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A CENTRAL EXTENSION OF THE ALGEBRA OF PSEUDODIFFERENTIAL SYMBOLS

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UDC 517.9

1. The Lie algebra of pseudodifferential symbols ($\Psi\mathcal{DS}$) on the circle is a natural generalization of the algebra of differential operators ($\mathcal{DO}$) on the circle, which, in turn, is a generalization of the Lie algebra of vector fields. In this note it is shown that the algebra of $\Psi\mathcal{DS}$ possesses a nontrivial central extension; moreover, the restriction of the cocycle constructed here to the algebra $\mathcal{DO}$ gives the cocycle found in Radul's work [1], while its restriction to the algebra of vector fields gives the Gel'fand–Fuks cocycle.

In defining the cocycle it turned out useful to introduce the action on the algebra of a new object. This object will be naturally referred to as the logarithm of the differentiation, $\ln \partial$. For arbitrary pseudodifferential symbols L and M the cocycle is given by the formula

$$c(L, M) = \int_{S^1} \operatorname{res}([\ln \partial, L]\, M), \tag{1}$$

where $[\ln \partial, *]$ is the action of $\ln \partial$ on the elements of $\Psi\mathcal{DS}$.

The recent growing interest in algebras of differential operators and pseudodifferential symbols is connected with their intense utilization in conformal field theory. In [1] it is shown that, like the vector fields, the algebra $\mathcal{DO}$ embeds in the Gel'fand–Dikii algebra, which is the classical limit of the so-called W_n-algebras (see [2, 3]). In [1] it was conjectured that the universal W_∞-algebra (which is the limit of W_n as $n \to \infty$) is a nontrivial central extension of the Lie algebra of differential operators on the circle. The existence and uniqueness of this central extension were established by Feigin [4]. This extension also arose in [5], as a generalization of the extension of Beilinson, Manin, and Shekhtman [6].

From the paper [7] it follows that the second cohomology space of the algebra $\Psi\mathcal{DS}$ on the circle is two-dimensional. As one of its generators we propose the 2-cocycle generated by the action $[\ln \partial, *]$. As indicated to us by I. S. Zakharevich, a second generator is provided, for example, by the action $[x, *]$, where x is a linear function on the covering space of the circle.

As A. O. Radul communicated to us, this construction of $\ln \partial$ can be generalized to the algebra $\Psi\mathcal{DS}$ on the n-dimensional torus by means of the notion of residue introduced by Wodzicky [8].

We are deeply grateful to I. S. Zakharevich, A. O. Radul, A. M. Semikhatov, and V. V. Fok for numerous useful discussions, and also to A. Yu. Vaintrob, who indicated to us the papers [5] and [7].

2. Let us turn to precise formulations.

Physics Institute, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 25, No. 2, pp. 83-85, April-June, 1991. Original article submitted May 3, 1990.

Definition 1. The ring $\mathcal{R}$ of pseudodifferential symbols is defined to be the ring consisting of the formal series of the form $A(x, \xi) = \sum_{i=-\infty}^{n} a_i(x) \xi^i$, where $a_i(x) \in C^\infty(S^1, k)$, $k = \mathbf{R}, \mathbf{C}$, and to the variable ξ there corresponds the operator d/dx. Then the multiplication rule in $\mathcal{R}$

$$A(x, \xi) \circ B(x, \xi) = \sum_{n \geqslant 0} \frac{1}{n!} \partial_\xi^n A(x, \xi)\, \partial_x^n B(x, \xi) \tag{2}$$

when restricted to polynomials in ξ, coincides with the multiplication (composition) rule for differential operators. This rule yields an associative operation on $\mathcal{R}$ [and also on any differential algebra in which the right-hand side of (2) converges] and hence a structure of a Lie algebra on $\mathcal{R}$: $[A, B] = A \circ B - B \circ A$.

On $\mathcal{R}$ there is defined the residue map res: $\mathcal{R} \to C^\infty(S^1)$, $\mathrm{res}(\Sigma a_i \xi^i) = a_{-1}$. The main property of the residue is that

$$\int \mathrm{res}\, [AB]\, dx = 0 \quad \text{for any} \quad A, B \in \mathcal{R}. \tag{3}$$

Here and in what follows the integral is taken over the circle S^1.

