CHAPTER VI

DYNAMICAL SYSTEMS WITH HYDRODYNAMICAL BACKGROUND

This chapter is a survey of several relevant systems to which the group-theoretic scheme of the preceding chapters or its modifications can be applied. The choice of topics for this chapter was intended to show different (but nevertheless, "hydrodynamical") features of a variety of dynamical systems and to emphasize suggestive points for further study and future results.

§1. The Korteweg–de Vries equation as an Euler equation

In Chapter I we discussed the common Eulerian nature of the equations of a three-dimensional rigid body and of an ideal incompressible fluid. The first equation is related to the Lie group $SO(3)$, while the second is related to the huge infinite-dimensional Lie group $S\mathrm{Diff}(M)$ of volume-preserving diffeomorphisms of M.

In this section we shall deal with an intermediate case: the Lie group of all diffeomorphisms of a one-dimensional object, the circle, or rather, with the one-dimensional extension of this group called the Virasoro group. In a sense it is the "simplest possible" example of an infinite-dimensional Lie group. It turns out that the corresponding Euler equation for the geodesic flow on the Virasoro group is well known in mathematical physics as the Korteweg–de Vries equation. This equation is widely regarded as a canonical example of an integrable Hamiltonian system with an infinite number of degrees of freedom.

1.A. Virasoro algebra. The Virasoro algebra is an object that is only one dimension larger than the Lie algebra $\mathrm{Vect}(S^1)$ of all smooth vector fields on the circle S^1 (in the physics literature, these vector fields are usually assumed to be trigonometric polynomials).

DEFINITION 1.1. The *Virasoro algebra* (denoted by *vir*) is the vector space

$\mathrm{Vect}(S^1) \oplus \mathbb{R}$ equipped with the following commutation operation:

$$[(f(x)\frac{\partial}{\partial x},\ a),\ (g(x)\frac{\partial}{\partial x},\ b)] = \left((f'(x)g(x) - f(x)g'(x))\frac{\partial}{\partial x},\ \int_{S^1} f'(x)g''(x)\,dx\right),$$

for any two elements $(f(x)\partial/\partial x,\ a)$ and $(g(x)\partial/\partial x,\ b)$ in vir.

The commutator is a pair consisting of a vector field and a number. The vector field is minus the Poisson bracket of the two given vector fields on the circle: $\{f\,\partial/\partial x,\ g\,\partial/\partial x\} = (fg' - f'g)\,\partial/\partial x$. The bilinear skew-symmetric expression $c(f,g) := \int_{S^1} f'(x)g''(x)dx$ is called the *Gelfand–Fuchs 2-cocycle*; see [GFu].

DEFINITION 1.2. A *real-valued two-cocycle* on an arbitrary Lie algebra $\mathfrak{g}$ is a bilinear skew-symmetric form $c(\cdot,\cdot)$ on the algebra satisfying the following identity:

$$\sum_{(f,g,h)} c([f,g],h) = 0, \qquad \text{for any three elements } f,g,h \in \mathfrak{g},$$

where the sum is considered over the three cyclic permutations of the elements (f,g,h).

The cocycle identity means that the extended space $\hat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{R}$ with the commutator defined by

$$[(f,a),\ (g,b)] = ([f,g],\ c(f,g)) \tag{1.1}$$

obeys the Jacobi identity of a Lie algebra. One can define $c(f,g)$ in (1.1) by setting $c(f,g) = 0$ for all pairs f,g and get a *trivial extension* of the Lie algebra $\mathfrak{g}$. An extension of the algebra $\mathfrak{g}$ is called *nontrivial* (or the corresponding 2-cocycle *is not a 2-coboundary*) if it cannot be reduced to the extension by means of zero cocycle via a linear change of coordinates in $\hat{\mathfrak{g}}$. We discuss cocycles on Lie algebras, as well as the geometric meaning of the Gelfand–Fuchs cocycle, in more detail in Section 1.D

The Virasoro algebra is the unique nontrivial one-dimensional central extension of the Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on the circle. There exists a Virasoro group whose Lie algebra is the Virasoro algebra vir; see, e.g., [Ner1].

DEFINITION 1.3. The *Virasoro* (or *Virasoro–Bott*) *group* is the set of pairs $(\varphi(x),a) \in \mathrm{Diff}(S^1) \oplus \mathbb{R}$ with the multiplication law

$$(\varphi(x),\ a) \circ (\psi(x),\ b) = \left(\varphi(\psi(x)),\ a + b + \int_{S^1} \log(\varphi \circ \psi(x))'\ d\log\psi'(x)\right).$$

Applying the general constructions of Chapter I to the Virasoro group, we equip this group with a (right-invariant) Riemannian metric. For this purpose we fix the energy-like quadratic form in the Lie algebra vir, i.e., on the tangent space to the group identity:

$$H(f(x)\frac{\partial}{\partial x},\ a) = \frac{1}{2}\left(\int_{S^1} f^2(x)\ dx + a^2\right).$$

Consider the corresponding Euler equation, i.e., the equation of the geodesic flow generated by this metric on the Virasoro group.

Definition 1.4. The *Korteweg–de Vries* (KdV) equation on the circle is the evolution equation

$$\partial_t u + uu' + u''' = 0$$

on a time-dependent function u on S^1, where $' = \partial/\partial x$ and $\partial_t = \partial/\partial t$; see [KdV].

Theorem 1.5 [OK1]. *The Euler equation corresponding to the geodesic flow (for the above right-invariant metric) on the Virasoro group is a one-parameter family of KdV equations.*

Proof. The equation for the geodesic flow on the Virasoro group corresponds to the Hamiltonian equation on the dual Virasoro algebra vir^*, with the linear Lie–Poisson bracket and the Hamiltonian function $-H$.

The space vir^* can be identified with the set of pairs

$$\left\{(u(x)(dx)^2, c)|\ u(x) \text{ is a smooth function on } S^1,\ c \in \mathbb{R}\right\}.$$

Indeed, it is natural to contract the quadratic differentials $u(x)(dx)^2$ with vector fields on the circle, while the constants are to be paired between themselves:

$$\langle(v(x)\frac{\partial}{\partial x},\ a), (u(x)(dx)^2, c)\rangle = \int_{S^1} v(x)\cdot u(x)\ dx + a\cdot c.$$

The coadjoint action of a Lie algebra element $(f\,\partial/\partial x, a) \in vir$ on an element $(u(x)(dx)^2, c)$ of the dual space vir^* is

$$\mathrm{ad}^*_{(f\,\partial/\partial x, a)}(u(dx)^2, c) = (2f'u + fu' + cf''',\ 0), \tag{1.2}$$

where $'$ stands for the x-derivative. It is obtained from the identity

$$\langle[(f\frac{\partial}{\partial x},\ a),\ (g\frac{\partial}{\partial x},\ b)], (u(dx)^2, c)\rangle = \langle(g\frac{\partial}{\partial x},\ b), \mathrm{ad}^*_{(f\frac{\partial}{\partial x},\ a)}(u(dx)^2, c)\rangle,$$

which holds for every pair $(g\frac{\partial}{\partial x},\ b) \in vir$.

The quadratic energy functional H on the Virasoro algebra determines the "tautological" inertia operator $A : vir \to vir^*$, which sends a pair $(u(x)\partial/\partial x, c) \in vir$ to $(u(x)(dx)^2, c) \in vir^*$.

