{G})\}$. (Note that $\mathscr{R}_{0,\mathscr{G}} = R_{+,\mathscr{G}}$.)

And, finally, the following two subalgebras are well known: $\tilde{\mathscr{G}} = \{B(x, D) = b(x)D^0,\ b \in C^\infty(S^1, \mathscr{G})\} \subset \mathscr{R}_{\mathscr{G}}$ is the current algebra corresponding to $\mathscr{G}$, and Vect $= \{V(x, D) = v(x)D^1,\ v \in C^\infty(S^1, \mathscr{G}\ell_n)$ such that $v(x)$ is a scalar matrix $\forall x\} \subset \mathscr{R}_{\mathscr{G}\ell_n}$ is isomorphic to the Lie algebra of vector fields on S^1; see Section 1.

THEOREM 7. *The restriction of the 2-cocycle $c(A, B) = \int res([\log D, A] \circ B)$ to each of the subalgebras $\mathscr{R}_{+,\mathscr{G}}$, $\mathscr{R}_{m,\mathscr{G}}$ (for any $m \in \mathbb{N}$), $\tilde{\mathscr{G}}$,* Vect *(for $\mathscr{G} = \mathscr{G}\ell_n$) of the Lie algebra $\mathscr{R}_{\mathscr{G}}$ defines a nontrivial central extension of this subalgebra.*

Remark 8. Note that this "logarithmic" central extension of $\tilde{\mathscr{G}}$ and Vect gives the Kac-Moody and Virasoro algebras respectively. Thus, these objects appear together in this general context.

Proof. The nontriviality of the Maurer-Cartan and Gelfand-Fuchs cocycles implies the nontriviality of the central extensions of the subalgebras $\mathscr{R}_{+,\mathscr{G}}$, $\mathscr{R}_{1,\mathscr{G}}$, containing $\tilde{\mathscr{G}}$ or Vect. Thus, we need to prove the theorem for the hierarchy $\mathscr{R}_{m,\mathscr{G}}$, $m \geqslant 2$ only, where this argument about the restriction is not applicable.

We recall that there exists a "Killing form" $K(A, B) = \int res(A \circ B)$ on the algebra $\mathscr{R}_{\mathscr{G}}$. This form is really nondegenerate, because "*res*" is the trace of the matrix product and, hence, is a nondegenerate bilinear form for a reductive algebra $\mathscr{G}$.

Using this "Killing form", we naturally identify the dual space $\mathscr{R}^*_{\mathscr{G}}$ with $\mathscr{R}_{\mathscr{G}}$ itself. A pseudodifferential symbol $L \in \mathscr{R}_{\mathscr{G}}$ can be considered as a linear functional on $\mathscr{R}_{\mathscr{G}}$ (i.e., $L \in \mathscr{R}_{\mathscr{G}}$): $\langle L, A\rangle = \int res(A \circ L)$. Then the dual spaces for the subalgebras $\mathscr{R}_{m,\mathscr{G}}$ can be viewed as subspaces in $\mathscr{R}^*_{\mathscr{G}} \simeq \mathscr{R}_{\mathscr{G}}$: $\mathscr{R}^*_{m,\mathscr{G}} \simeq \{W(x, D) = \sum_{i \leqslant -m-1} w_i(x)D^i,\ w \in C^\infty(S^1, \mathscr{G})\}$. (We identify $\mathscr{G}^* = \mathscr{G}$.)

Let A and B be elements of $\mathscr{R}_{m,\mathscr{G}}$: $A(x, D) = a_m(x)D^m + a_{m+1}(x)D^{m+1} + \cdots$, $B(x, D) = b_m(x)D^m + b_{m+1}(x)D^{m+1} + \cdots$. Assume that the cocycle $c(A, B)$ is a coboundary: $c(A, B) = \langle L, [A, B]\rangle = \int res([A, B] \circ L)$ for some $L \in \mathscr{R}^*_{m,\mathscr{G}}(L(x, D) = l_{-m-1}(x)D^{-m-1} + l_{-m-2}(x)D^{-m-2} + \cdots)$.

Calculating the residue, $c(A, B) = \int res([a_m D^m + a_{m+1}D^{m+1} + \cdots, b_m D^m + b_{m+1}D^{m+1} + \cdots] \circ (l_{-m-1}D^{-m-1} + l_{-m-2}D^{-m-2} + \cdots))$, we get a linear combination of the terms $\int a_p^{(s)} b_q^{(t)} l_r^{(n)}\, dx$, where the integers p, q, r, s, t, u satisfy

$$(p + q + r) - (s + t + u) = -1, \qquad (4)$$

$s, t, u \geqslant 0$, $p, q \geqslant m$, $r \leqslant -m-1$. We recall that the functions $l_r(x)$ (and, hence, all derivatives $l_r^{(u)}(x)$) are fixed by our assumption.

On the other hand, the value

$$c(A, B) = \int res([\log D, A] \circ B)$$

$$= \int res([\log D, a_m D^m - a_{m+1} D^{m+1} + \cdots] \circ (b_m D^m + b_{m+1} D^{m+1} + \cdots))$$

$$= \int res((+a_m' D^{m-1} - a_m'' D^{m-2}/2 - \cdots + a_{m+1}' D^m - a_{m+1}'' D^{m-1}/2 - \cdots)$$

$$\circ (b_m D^m + b_{m+1} D^{m+1} + \cdots))$$

is a linear combination of the terms $\int a_p^{(s)} b_q^{(t)}\, dx$ such that

$$(5) \qquad (p+q) - (s+t) = -1.$$

Comparing these terms for arbitrary A and B with the linear combination above and, in particular, comparing the conditions (4) and (5), we obtain $r = u$. This contradicts the condition set $u \geqslant 0$ and $r \leqslant -m-1$. Indeed, for nonnegative integer m, the number $u = r \leqslant -1$. Thus, we have proved the nontriviality of the cocycle for any $m \geqslant 0$ (not only for $m \geqslant 2$). Q.E.D.

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