Definition 2. The extension $\hat{\mathcal{R}}$ of the algebra $\mathcal{R}$ is defined to be the differential algebra consisting of the formal series of the form $\sum_{i=-\infty}^{n} a_i \xi^i + \lambda \ln \xi$. As a linear space, $\hat{\mathcal{R}} = \mathcal{R} \oplus \mathbf{C}$. It is readily verified that $[\ln \xi, R] \in \mathcal{R}$ for any $R \in \mathcal{R}\left(R = \sum_{i=-\infty}^{n} r_i(x) \xi^i\right)$, more precisely,

$$[\ln \xi, R] = \sum_{k \geqslant 1} \frac{(-1)^k}{k} R_x^{(k)} \xi^{-k}. \tag{4}$$

Here, $R_x^{(k)}$ denotes the symbol $R_x^{(k)} = \sum_{i=-\infty}^{n} r_i^{(k)} \xi^i$, where $r_i^{(k)} = \partial^k r_i / \partial x^k$, or, equivalently, $R_x^{(k)} = \underbrace{[\xi, [\xi \ldots [\xi, R] \ldots]]}_{k \text{ times}}$.

THEOREM. The nontrivial central extension $\hat{\mathcal{R}}$ is given by the following cocycle:

$$c(L, M) = \int \mathrm{res}\,([\ln \xi, L]\, M) = \int \mathrm{res}\left(\sum_{k \geqslant 1} \frac{(-1)^k}{k} L^{(k)} \xi^{-k} M\right), \tag{5}$$

where L and M are arbitrary pseudodifferential symbols on the circle.

Proof. Skew-symmetry follows from the two equalities

$$[\ln \xi, LM] = [\ln \xi, L]\, M + L\, [\ln \xi, M], \quad \int \mathrm{res}\, [\ln \xi, A] = 0,$$

which hold for all $L, M, A \in \mathcal{R}$. These equalities follow directly from the definitions of the associative multiplication, of the commutator, and of $\ln \xi$ (2-4).

The cocycle property

$$\int \mathrm{res}\,([\ln \xi, L]\, [M, N] + [\ln \xi, N]\, [L, M] + [\ln \xi, M]\, [N, L]) = 0 \tag{6}$$

follows from the chain of equalities

$$\begin{aligned} \int \mathrm{res}\,([\ln \xi, L]\, [M, N]) &= -\int \mathrm{res}\,([\ln \xi, [M, N]\, L) = \\ &= \int \mathrm{res}\,([M, [\ln \xi, N]]\, L) - \int \mathrm{res}\,([N, [\ln \xi, M]]\, L) = \\ &= \int \mathrm{res}\,([\ln \xi, N]\, [L, M]) + \int \mathrm{res}\,([\ln \xi, M]\, [N, L]). \end{aligned}$$

The first and last of these equalities is a consequence of skew-symmetry, while the one in the middle follows from the Jacobi identity. Similarly, each term in (6) turns out to be the sum of two others, so that adding all three terms we obtain zero.

Remark. The value $c(f\partial^m, g\partial^n)$, $n, m \in \mathbb{N}$ of the cocycle on homogeneous generators of the algebra is equal to zero for $n + m + 1 < 0$ and, in general, is different from zero for $n + m + 1 \geq 0$. The restriction of the cocycle to the differential operators coincides with Radul's cocycle [1]:

$$c(f\partial^m, g\partial^n) = \frac{m!n!}{(m+n+1)!}\int f^{(n)}g^{(m+1)}\,dx, \tag{7}$$

which can be verified, for example, by induction on m.

Thus, we obtained a central extension of the algebra of pseudodifferential operators by means of the number field:

$$0 \to k \to \Psi\widehat{\mathscr{D}\mathscr{S}}(S^1) \to \Psi\mathscr{D}\mathscr{S}(S^1) \to 0. \tag{8}$$

3. As an application of formula (5), let us give a simple way of writing the coadjoint action for the central extension of the Lie algebra of differential operators $\widehat{\mathscr{D}\mathscr{O}}$ on its dual space (the space of integral operators) $\widehat{\mathscr{I}\mathscr{O}}^*$:

$$\mathrm{ad}^*_{(L,\alpha)}(Q, \nu) = ([\nu \ln \partial + Q, L]_-, 0), \tag{9}$$

where L is a differential operator, Q is an integral operator, α and ν are numbers ($(L, \alpha) \in \widehat{\mathscr{D}\mathscr{O}}$, $(Q, \nu) \in \widehat{\mathscr{I}\mathscr{O}}^*$), and the "minus' appended to the commutator denotes the operation of "forgetting" the differential part.

It would be interesting to describe the orbits of this action and find a pseudodifferential version of Floquet's theorem. The traditional formulation of this theorem asserts that the unique invariant of the gauge action on the differential operators is their monodromy operator [9].

Another intriguing problem is to construct a Hamiltonian reduction of the type of the Drinfel'd–Sokolov reduction [10] on the algebra of pseudodifferential symbols $\Psi\widehat{\mathscr{D}\mathscr{S}}$. In the case of success it is precisely the algebra $\Psi\widehat{\mathscr{D}\mathscr{S}}$ that would be candidate for role of universal algebra W_∞.

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