In particular, it defines the quadratic Hamiltonian on the dual space vir^*,

$$\begin{aligned} H(u(dx)^2, c) =& \frac{1}{2}\Big(\int u^2 \, dx + c^2\Big) \\ =& \frac{1}{2}\langle (u\frac{\partial}{\partial x}, c),\ (u(dx)^2, c)\rangle = \frac{1}{2}\langle (u\frac{\partial}{\partial x}, c),\ A(u\frac{\partial}{\partial x}, c)\rangle. \end{aligned}$$

The corresponding Euler equation for the right-invariant metric on the group (according to the general formula (I.6.4), Theorem I.6.15) is given by

$$\frac{\partial}{\partial t}(u(dx)^2, c) = -\mathrm{ad}^*_{A^{-1}(u(dx)^2,c)}(u(dx)^2, c).$$

Making use of the explicit formula for the coadjoint action (1.2) with

$$(f\,\partial/\partial x, a) = A^{-1}(u(dx)^2, c) = (u\,\partial/\partial x, c),$$

we get the required Euler equation:

$$\begin{cases} \partial_t u = -2u'u - uu' - cu''' = -3uu' - cu''', \\ \partial_t c = 0. \end{cases}$$

The coefficient c is preserved in time, and the function u satisfies the KdV equation (with different coefficients). □

REMARK 1.6. Without the central extension, the Euler equation on the group of diffeomorphisms of the circle has the form

$$\partial_t u = -3uu'.$$

(called a nonviscous Burgers equation). Rescaling time, this equation can be reduced to the equation on the velocity distribution u of freely moving noninteracting particles on the circle. It develops completely different properties as compared to the KdV equation (see, e.g., [Arn15]).

If $u(x,t)$ is the velocity of a particle x at moment t, then the substantial derivative of u is equal to zero: $\partial_t u + uu' = 0$. In Fig.71 one sees a typical perestroika of the velocity field u in time. Since every particle keeps its own velocity, fast particles pass by slow ones. Every point of inflection in the initial velocity profile $u(x, 0)$ generates a shock wave.

Thus, in finite time, solutions of the corresponding Euler equation define a multivalued, rather than univalued, vector field of the circle. In other words, the geodesic

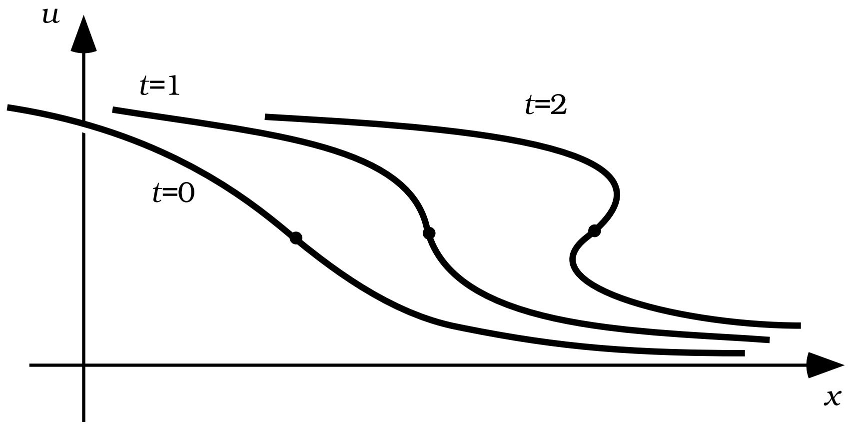

FIGURE 71. The shock wave generated by freely moving noninteracting particles.

flow on the group Diff(S^1), with respect to the right-invariant metric generated by the quadratic form $\int u^2 dx$ on the Lie algebra, is incomplete.

Note also that in the case of the Burgers equation with small viscosity, shock waves appear as well. Initial series of typical bifurcations of shock waves were described in [Bog] (see also [SAF, Si2]). Typical singularities of projections of the solutions on the plane of independent variables for 2×2 quasilinear systems are classified in [Ra].

On the other hand, the solutions of the KdV equation exist and remain smooth for all t, and do not develop shock waves. In the interpretation of the KdV as the shallow water equation, the parameter c measures dispersion of the medium.

REMARK 1.7. Differential geometry of the Virasoro group with respect to the above right-invariant metric is discussed in [Mis3]. In particular, the sectional curvatures in the two-dimensional directions containing the central direction are nonnegative (cf. Remark IV.2.4). For the relation between the geometry of the KdV equation and the Kähler geometry of the Virasoro coadjoint orbits see [Seg, STZ].

One can extend this Eulerian viewpoint to the super-KdV equation (introduced in [Kup]) and describe the latter as the equation of geodesics on the super-analogues of the Virasoro group, corresponding to the Neveu–Schwarz and Ramond superalgebras; see [OK1]. Another elaboration of this viewpoint is the passage from the L^2-metric on the Virasoro algebra to another one, say, the H^1-metric:

$$\|(f(x)\frac{\partial}{\partial x},\ a)\|^2_{H^1} = \frac{1}{2}\left(\int_{S^1} f^2(x)\ dx + \int_{S^1} (f'(x))^2\ dx + a^2\right).$$

PROPOSITION 1.8 [Mis4]. *The H^1-metric on the central extension of the Lie* *algebra* $\mathrm{Vect}(S^1)$ *of vector fields on the circle given by the (trivial) 2-cocycle*

$$c(f,g) := \int_{S^1} f'(x)g(x)\,dx$$

generates the shallow water equation

$$\begin{cases} \partial_t u - \partial_t u'' = uu''' + 2u'u'' - 3uu' - cu', \\ \partial_t c = 0 \end{cases}$$

(introduced in [C-H]*). Here the prime* $'$ *stands for* $\frac{\partial}{\partial x}$*, and* ∂_t *denotes* $\frac{\partial}{\partial t}$*.*

1.B. The translation argument principle and integrability of the high-dimensional rigid body.

Definitions 1.9. A function F on a symplectic manifold is a *first integral of a Hamiltonian system* with Hamiltonian H if and only if the Poisson bracket of H with F is equal to zero. Functions whose Poisson bracket is equal to zero are said to be *in involution* with respect to this bracket.

A Hamiltonian system on a symplectic $2n$-dimensional manifold M^{2n} is called *completely integrable* if it has n integrals in involution that are functionally independent almost everywhere on M^{2n}.

A theorem attributed to Liouville states that connected components of non-critical common level sets of n first integrals on a compact manifold are the n-dimensional tori. The Hamiltonian system defines a quasiperiodic motion $\dot{\phi} =$ const in appropriate angular coordinates $\phi = (\phi_1, \ldots, \phi_n)$ on each of the tori; see [Arn16].

Example 1.10. Every Hamiltonian system with one degree of freedom is completely integrable, since it always possesses one first integral, the Hamiltonian function itself.

In particular, the Euler equation of a three-dimensional rigid body is a completely integrable Hamiltonian system on the coadjoint orbits of the Lie group $SO(3)$. These orbits are the two-dimensional spheres centered at the origin and the origin itself, while the Hamiltonian function is given by the kinetic energy of the system.

Example 1.11. A consideration of dimensions is not enough to argue the complete integrability of the equation of an n-dimensional rigid body for $n > 3$. Free motions of a body with a fixed point are described by the geodesic flow on the group $SO(n)$ of all rotations of Euclidean space $\mathbb{R}^n$.

The group $SO(n)$ is equipped with a particular left-invariant Riemannian metric defined by the inertia quadratic form in the body's internal coordinates. On the

Lie algebra $\mathfrak{so}(n)$ of skew-symmetric $n \times n$ matrices this quadratic form is given by $-tr(\omega D \omega)$, where

$$\omega \in \mathfrak{so}(n), \quad D = \operatorname{diag}(d_1, \ldots, d_n), \quad d_k = \frac{1}{2} \int \rho(x) x_k^2 \, d^n x,$$

and where $\rho(x)$ is the density of the body at the point $x = (x_1, \ldots, x_n)$. The inertia operator $A : \mathfrak{so}(n) \to \mathfrak{so}(n)^*$ defining this quadratic form sends a matrix ω to the matrix $A(\omega) = D\omega + \omega D$.

