Versal deformations of intersections of invariant submanifolds of dynamical systems

B.A. Khesin

On the boundary of the space of Morse-Smale systems there are systems with non-transversal intersection of the incoming and outgoing invariant manifolds. In this note we study the structure of bifurcation diagrams in the space of systems in a neighbourhood of a system with such a non-transversal intersection. It is shown that this diagram contains as fragments the standard bifurcation sets of V-versal deformations of germs of maps from $\mathbf{R}^n$ to $\mathbf{R}^m$.

The author expresses his acknowledgement to V.I. Arnol'd for posing this problem and useful discussions.

Definition. By an *embedded pair* we mean a smooth multigerm $(f_1, f_2) : (\mathbf{R}^k, 0;\ \mathbf{R}^l, 0) \to (\mathbf{R}^n, 0)$ with derivatives of maximal rank, that is, $\mathrm{rk}\, df_1 \mid_0 = k$, $\mathrm{rk}\, df_2 \mid_0 = l$ (here and in what follows, $k, l < n$). Two embedded pairs are *equivalent* if there exist germs of diffeomorphisms $h_1 : (\mathbf{R}^k, 0) \to (\mathbf{R}^k, 0)$, $h_2 : (\mathbf{R}^l, 0) \to (\mathbf{R}^l, 0)$, $g : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, such that $g \circ (f_1, f_2) = (f_1 \circ h_1, f_2 \circ h_2)$. A deformation of an embedded pair (the multigerm of the map $(\mathbf{R}^k, 0;\ \mathbf{R}^l, 0) \times (\mathbf{R}^\mu, 0) \to (\mathbf{R}^n, 0)$ at a point of the direct product of the inverse image space with the parameter space) is said to be *versal* if any deformation is equivalent to that induced from it.

We associate with an embedded pair the germ of the map $\varphi : (\mathbf{R}^k, 0) \to (\mathbf{R}^{n-l}, 0)$ in the following way. Let Φ be an arbitrary germ $(\mathbf{R}^n, 0) \to (\mathbf{R}^{n-l}, 0)$ for which zero is a regular value and the inverse image of zero is the germ of a smooth surface $f_2(\mathbf{R}^l)$. The germ of φ is the restriction of Φ to the first surface of the pair: $\varphi = \Phi \mid_{f_1(\mathbf{R}^k)}$. The construction of the germ φ, corresponding to the deformation of an embedded pair is similar.

We recall that two germs α and $\widetilde{\alpha} : (\mathbf{R}^s, 0) \to (\mathbf{R}^t, 0)$ are said to be *V-equivalent* if there exist germs $h : (\mathbf{R}^s, 0) \to (\mathbf{R}^s, 0)$ and $M : (\mathbf{R}^s, 0) \to GL(\mathbf{R}^t)$ such that $\widetilde{\alpha}(x) \equiv M(x)\alpha(h(x))$. A V-versal deformation is defined correspondingly (see [1]).

Theorem. *The following statements are equivalent*:

a) *the deformation of an embedded pair is versal*;

b) *there exists a system of coordinates in a neighbourhood of the origin of* $\mathbf{R}^n$ *that is smoothly dependent on the parameter* ε *and in which the image of* $(f_1, f_2)_\varepsilon$ *is given by the equations*

$$f_1(\mathbf{R}^k) = \{x_j = 0,\ k + 1 \leqslant j \leqslant n\},$$

$$f_2(\mathbf{R}^l) = \{x_j = 0,\ 1 \leqslant j \leqslant k - i + 1,\ (x_{k+l+1-i}, \ldots, x_n) = u_\varepsilon(x_{k+1-i}, \ldots, x_k)\}$$

for some i, *where* u_ε *is a V-versal deformation of the map* $u_0 : (\mathbf{R}^i, 0) \to (\mathbf{R}^{n-(k+l)+i}, 0)$;

c) *the germ* φ_ε *corresponding to the deformation of an embedded pair is a V-versal deformation of the germ of* φ.

