Invariants of Hamiltonian KdV-structures

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The Hamiltonian structures for KdV equations (also called *Gel'fand–Dikii brackets* or *classical* W-*algebras*) that are associated with an arbitrary semisimple Lie group are defined on gauge-equivalence classes of matrix differential operators (DO) on a circle (see [1]). The aim of this note is to describe continuous invariants of symplectic sheets (or maximal non-degenerate submanifolds) of these Poisson brackets, that is, to describe a complete set of functions that are in involution with arbitrary Hamiltonian. These functions are invariants of the corresponding KdV equations connected with the bracket structure and are independent of the concrete Hamiltonian equations. All necessary definitions are given in section 1.

***Theorem* 1.** *The only continuous invariant of symplectic sheets of the second KdV-structure is the monodromy operator for the gauge class DO. The contiguity of sheets in this structure associated with a semisimple Lie group G is that of classes of conjugate elements in G.*

Corollary. *The codimensions of all symplectic sheets of this bracket are finite and are the same as the codimensions of the monodromy conjugacy classes in the corresponding group G.*

We note that the number of symplectic sheets corresponding to the same monodromy operator is different in general for different operators.

In the simplest case, that of $SL_2(\mathbb{R})$, the problem of classifying symplectic sheets of a second KdV-structure is equivalent to describing the co-adjoint orbits of the Virasoro group. There are independent solutions, in different terms, by Cooper, Lazutkin and Pankratova, Kirillov, and Segal (see [2]). The study of stratifications into symplectic sheets is useful in quantizing these Poisson brackets (cf. [2]). In accordance with the method of orbits, the structure of contiguity of sheets gives information about degeneracy of mutually irreducible representations corresponding to different symplectic sheets.

1. Definitions and necessary facts.

1) Let G be a semisimple Lie group of matrices or GL_n, $\mathfrak{G}$ its Lie algebra, $\widetilde{G} = C^\infty(S^1, G)$ the group of loops, and $\widehat{\mathfrak{G}}$ the corresponding Kac–Moody algebra on a circle. We identify the space $\widehat{\mathfrak{G}}^*$ with the space of matrix DO of the form $a\cdot d/dx + q(x)$, $q \in \widetilde{\mathfrak{G}}^* = C^\infty(S^1, \mathfrak{G}^*)$, $a \in k$ ($\mathbb{R}$ or $\mathbb{C}$) (see [1], [3]). A co-adjoint action consists of a gauge action of $\widetilde{G}$ on DO.

Let $\mathcal{M}$ be a hyperplane in $\widehat{\mathfrak{G}}^*$ with condition $a = 1$. A *second Hamiltonian structure on* $\mathcal{M}$ is a bounded linear Poisson structure with co-algebra $\widehat{\mathfrak{G}}^*$ on $\mathcal{M}$. Symplectic sheets of this structure are orbits of the co-adjoint (that is, gauge) action of $\widetilde{G}$ on $\mathcal{M}$.

2) A generalized KdV-structure is a Hamiltonian reductive structure on $\mathcal{M}$ relative to a unipotent subgroup $\widetilde{N}$ of $\widetilde{G}$.

For $G = GL_n$, we consider the subset $\mathcal{P}$ of $\mathcal{M}$ consisting of DO of the form $P = d/dx + \Lambda + p(x)$, where $\Lambda = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & 0 \\ 0 & & 1 & \end{pmatrix}$, $p \in C^\infty(S^1, \mathfrak{B})$, $\mathfrak{B}$ being the subalgebra consisting of the upper-triangular matrices. For a semisimple group G, we consider a $\mathbb{Z}$-grading of $\mathfrak{G}$ for which the (one-dimensional) subspaces corresponding to the chosen simple roots have degree 1. Then $\mathfrak{B}$ is the Borel subalgebra of $\mathfrak{G}$ consisting of elements with non-negative grading, and Λ is an element in general position in the graded component of degree -1 (see [1]).

Finally, we factor $\mathcal{P}$ by the gauge action of the unipotent subgroup $\widetilde{N}$ of $\widetilde{G}$ corresponding to the chosen Borel subgroup ($N \subset B$). The Hamiltonian structure thus introduced is well defined on the quotient-space $\mathcal{L} = \mathcal{P}/\widetilde{N}$ and is called *a second Hamiltonian KdV-structure.*

3) We associate with a matrix differential operator P the monodromy operator $T(P)$ of the differential equation $Py = 0$ with periodic coefficients. The monodromy $T(P)$ defines a conjugacy class of elements in the corresponding Lie group G. Here, gauge-equivalent DO correspond to conjugate monodromy operators.

2. We have the following sharpening of Theorem 1.

Theorem **1′.** *For every smooth function $\varphi(L)$ defined on classes of operators L in $\mathscr{L}$ in the neighbourhood of a given symplectic sheet with second KdV-structure constant on sheets, there exists a smooth central function ψ on G such that $\varphi(L) = \psi(T(L))$. The algebra of invariants of these sheets is in one-to-one correspondence with the algebra of central functions on G.*

Remark. In the case of GL_n, SL_n, Sp_{2k}, and SO_{2k+1}, Theorem 1 was proved in [4], with different terminology. For these groups, the quotient-space $\mathscr{L}$ can be identified with the submanifold of $\mathscr{P}$ consisting of the scalar DO of n-th order on a circle (see [1]). The proof offered below of the general case is essentially geometric in nature.

Proof. The monodromy conjugacy class is the only invariant of the gauge action of $\widetilde{G}$ on $\mathscr{M}$ (see [3]), so that Theorem 1′ follows immediately from the following lemma.

Lemma. *The $\widetilde{G}$-orbits of all DO in $\mathscr{M}$ are transversal to $\mathscr{P}$.*

Proof of the Lemma. To prove that the $\widetilde{G}$-orbits are transversal to $\mathscr{P}$, it is enough to establish that the map $P \to T(P)$ for $P \in \mathscr{P}$ is locally epimorphic, since monodromy is the only invariant of a $\widetilde{G}$-orbit. Let $g_P^t : \mathbb{R} \to G$ be a phase flow of the differential equation corresponding to the DO P (viz. $g_P^0 = 1$, $g_P^{2\pi} = T(P)$). For $P \in \mathscr{P}$, the flow g_P is tangent to the right-invariant completely non-integrable distribution $\mathscr{K}$ on G defined at the identity 1 of the group as the linear span of the vectors in $\Lambda + \mathfrak{B}$. The fact that $\mathscr{K}$ is completely non-integrable follows from the fact that the derived set of $\Lambda + \mathfrak{B}$ is exactly $\mathfrak{G}$.

Every pair of close points on G are connected by a C^0-small path subject to the non-integrable distribution $\mathscr{K}$ (the Chow–Rashevskii theorem). This path can be chosen to be C^1-small, by applying this theorem to lifting the given distribution to the space of jets. The paths C^1-close to some g_P^t satsify constraints of strict inequality type on the linear span of $\Lambda + \mathfrak{B}$, which means that they are flows of certain operators in $\mathscr{P}$. Thus, for arbitrary $P \in \mathscr{P}$, there exists a (dim G)-parametric deformation $P_\varepsilon \in \mathscr{P}$ such that $T(P_\varepsilon)$ covers a neighbourhood of the point $T(P)$ of G. Q.E.D.

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