REMARK 1.12. For $n = 3$ this formula implies the triangle inequality for the principal momenta d_k. Operators satisfying these inequalities form an open set in the space of symmetric 3×3 matrices. In higher dimension ($n > 3$) the symmetric matrices representing the inertia operators of the rigid bodies are very special. They form a variety of dimension n in the $n(n-1)/2$-dimensional space of equivalence classes of symmetric matrices on the Lie algebra.

THEOREM 1.13 ([Mish] FOR $n = 4$, [Man] FOR ALL n). *The Euler equation $\dot{m} = ad_\omega^* m$ of an n-dimensional rigid body, where $\omega = A^{-1} m$ and the inertia operator A is defined above, is completely integrable. The functions*

$$H_{\lambda,\nu} = \det(m + \lambda D^2 + \nu E) \tag{1.3}$$

on the dual space $\mathfrak{so}(n)^*$ provide a complete family of integrals in involution.

The involutivity of the quantities $H_{\lambda,\nu}$ can be proved by the method of Poisson pairs and translation of the argument, which we discuss below (see [Man]). Note that the physically meaningful inertia operators $A(\omega) = D\omega + \omega D$ (i.e., those with entries $a_{ij} = d_i + d_j$) form a very special subset in the space of all symmetric operators $A : \mathfrak{so}(n) \to \mathfrak{so}(n)^*$. According to Manakov, a sufficient condition for integrability is that

$$a_{ij} = \frac{p_i - p_j}{q_i - q_j}$$

(which for $p_i = q_i^2$ becomes the physical case above). The limit $n \to \infty$ of the integrable cases on $SO(n)$ was considered in [War].

The geodesic flow on the group $SO(n)$ equipped with an arbitrary left-invariant Riemannian metric is, in general, nonintegrable.

Usually, integrability of an infinite-dimensional Hamiltonian system is related to the existence of two independent Poisson structures forming a so-called Poisson pair, such that the system is Hamiltonian with respect to both structures.

DEFINITIONS 1.14. Assume that a manifold M is equipped with two Poisson structures $\{.,.\}_0$ and $\{.,.\}_1$. They are said to *form a Poisson pair* (or to *be compatible*) if all of their linear combinations $\lambda\{.,.\}_0 + \nu\{.,.\}_1$ are also Poisson structures.

A dynamical system $\dot{x} = v(x)$ on M is called *bi-Hamiltonian* if the vector field v is Hamiltonian with respect to both structures $\{.,.\}_0$ and $\{.,.\}_1$.

REMARK 1.15. The condition on $\{.,.\}_0$ and $\{.,.\}_1$ to form a Poisson pair is equivalent to the identity

$$\sum_{(f,g,h)} \{\{f,g\}_0, h\}_1 + \{\{f,g\}_1, h\}_0 = 0 \tag{1.4}$$

for any triple of smooth functions f, g, h on M, where the sum is taken over all three cyclic permutations of the triple.

In the next theorem we assume, for the sake of simplicity, that M is simply connected and that the Poisson structures $\{.,.\}_0$ and $\{.,.\}_1$ are everywhere nondegenerate, i.e., they are inverses of some symplectic structures on M.

THEOREM 1.16 [GDo]. *Let v be a bi-Hamiltonian vector field with respect to the structures of a Poisson pair $\{.,.\}_0$, $\{.,.\}_1$. Then there exists a sequence of smooth functions H_k, $k = 0, 1, \dots$, on M such that*

(1) *H_0 is a Hamiltonian of the field $v_0 := v$ with respect to the structure $\{.,.\}_0$;*
(2) *the field v_k of the 0-Hamiltonian H_k coincides with the field of the 1-Hamiltonian H_{k+1};*
(3) *the functions H_k, $k = 0, 1, \dots$, are in involution with respect to both Poisson brackets.*

The algorithm for generating the Hamiltonians H_k is called the *Lenard scheme* and is shown in Fig.72.

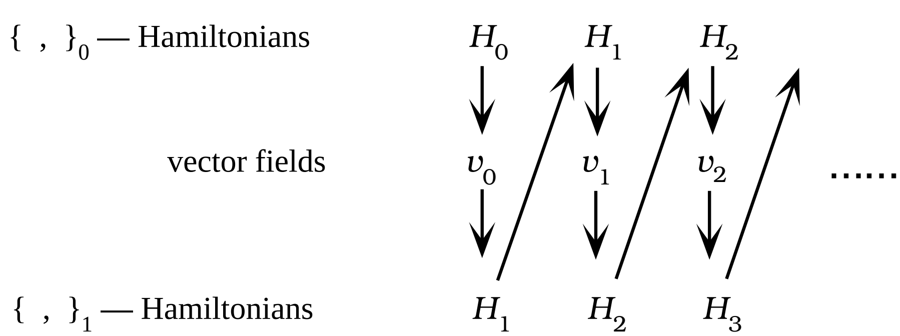

FIGURE 72. Generation of a sequence of the Hamiltonians H_k for a bi-Hamiltonian vector field.

Although this theorem is formulated and proven for the case of nondegenerate brackets only, the procedure is usually applied in a more general context. Namely, let the field $v_0 := v$ be Hamiltonian with Hamiltonian functions H_0 and H_1 relative to the structures $\{.,.\}_0$ and $\{.,.\}_1$, respectively. Consider the function H_1 as the Hamiltonian with respect to the bracket $\{.,.\}_0$ and generate the next Hamiltonian field v_1. One readily shows that the field v_1 *preserves* the Poisson bracket $\{.,.\}_0$, provided that the two brackets form a Poisson pair (a formal application of the identity (1.4)). However, this does not imply, in general, that the field v_1 is *Hamiltonian* with respect to the bracket $\{.,.\}_0$. (Example: The vertical field $\partial/\partial z$ preserves the Poisson structure in $\mathbb{R}^3_{x,y,z}$ given by the bivector field $\partial/\partial x \wedge \partial/\partial y$, but it is not defined by any Hamiltonian function. Every Hamiltonian field for this structure would be horizontal.) If we are lucky, and the field v_1 is indeed Hamiltonian, we continue the process to the next step, and so on.

To apply the technique of Poisson pairs in the Lie-algebraic situation, recall that on the dual space $\mathfrak{g}^*$ to any Lie algebra $\mathfrak{g}$ there exists a natural linear Lie–Poisson structure (see Section I.6)

$$\{f, g\}(m) := \langle [df, dg], m\rangle$$

for any two smooth functions f, g on $\mathfrak{g}^*$, and $m \in \mathfrak{g}^*$. In other words, the Poisson bracket of two linear functions on $\mathfrak{g}^*$ is equal to their commutator in the Lie algebra $\mathfrak{g}$ itself. The symplectic leaves of this Poisson structure are the coadjoint orbits of the group action on $\mathfrak{g}^*$, while the Casimir functions are invariant under the coadjoint action. The following method of constructing functions in involution on the orbits is called the *method of translation of the argument*, and originally appeared in Manakov's paper [Man] to describe the integrable cases of higher-dimensional rigid bodies (see further generalizations in, e.g., [A-G, T-F]).

Fix a point m_0 in the dual space to a Lie algebra. One can associate to this element a new Poisson bracket on $\mathfrak{g}^*$.