Proof. Let Γ_1, Γ_2 be the tangent planes to $f_1(\mathbf{R}^k)$, $f_2(\mathbf{R}^l)$ at the zero value of the parameter, and let $i = \dim(\Gamma_1 \cap \Gamma_2)$. Then the equivalence a) $\Leftrightarrow$ b) is a consequence of the implicit function theorem, which enables us to select directions $x_1, \ldots, x_{k-i}$ in $\Gamma_1/(\Gamma_1 \cap \Gamma_2)$ and $x_{k+1}, \ldots, x_{k+l-i}$ in $\Gamma_2/(\Gamma_1 \cap \Gamma_2)$ along which no bifurcation occurs, and the theorem that states that V, the class of the germ $\beta : \mathbf{R}^i \to \mathbf{R}^m = \mathbf{R}^n/(\Gamma_1 + \Gamma_2)$ (here $\Gamma_1 + \Gamma_2$ is the linear span of the vectors in Γ_1 and Γ_2), is defined by the equivalence class of the embedded pair of germs of i-dimensional submanifolds of the direct product $\mathbf{R}^i \times \mathbf{R}^m$: (horizontal plane, graph of β) (see [2]). The equivalence b) $\Leftrightarrow$ c) follows from the fact that $\mathrm{rk}\, \varphi = k - i$ and in the coordinate system b) the map φ_ε has the form $\varphi_\varepsilon(x_1, \ldots, x_k) = (x_1, \ldots, x_{k-i}, u_\varepsilon(x_{k-i+1}, \ldots, x_k))$, that is, the versality of φ_ε is equivalent to that of u_ε.

***Corollary* 1** (see [3]). *The normal form of a simple tangency* (*a non-degenerate tangency by a surface with the condition* $\dim \Gamma_1 + \dim \Gamma_2 \geqslant n$, $\dim(\Gamma_1 + \Gamma_2) = n - 1$) *is a combination of this theorem and the Morse lemma*: *in the canonical coordinates of part* b), *the function*

$$u_0(x) = \pm x_{k-i+1}^2 \pm \ldots \pm x_k^2.$$

***Corollary* 2.** *Suppose that a vector field on a manifold has a non-transversally intersecting pair of invariant submanifolds, and that it has a finite-dimensional versal deformation. Then the bifurcation diagram* (BD) *of this deformation* (*the set of values of the parameter corresponding to the fields that are not Morse-Smale systems* (*see* [4]) *contains as a fragment the cylinder over the bifurcation diagram of the V-versal deformation corresponding to a singularity of* φ.

By the *bifurcation diagram* of the deformation of a *germ* $(\mathbf{R}^s, 0) \to (\mathbf{R}^t, 0)$ we mean the set of values of the parameter corresponding to the maps with critical value 0 (for $s \geqslant t$), or maps with non-empty inverse image of the origin (for $s < t$).

In fact, intersecting invariant manifolds of a vector field have an entire intersection trajectory; the restriction of manifolds to a transversal to this trajectory at any point reduces the problem on non-transversal intersections of invariant surfaces to the description of the mutual disposition of their traces as given by the theorem. The above definition of the diagram of the germ of φ_ε corresponds exactly to the definition of the diagram of the deformation of an embedded pair as the sets of values of the parameter corresponding to the non-transverse intersections of the surfaces of the pair.