DEFINITION 1.17. The *constant Poisson bracket associated to a point* $m_0 \in \mathfrak{g}^*$ is the bracket $\{.,.\}_0$ on the dual space $\mathfrak{g}^*$ defined by

$$\{f, g\}_0(m) := \langle [df, dg], m_0\rangle$$

for any two smooth functions f, g on the dual space, and any $m \in \mathfrak{g}^*$. The differentials df, dg of the functions f, g are taken at a current point m, and, as above, are regarded as elements of the Lie algebra itself.

The brackets $\{.,.\}$ and $\{.,.\}_0$ coincide at the point m_0 itself. Moreover, the bivector defining the constant bracket $\{.,.\}_0$ does not depend on the current point.

The symplectic leaves of the bracket are the tangent plane to the group coadjoint orbit at the point m_0, as well as all the planes in $\mathfrak{g}^*$ parallel to this tangent plane (Fig.73).

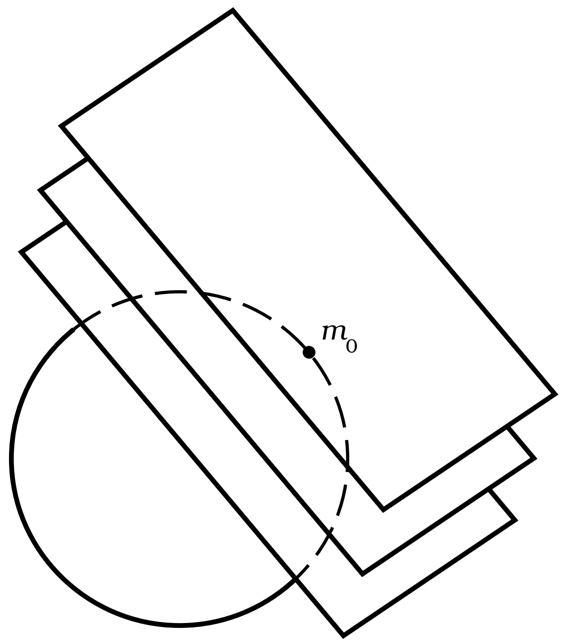

FIGURE 73. Symplectic leaves of the constant bracket are the planes parallel to the tangent plane to the coadjoint orbit at m_0.

PROPOSITION 1.18. *The brackets $\{.,.\}$ and $\{.,.\}_0$ form a Poisson pair for every fixed point m_0.*

PROOF. The linear combination $\{.,.\}_\lambda := \{.,.\} + \lambda\{.,.\}_0$ is a Poisson bracket, being the linear Lie–Poisson structure $\{.,.\}$ translated from the origin to the point $-\lambda m_0$. $\square$

COROLLARY 1.19. *Let $f, g : \mathfrak{g}^* \to \mathbb{R}$ be invariants of the group coadjoint action, and let $m_0 \in \mathfrak{g}^*$. Then the functions $f(m+\lambda m_0)$, $g(m+\nu m_0)$ of the point $m \in \mathfrak{g}^*$ are in involution for any $\lambda, \nu \in \mathbb{R}$ on each coadjoint orbit.*

PROOF. This is an immediate consequence of the fact that Casimir functions for all linear combinations of compatible Poisson brackets are in involution with each other. The latter holds by virtue of the definition of a Poisson pair. $\square$

We leave it to the reader to adjust this Corollary to produce the family (1.3) of first integrals providing the integrability of the higher-dimensional rigid body (see [Man]). Below, we show how this scheme works for the KdV equation.

REMARK 1.20. A Hamiltonian function f and the Poisson structure $\{.,.\}_0$ generate the following Hamiltonian vector field on the dual space $\mathfrak{g}^*$:

$$v(m) = \mathrm{ad}^*_{df}\, m_0, \tag{1.5}$$

where the differential df is taken at the point m. Indeed, for an arbitrary function g one has

$$\{f,g\}_0(m) = \langle \mathrm{ad}_{df}(dg), m_0\rangle = \langle (dg), \mathrm{ad}^*_{df}\, m_0\rangle,$$

and the latter pairing is the Lie derivative $L_v g$ (at the point $m \in \mathfrak{g}^*$) of the function g along the field v defined by (1.5). Hence, this vector field v is Hamiltonian with Hamiltonian function f.

1.C. Integrability of the KdV equation. The existence of an infinite number of conserved charges for the flow determined by the KdV equation was discovered in the late 1960s, and in a sense this discovery launched the modern theory of infinite-dimensional integrable Hamiltonian systems (see [Ma, Miu] for an intriguing historical survey).

The first of the KdV conservation laws were found via calculations with undetermined coefficients, but this method stopped at the 9^{th} invariant. Miura describes in [Miu] how, in the summer of 1966, a rumor circulated that there were exactly 9 conservation laws in this case. Miura spent a week of his summer vacation and succeeded in finding the 10^{th} one. Later code was written computing the 11^{th} law. After that the specialists were convinced that there should be an infinite series of conservation laws.

In this section, we shall see how these laws can be extracted via the recursive Lenard scheme, or equivalently, via Manakov's method of the translation of argument from the preceding section.

The KdV equation is an example of a bi-Hamiltonian system. First, as we discussed in Section 1.A, it is Hamiltonian on the dual space $vir^* = \{(u(dx)^2, c)\}$ of the Virasoro algebra with the quadratic Hamiltonian function

$$-H(u(dx)^2, c) = -\frac{1}{2}\left(\int u^2\; dx + c^2\right)$$

relative to the linear Poisson structure. This Poisson structure is called the *second KdV Hamiltonian structure* and is sometimes referred to as the Magri bracket; see [Mag].

Moreover, one can specify a point in the space vir^* such that the KdV equation will also be Hamiltonian with respect to the constant Poisson structure associated to this point. Namely, let the pair $(u_0(x)(dx)^2, c_0)$ consist of the function $u_0(x) \equiv 1/2$ and $c_0 = 0$.

DEFINITION 1.21. Let F be a function on the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ and $m \in \mathfrak{g}^*$. In the case of an infinite-dimensional space $\mathfrak{g}^*$, the differential $dF|_m$ (regarded as a vector of the Lie algebra itself) is called the *variational derivative* $\frac{\delta F}{\delta m}$, and it is defined by the relation

$$\frac{d}{d\epsilon} F(m+\epsilon w)\Big|_{\epsilon=0} = \langle \, \frac{\delta F}{\delta m}, \ w \, \rangle.$$

For instance, in the case of the Virasoro algebra, a functional F is defined on the set of pairs $(u(x)(dx)^2, c)$. The variational derivative

$$\left(\left(\frac{\delta F}{\delta u} \right) \frac{\partial}{\partial x} \, , \ \frac{\delta F}{\delta c} \right)$$

is the pair consisting of a vector field and a number such that

$$\begin{aligned} \frac{d}{d\epsilon} F((u+\epsilon w)(dx)^2, \ c+\epsilon b)\Big|_{\epsilon=0} &= \langle \ \left(\left(\frac{\delta F}{\delta u} \right) \frac{\partial}{\partial x} \, , \ \frac{\delta F}{\delta c} \right), \ (w(dx)^2, b) \ \rangle \\ &= \int \left(\frac{\delta F}{\delta u}(x) \cdot w(x) \right) dx + \frac{\delta F}{\delta c} \cdot b. \end{aligned}$$

(To specify the class of functionals, one usually considers differential polynomials on vir^*, i.e., integrals of polynomials in u and in its derivatives; see [GDo]).

PROPOSITION 1.22. *(i) The Poisson structure* $\{.,.\}_0$ *associated to the point*

$$\left(\frac{1}{2}(dx)^2, \ 0 \right) \in vir^*$$

sends every Hamiltonian function F on the dual space vir^* to the Hamiltonian vector field on vir^* whose value at a point $(u(dx)^2, c)$ is the pair

$$\left(\left(\frac{\delta F}{\delta u} \right)' (x)(dx)^2, 0 \right).$$

(ii) The Korteweg–de Vries equation is Hamiltonian with respect to the constant Poisson structure $\{.,.\}_0$ with the Hamiltonian function

$$Q(u(dx)^2, c) = \frac{1}{2} \int_{S^1} \left(-u^3(x) + c(u')^2(x) \right) \ dx. \tag{1.6}$$

The Poisson structure $\{.,.\}_0$ is called the *first KdV Hamiltonian structure*; see, e.g., [LeM]. The Hamiltonians $H_2 = H$ and $H_3 = Q$ of the KdV equation with respect to the Poisson pair $\{.,.\}$ and $\{.,.\}_0$ start the series of conservation laws generated by the Lenard iteration scheme. One readily shows that at each step the Hamiltonian functional H_k is a differential polynomial of order k in $u(x)$. Usually, this series of first integrals for the KdV starts with the Hamiltonian function $H_1(u) := \int u(x)\, dx$.

PROOF. Item (i) is a straightforward application of the notion of variational derivative to formula (1.5). Indeed, one obtains the Hamiltonian vector field for a functional F on the space vir^* by freezing the values of $u(x)$ and c as $u_0(x) \equiv 1/2$ and $c_0 = 0$ in (1.2):

$$\mathrm{ad}^*_{(f\,\partial/\partial x, a)}(u_0(dx)^2, c_0) = \left((2f'u_0 + fu_0' + c_0 f''')(dx)^2,\ 0\right) = (f'(dx)^2,\ 0),$$

where $f := \left(\frac{\delta F}{\delta u}\right)$, and $a := \left(\frac{\delta F}{\delta c}\right)$.

(ii) The variational derivative of the functional Q given by (1.6) is

$$\left(\frac{\delta Q}{\delta u}\right) = -\frac{3}{2}u^2 - cu''. \tag{1.7}$$

Indeed, this follows from the equality

$$\frac{d}{d\epsilon}\,\frac{1}{2}\int_{S^1}\left(-(u+\epsilon w)^3 + c\left((u+\epsilon w)'\right)^2\right)\, dx = -\int_{S^1}\left(\frac{3}{2}u^2 + cu''\right)\cdot w\, dx.$$

Then, substituting the variational derivative $f = \delta Q/\delta u$ from (1.7) into (1.2), we get the following Hamiltonian vector field on the dual space vir^*:

$$\begin{cases} \partial_t u = f' = -(\frac{3}{2}u^2 + cu'')' = -3uu' - cu''', \\ \partial_t c = 0, \end{cases}$$

that is, the KdV equation. □

REMARK 1.23. The KdV flow is tangent to the coadjoint orbits of the Virasoro algebra (as is the flow of every Euler equation on the dual space to any Lie algebra). Note that none of the above first integrals of the KdV equation are invariants of the Virasoro coadjoint action, and therefore their meaning is completely different from the Casimir functions of two-dimensional hydrodynamics (cf. Remark I.9.8). The description of the Virasoro orbits (or Casimir functions), besides being evident information on the behavior of KdV solutions, is an interesting question in its own right.

The classification problem for the Virasoro coadjoint orbits is also known as the classification of Hill's operators

$$\left\{\frac{d^2}{dx^2} + u(x) \;\Big|\; u \in C^\infty(S^1)\right\},$$

or of projective structures on the circle, and it has been solved independently in different terms and at different times (see [Kui, LPa, Seg, Ki2]). The orbits are enumerated by one discrete parameter and one continuous parameter. Generalization of this problem to the classification of symplectic leaves of the so-called Gelfand–Dickey brackets, which are certain natural Poisson brackets on differential operators of higher order on the circle, as well as the relation of this problem to enumeration of homotopy types of nonflattening curves on spheres, is given in [OK2] (see also [KhS, Sha, E-K] for relevant problems).

The Lenard scheme generates a series of Hamiltonian equations called the KdV hierarchy. A similar construction exists for higher KdV hierarchies, which are Hamiltonian flows on coefficients of differential operators of higher order on the circle; see [Adl, GDi, SeW, PrS].

1.D. Digression on Lie algebra cohomology and the Gelfand–Fuchs cocycle. The theory of Lie algebra cohomology is an algebraic generalization of the following geometric construction from Lie group theory.

Let G be a compact connected simply connected Lie group equipped with a two-sided invariant metric (a typical example: the group of unit quaternions $SU(2) \approx S^3$ or the group $SU(n)$ of unitary matrices with unit determinant).

One can calculate the cohomology groups and, furthermore, their exterior algebra for the group G as follows.

THEOREM 1.24. *The exterior algebra of two-sided invariant differential forms on G is isomorphic to the cohomology exterior algebra of the manifold G. The isomorphism is defined by assigning to each differential form its cohomology class.*

The proof is based on two facts: *i*) every two-sided invariant form is closed (see (1.8) below); *ii*) every closed 2-form is cohomologous to a two-sided invariant form, namely, to the average value of all its shifts.

This classical theorem reduces all calculations to a purely algebraic consideration in terms of the commutator of the Lie algebra. Indeed, any one-sided invariant form is determined by its value on the Lie algebra. The exterior differential of this form is also an invariant form, and hence it is also determined by its value on the Lie algebra.

Given an invariant 1-form ω, its differential is a 2-form defined at the identity by the following *Maurer–Cartan formula*:

$$(d\omega)(\xi, \eta) = \mp\omega([\xi, \eta])$$

(the sign is defined by whether the form is left- or right-invariant). This formula allows one to write algebraically the closedness condition (and to verify that it coincides with the condition of two-sided invariance). More generally, one has the following

THEOREM 1.25. *Given a one-sided invariant n-form ω whose value on the Lie algebra $\mathfrak{g}$ is $\omega(\xi_1, \dots, \xi_n)$, $\xi_i \in \mathfrak{g}$, its exterior differential is the invariant $(n+1)$-form $d\omega$ whose value on the Lie algebra is*

$$d\omega(\xi_0, \dots, \xi_n) = \pm \sum_{0 \le i < j \le n} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_n), \tag{1.8}$$

where the sign is determined by whether the form ω is left- or right-invariant, and the hat $\hat{}$ means that the corresponding vector is missing.

EXAMPLE 1.26. For a 1-form ω we have $\mp(d\omega)(\xi, \eta) = \omega([\xi, \eta])$. The differential of a 2-form ω is given by the formula

$$\mp d\omega(\xi, \eta, \zeta) = \omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta).$$

The algebraic generalization mentioned above, which allows one to avoid calculations on the Lie group, proceeds as follows.

DEFINITION 1.27. The *cohomology complex of a Lie algebra* $\mathfrak{g}$ is the complex

$$\Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \xrightarrow{d_2} \dots,$$

where Ω^n is a vector space of exterior n-forms on the Lie algebra $\mathfrak{g}$, and the differential d_n is given by formula (1.8).

The n^{th}-cohomology group (or space) of the Lie algebra $\mathfrak{g}$ is the vector space

$$\frac{\operatorname{Ker} d_n : \Omega^n \to \Omega^{n+1}}{\operatorname{Im} d_{n-1} : \Omega^{n-1} \to \Omega^n},$$

that is, the quotient of the space of all closed n-forms over the subspace of all exact forms. The elements of the space $\operatorname{Ker} d_n : \Omega^n \to \Omega^{n+1}$ are called *n-cocycles*, while the elements of the subspace $\operatorname{Im} d_{n-1} : \Omega^{n-1} \to \Omega^n$ are *n-coboundaries*, or the cocycles cohomologous to zero.