Examples. First we consider germs $\varphi\colon (\mathbf{R}^s, 0) \to (\mathbf{R}^t, 0)$ subject to the condition $s \geqslant t$. In this case, the V-simple germs are described by the real variant of the Giusti list (see [1]). For the case of one, two, and three parameters, one encounters only singularities of functions of types A_1, A_2, A_3 and their trivial extensions. (The germ φ with singularity A_μ can be represented in some coordinate system in the form

$$\varphi(x) = (x_1, \ldots, x_{t-1}, x_t^{\mu+1}, \pm x_{t+1}^2 \pm \cdots \pm x_s^2).)$$

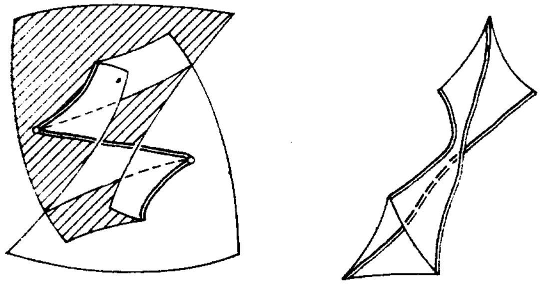
The singularity A_1 corresponds to a simple tangency (see Corollary 1), the diagrams for A_2 and A_3 are a semicubical parabola and a swallow-tail (and cylinders over them). Starting with four parameters, apart from the singularity A_4 singularities of the functions $D_4^\pm$ occur for $s > t$ (the series

$$D_\mu : \varphi(x) = (x_1, \ldots, x_{t-1}, x_t^{\mu-1} \pm x_t x_{t+1}^2 \pm x_{t+2}^2 \pm \cdots \pm x_s^2)),$$

while for $s = t$ there are singularities of maps of type $I_{2,2}^\pm$ (having normal form

$$\varphi(x) = (x_1, \ldots, x_{t-2}, x_{t-1}, x_t, x_{t-1}^2 \pm x_t^2)).$$

The traces of the BD of the singularities $I_{2,2}^+$ and $I_{2,2}^-$ on three-dimensional sections of general position are depicted in Fig. 1a and 1b respectively. On passing through the deformable singularities, the cross sections are diffeomorphic to a "purse" and a "pyramid" (see [1]), and are then converted into mirror images of the manifolds depicted in Fig. 1a and 1b.



a b

Fig. 1

When there is tangency of surfaces of dimensions k and l ($k+l \geqslant n$, here $k = s$, $l = n-t$) there occur singularities of the functions if $\dim(\Gamma_1 \cap \Gamma_2) = k+l-n+1$ (here it is necessary that $k+l > n$ for D_μ). Singularities of type I occur for $k+l = n$ and $\dim(\Gamma_1 \cap \Gamma_2) = 2$.

Now let $s < t$. In this case surfaces of dimensions s and $n-t$ are transversal only if they are non-intersecting. It is easy to see that the BD (the values of the parameter for which the inverse image of the origin for φ is non-empty) is a plane of codimension $t-s$. In this plane there are the values of the parameter for which the surfaces not only intersect but also touch one another and hence there exists a natural stratification in this plane (according to the degree of tangency), which is the same as for the case $s = t$ considered above.

References

[1] V.I. Arnol'd, A.N. Varchenko, and S.M. Gusein-Zade, *Osobennosti differentsiruemykh otobrazhenii*, Vol. I, Nauka, Moscow 1982. MR **84m**:58016.
Translation: Singularities of differentiable maps, Vol. I, Birkhäuser, Boston, MA, 1985. MR **86f**:58018.

[2] M. Golubitsky and V. Guillemin, Stable mappings and their singularities, Springer-Verlag, New York-Heidelberg 1973. MR **49** # 6269.
Translation: *Ustoichivye otobrazheniya i ikh osobennosti*, Mir, Moscow 1977.

[3] S.E. Newhouse, J. Palis, and F. Takens, Bifurcation and stability of families of diffeomorphisms, Publ. Math. IHES **57** (1983), 5–71. MR **84g**:5808.

[4] V.I. Arnol'd, V.S. Afraimovich, Yu.S. Il'yashenko, and L.P. Shil'nikov, *Teoriya bifurkatsii* (Bifurcation theory), Itogi Nauki i Tekhniki VINITI, Sovrem. Problemy Mat. Fund. Napravleniya **5** (1986), 5–218.

Moscow State University

Received by the Board of Governors 27 December 1988