REMARK 1.28. The fact that $d_n d_{n-1} = 0$ readily follows from formula (1.8) and the Jacobi identity. Geometrically, it means that the boundary of any simplex boundary (say, for a triangle or a tetrahedron) is zero.

EXAMPLE 1.29. Let $a \in \mathfrak{g}^*$ be any element of the dual space to the Lie algebra $\mathfrak{g}$ and set

$$\omega(\xi, \eta) := a([\xi, \eta]).$$

This function is a 2-cocycle, and even a 2-coboundary, on $\mathfrak{g}$.

For instance, for the Lie algebra $\mathfrak{g} = \mathrm{Vect}(S^1)$ of vector fields on the circle, and the point $a = (dx)^2/2 \in \mathrm{Vect}^*$, we get the following 2-cocycle cohomologous to zero:

$$\omega(g(x)\frac{\partial}{\partial x}, h(x)\frac{\partial}{\partial x}) = \frac{1}{2}\int_{S^1}(g'h - gh')\,dx = \int_{S^1} g'h\,dx.$$

REMARK 1.30. The 2-cocycle on $\mathrm{Vect}(S^1)$ defining the Virasoro algebra is of a more subtle nature, and is related to the projective structures on the circle (we follow [Tab3] below).

Note that every 2-cocycle on a Lie algebra is a linear map from this Lie algebra to its dual. We construct a natural map from the Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on the circle to its dual, the space of quadratic differentials $\mathrm{Vect}(S^1)^* = \{u(x)(dx)^2\}$, fixing first a projective structure on S^1.

Consider four points $x, x+t, x+2t, x+3t$ in an affine coordinate system, where t is very small. A diffeomorphism $f : S^1 \to S^1$ sends them to four points whose cross ratio is of order t^2 (not of order t!). The principal part of this cross ratio at the point x is (up to a constant factor) the *Schwarzian derivative* $S(x)$ of the diffeomorphism f:

$$S(x) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The corresponding quadratic differential $S(x)(dx)^2$ is independent of the (projective) choice of the coordinate x and measures the "nonprojectivity" of the map f. It is a cocycle of the diffeomorphism group of the circle with values in the quadratic differentials.

Now consider the Lie algebra of vector fields. Let f be a diffeomorphism of S^1 close to the identity, $f(x) = x + sv(x)$, where s is small, and let $v(x)d/dx$ be a vector field of the algebra $\mathrm{Vect}(S^1)$. Then $S(x)$ is $sv'''(x) + O(s^2)$, where $'$ stands for d/dx. Neglecting higher-order terms, we get the desired mapping, which sends the field $v(x)\,d/dx$ to the quadratic differential $v'''(x)(dx)^2$.

For the angular coordinate q on the circle, the field $w(q)\partial/\partial q$ is sent to $(d^3w/dq^3 + dw/dq)(dq)^2$. This can be deduced with virtually no calculations from the description of the Lie algebra spanned by the generators of the projective group: $\partial/\partial q, (\cos q)\partial/\partial q, (\sin q)\partial/\partial q$ (accompanied by the change of variables normalizing the first coefficient).

We have now obtained the cocycle whose value on two vector fields $v(q)\partial/\partial q$ and $w(q)\partial/\partial q$ is given by the expression

$$\int_{S^1}(vw''' + vw')dq,$$

where $'$ is the derivative d/dq along the angular coordinate. The second term is cohomologous to zero, as we have seen above (see Example 1.29). Integrating by parts the first monomial, we obtain the Gelfand–Fuchs cocycle. Thus the Gelfand–Fuchs cocycle (and hence, the Virasoro algebra) measures the deformation of the projective structure on $S^1 = \mathbb{R}P^1$ by diffeomorphisms.

§2. Equations of gas dynamics and compressible fluids

The evolution of a compressible fluid naturally extends the motion of an ideal incompressible fluid: Instead of the incompressibility condition, one assumes now that the pressure term of the Euler equation is determined by the intrinsic degrees of freedom of the fluid. Usually these internal parameters are the density and entropy of the fluid.

2.A. Barotropic fluids and gas dynamics. Barotropic fluids (or gas dynamics) are simplified models of compressible fluids in which the only intrinsic degree of freedom is the density of the fluid or of the gas.

DEFINITION 2.1. A (compressible) fluid is *barotropic* (or *isentropic*) if the pressure term in the evolution equation is defined solely by the fluid's density. The fluid motion is described by the following system of equations:

$$\left\{ \begin{array}{l} \rho\,\dot v = -\rho\,(v,\nabla)v - \nabla h(\rho), \\ \dot\rho + \ \mathrm{div}(\rho v) = 0, \end{array} \right. \tag{2.1}$$

where v and ρ are respectively the velocity vector field and the density function of the fluid. The pressure function $h(\rho)$ depends on the physical properties of the fluid, and is assumed to be given. For instance, the equation of *gas dynamics* on a line corresponds to the choice $h(\rho) = \rho^\nu$ (for the motion of air $\nu \approx 1.4$).

Equations (2.1) make sense for an arbitrary Riemannian manifold M, provided that (v, ∇) stands for the covariant derivative along the field v (see Chapter I) and the divergence is taken with respect to the volume form induced by the metric. The first equation is similar to the Euler dynamics of an incompressible fluid, but the velocity field $v \in \mathrm{Vect}(M)$ is no longer divergence-free. The second equation is the continuity equation for the function ρ. Thus the phase space of the system consists of all pairs $\{(v, \rho) \mid v \in \mathrm{Vect}(M),\ \rho \in C^\infty(M)\}$.

The configuration space of the barotropic fluid on a manifold M is the group

$$P := \mathrm{Diff}\ M \ltimes C^\infty(M),$$

defined as the semidirect product of the group of *all* diffeomorphisms of M and the space $C^\infty(M)$ of all smooth functions on the manifold considered (see [HMRW] for a derivation of the equation via reduction in the Lagrangian representation).

Recall (cf. Section I.10 on the magnetic extension of a group) that the group structure on the *semidirect product* P is defined by the formula

$$(\varphi, a) \circ (\psi, b) = (\varphi \circ \psi, \psi_* a + b),$$

where $\psi_* a$ is the natural action of the diffeomorphism ψ on the function a: $\psi_* a = a(\psi^{-1}(x))$. The commutator in the corresponding Lie algebra

$$\mathfrak{p} = \mathrm{Vect}(M) \ltimes C^\infty(M)$$

is also defined via the semidirect product of the Lie algebras involved:

$$[(v, a), (w, b)] = ([v, w], L_w a - L_v b),$$

where $\varphi, \psi \in \mathrm{Diff}(M)$; $a, b \in C^\infty(M)$; $v, w \in \mathrm{Vect}(M)$; and $[v, w]$ denotes the commutator, i.e., minus the Poisson bracket, of the two vector fields on M ($[v, w] = -\{v, w\}$); see Section I.2.

REMARK 2.2. The Lie algebra $\mathfrak{p} = \mathrm{Vect}\ (M) \ltimes C^\infty(M)$ has a simple geometric meaning: It is the Lie algebra of differential operators of the first order on M. Such an operator is always the sum $L_v + \rho$, where L_v is the operator of the Lie derivative along the field v on M, and ρ is regarded as the operator of the 0^{th} order, namely the operator of multiplication by the function ρ.

PROPOSITION 2.3 [GS1, MRW, Nov2]. *The equation of a barotropic fluid is a Hamiltonian equation on $\mathfrak{p}^*$ with respect to the linear Poisson–Lie structure and Hamiltonian function*

$$H(v, \rho) = -\int_M \left(\frac{1}{2}\rho v^2 + \Phi(\rho)\right)\ \mu,$$

where $\frac{d}{d\rho}\Phi(\rho) = h(\rho)$.

REMARK 2.4. In contrast with the Euler dynamics (both of the rigid body and the ideal fluid), the total energy of a barotropic fluid is not a quadratic form, and it no longer has the meaning of a Riemannian metric on an appropriate group. However, one still has a variational problem on the cotangent space T^*P of the Lie group P, such that its extremals are the solutions of Equations (2.1). The group-theoretical interpretation and all the Hamiltonian properties of the equations described earlier will be valid for the barotropic fluid (or gas dynamics) with merely cosmetic changes.

Note that for the one-dimensional manifold $M = \mathbb{R}$ or S^1 the equations of gas dynamics (2.1) for the algebra $\mathfrak{p} = \text{Vect}\,(M) \ltimes C^\infty(M)$ are integrable (see Section 3.B). Note that this Lie algebra has three independent nontrivial 2-cocycles (one of them being the Gelfand–Fuchs cocycle of the Virasoro algebra).

PROOF SKETCH OF PROPOSITION 2.3. One readily verifies the following

PROPOSITION 2.5. *The dual to the space of vector fields Vect(M) on an n-dimensional manifold M is the space $\Omega^1(M) \otimes_f \Omega^n(M)$, where $\otimes_f$ means that the tensor product is taken over functions on M.*

In other words, elements of $\Omega^1(M) \otimes_f \Omega^n(M)$ are pairs $\beta \otimes \mu$, $\beta \in \Omega^1$, $\mu \in \Omega^n$, and we do not distinguish between the pairs $f\beta \otimes \mu$ or $\beta \otimes f\mu$ for all functions f.

The pairing between $v \in \text{Vect}(M)$ and $\bar{\beta} = \beta \otimes_f \nu \in \Omega^1(M) \otimes_f \Omega^n(M)$ is as follows:

$$\langle v,\ \beta \otimes \mu \rangle = \int_M (i_v \beta)\ \mu$$

(the vector field v is contracted with the 1-form β, and the obtained n-form $(i_v\beta)\mu$ is integrated over M). That this choice of dual space is natural is due to the (readily verified) fact that the coadjoint action of the Lie algebra Vect(M) is geometric:

$$\text{Ad}^*_\varphi(\beta \otimes_f \mu) = \varphi_*\beta \otimes_f \varphi_*\mu; \tag{2.2}$$

i.e., it is given by a change of coordinates in both of the 1-form β and the n-form μ.

In case of the Lie algebra $\mathfrak{p}^* = \text{Vect}\,(M) \ltimes C^\infty(M)$, elements of the corresponding dual space $\mathfrak{p}^*$ are pairs $(\bar{\beta}, \theta)$, where $\bar{\beta} \in \Omega^1(M) \otimes \Omega^n(M)$ and $\theta \in \Omega^n(M)$. We leave it to the reader to check that the coadjoint action of an element $(\varphi, a) \in \text{Diff}\,(M) \ltimes C^\infty(M)$ is

$$\text{Ad}^*_{(\varphi,a)}(\bar{\beta}, \theta) = (\varphi_*\bar{\beta} + da \otimes \varphi_*\theta, \varphi_*\theta)$$

(see, e.g., [MRW]).

Once the coadjoint action is known, it is routine to find the variational derivative of the Hamiltonian function (see Definition 1.21) and the corresponding Euler equation, according to the general rule

$$\dot{m} = \text{ad}^*_{\delta H/\delta m} m.$$

$\square$

It turns out that the equations of barotropic fluid or gas dynamics have plenty of similarities with the incompressible case (e.g., the structure of conservation laws in even and odd dimensions is the same). This phenomenon is due to "incompressibility" of the barotropic fluid in coordinates moving with density.

Namely, let μ be the volume form on M induced by the Riemannian metric. Assign to the density function ρ the *density n-form* $\theta := \rho\mu \in \Omega^n(M)$.

THEOREM 2.6 [KhC]. *The barotropic fluid equations (2.1) admit first integrals*

$$I(v) := \int_M u \wedge (du)^m \quad \text{and} \quad I_f(v) := \int_M f\left(\frac{(du)^m}{\theta}\right)\theta$$

according to the parity of $n = \dim(M)$ ($n = 2m+1$ and $n = 2m$, respectively), where the vector field v and the 1-form u are related by means of the metric, and $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function.

The integrals above can be read off from (I.9.2) if one replaces the n-form μ by the density form $\theta = \rho\mu \in \Omega^n(M)$, with ρ being the density function. We shall show that these invariants are Casimir functions on the dual space to the Lie algebra $\mathfrak{p}$. Another (though trivial) conservation law of the same nature is given by the total mass of the fluid, that is, by the integral of the density form θ over the manifold M. The Hamiltonian function H is also a first integral of the equation, but it is not a Casimir function.

REMARK 2.7. The equations of a barotropic fluid with a nearly constant density ρ approximate the Euler equation of an incompressible fluid [Eb]. One can think of the condition of incompressibility within the general framework of systems with constraints (see [Arn16]). A dynamical system confined to a submanifold can be regarded as a subsystem in an ambient manifold with a strong "returning force" directed towards the submanifold.

For instance, consider a point mass that is constrained to move in the unit circle in the plane without forces. It can be thought of as a point attached to the center by a rigid rod. The latter is the limiting case of a point attached to the center by an elastic spring, where the elasticity coefficient of the spring tends to infinity, and in equilibrium the spring has length 1. While a point on a rod is confined to a circle, a point on a spring oscillates out from this circle. In the limit, the position and velocity of the "elastic pendulum" tend to those of the "rigid pendulum," but the acceleration does not.

Similarly, for the group of all diffeomorphisms of a manifold, one can introduce a "returning force" directed towards the subgroup of all volume-preserving diffeomorphisms (see [Eb]). Then the velocity and its first partial derivatives of a barotropic (weakly compressible) fluid tend to those of an ideal fluid. In particular, the above conservation laws for a barotropic fluid become the conserved charges (I.9.2) for an ideal fluid as $\rho \to 1$. Indeed, their explicit form involves only the fluid velocity v and its first derivatives $\partial v/\partial x$ (or the corresponding 1-form u and its differential du, where u is related to v by means of the Riemannian metric, i.e., without any differentiation). The conservation laws do not contain time derivatives of the velocity (i.e., do not contain the acceleration), and hence the limiting procedure is harmless for them.

Proof of Theorem 2.6. A heuristic argument is based on the fact that the density ρ is transported by the flow and the fluid is incompressible with respect to the new volume form θ (depending on time and on the initial conditions). Thus, we can apply Theorem I.9.2, whose assumptions require no relation between the metric and the volume form.

More precisely, the trajectories of the barotropic fluid equations are tangent to the orbits of the coadjoint representation of the group $P = \text{Diff}\, M \ltimes C^\infty(M)$, and the statement follows from

Proposition 2.8. *The functional*

$$I(\bar{\beta}, \theta) = \int_M u \wedge (du)^m$$

in the case of an odd $n = 2m+1$ and the functionals

$$I_f(\bar{\beta}, \theta) = \int f\left(\frac{(du)^m}{\theta}\right)\theta$$

in the case of an even $n = 2m$ (where the 1-form u is defined by $u := \bar{\beta}/\theta \in \Omega^1(M)$) are invariant under the coadjoint action of the group P on the dual space $\mathfrak{p}^*$.

Proof. Note that the ratio $u = \bar{\beta}/\theta$ has the geometric meaning of a differential 1-form (see (2.2)). Explicitly, one has the following action on this form:

$$\begin{aligned}\text{Ad}^*_{(\varphi,a)} u = \text{Ad}^*_{(\varphi,a)}\left(\frac{\bar{\beta}}{\theta}\right) &= \frac{\varphi_*\bar{\beta} + da \otimes \varphi_*\theta}{\varphi_*\theta} \\ &= \varphi_*\left(\frac{\bar{\beta}}{\theta}\right) + da = \varphi_* u + da,\end{aligned}$$

i.e., the 1-form u is transported by the flow modulo da, the differential of a function. Hence, the P-action on the coset $[u] \in \Omega^1/d\Omega^0$ of 1-forms on M, as well as on the n-form $\theta \in \Omega^n$, is geometric: It is nothing but a change of variables. Now Proposition

2.7 (as well as Proposition I.9.3) follows from the coordinate-free definition of the functionals I and I_f. □

To complete the proof of the theorem, recall that the inertia operator $\tilde{A} : \mathfrak{p} \to \mathfrak{p}^*$ defined by the Riemannian metric on the manifold M is the map $(v, \rho) \mapsto (u \otimes \theta, \theta)$, where $\theta = \rho\mu$ is the density form on M, and the 1-form u is obtained from the velocity v by the metric "lifting indices." Theorem 2.6 follows. □

2.B. Other conservative fluid systems. We refer to the surveys [GS2, HMRW, MRW, Nov2, DKN, VlM] for extended treatments of the Hamiltonian formalism related to the variety of different types of fluids, and in particular for applications of the techniques of semidirect products and Hamiltonian reductions.

We mention just a few examples:

— A general inviscid compressible fluid is regarded as having two internal degrees of freedom: The pressure term is defined by both the mass density and the entropy (unlike the barotropic case with density only); see [Nov2]. The corresponding Euler equation is related to the semidirect product Lie algebra
$$\tilde{\mathfrak{p}} := \text{ Vect } (M) \ltimes [C_\epsilon^\infty(M) \oplus C_\rho^\infty(M)].$$

— Anisotropic liquids (say, superfluid ^{4}He) require the introduction of a vector field for the internal degrees of freedom [Nov2, KhL].

— Magnetohydrodynamics in a compressible perfectly conducting fluid is constructed as the semidirect product of the magnetic extension of the diffeomorphism group (considered in Section I.10) with the space of smooth functions on the manifold M; see [MRW].

— The motion of an ideal incompressible fluid with a free boundary does not have an explicit group structure: One cannot compose two flow transformations with different shapes of the boundary. The Hamiltonian formalism for this problem, as well as the Hamiltonian form for the equations of a liquid drop with surface tension, is presented in [LMMR].

— A rigid body in a fluid is described by the Kirchhoff equations in $\mathbb{R}^6$ (see Section I.10). However, the whole "body–fluid" system is already an infinite-dimensional system. The body floating in the fluid is described by its impulse and angular momentum, while the fluid can be regarded as an infinite-dimensional system of the above type (having one fixed boundary component and the other a "free" one). A fluid filling a cavity M in a body is another, similar, system. Its dynamics is associated to the semidirect product of the group $E(3)$ (the motion of the body) and $S\,\mathrm{Diff}(M)$ (the motion of the fluid filling the cavity). See [VlI] for the stability analysis corresponding to the systems of both types.

— Various equations related to two-dimensional hydrodynamics manifest some features of integrability. For instance, the Kadomtsev–Petviashvili equation $(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$ is an integrable infinite-dimensional Hamiltonian system related to shallow water.

— The equations of infinite conductivity (or those of the β-plane in meteorology: $\Delta\psi_t + \beta\psi_x + \{\psi, \Delta\psi\} = 0$) differ from the standard incompressible 2D or 3D hydrodynamics by a Coriolis-type term; see [Fey].

— The equation $\{\psi^s + cy, \Delta\psi^s + \beta y\} = 0$ for steady waves in two dimensions, which is obtained from the β-plane equation by substituting $\psi(x, y, t) = \psi^s(x-ct, y)$, admits interesting solutions of steadily traveling dipole vortices [LaR] (see Section I.11.A for $\beta = 0$).

— Many dynamical systems on the sine-algebra, being the "quantum" version of the algebra of Hamiltonian fields on the two-torus (see Remark I.11.6), are described in [HOT].

— General Poisson brackets of hydrodynamic type [D-N, DKN] provide a general Hamiltonian formalism for first-order quasilinear equations on manifolds. The properties of these brackets impose very restrictive conditions on the Riemannian structure of the underlying manifold.

One more advantage of the Hamiltonian approach is a simple geometric interpretation of the so-called *Clebsch variables* in many physically interesting systems. These variables appeared in a hydrodynamical setting as a set of an excessive number of coordinates (with additional constraints between them) in which the Euler equation acquires the canonical Hamiltonian form; see [Lam]. A general framework for symplectic (or "Clebsch") variables from the Poisson point of view can be found in [M-W] (see also [Zak, MRW]).

Definition 2.9 [M-W, MRW]. If P is a Poisson manifold, then *symplectic variables for* P is a map $J : M \to P$ of a symplectic manifold M into P that respects the Poisson brackets (i.e., the pullback of the Poisson bracket of two functions f, g on P is the Poisson bracket on M of their pullbacks $f \circ J$, $g \circ J$). Any canonical symplectic coordinates on M are said to be *canonical coordinates* on the Poisson manifold P.

A Hamiltonian function $H : P \to \mathbb{R}$ determines a Hamiltonian function on M by $H_M := H \circ J$, and the integral curves of the "canonical" Hamiltonian system on M with the Hamiltonian H_M cover those for the Poisson "noncanonical" Hamiltonian system on P.

EXAMPLE 2.10. The construction of the manifold M and the map J in the case of the dual space $P=\mathfrak{g}^*$ to an arbitrary Lie algebra $\mathfrak{g}$ equipped with the Lie–Poisson bracket is very explicit. The symplectic manifold becomes the cotangent bundle $M=T^*G$ to the Lie group G, while the map J is the left shift L_g^* of any covector $\xi\in T_g^*G$ at a point $g\in G$ to the cotangent space at the identity: $T_e^*G=\mathfrak{g}^*$. The natural coordinates (p,q) in the cotangent bundle T^*G are canonical for $\mathfrak{g}^*$, since the symplectic structure has the form $dp\wedge dq$.

A linear version of the variables on T^*G is the set of canonical coordinates on $\bar{M}=\mathfrak{g}^*\oplus\mathfrak{g}$ with the map

$$\bar{J}:\ \mathfrak{g}^*\oplus\mathfrak{g}\to\mathfrak{g}^*$$

such that $(p,q)\mapsto \operatorname{ad}_q^* p$. We refer to [M-W, MRW, Zak] for a detailed description and numerous applications of this construction of Clebsch variables to dynamical systems and their conservation laws.

2.C. Infinite conductivity equation. The infinite conductivity equation possesses many properties inherent in ideal hydrodynamics. Its relationship to the equation of an incompressible fluid is due to the fact that at a high density, an electron gas is similar to a fluid. Indeed, the repelling of particles in electron clusters makes the gas incompressible.

DEFINITION 2.11 (SEE, E.G., [Fey]). The equation of (nonrelativistic) *infinite conductivity* in a domain of $\mathbb{R}^3$ is

$$\dot{v}=-(v,\nabla)v-v\times\mathbf{B}-\nabla p, \tag{2.3}$$

where v denotes a divergence-free velocity field of the electron gas, $\mathbf{B}$ is a constant in time (but not in space) external divergence-free magnetic field, and the symbol $\times$ stands for the cross product in $\mathbb{R}^3$. One can define an analogue of this equation on an arbitrary Riemannian manifold M with volume form μ.

PROPOSITION 2.12 [KhC]. *The infinite conductivity Equation (2.3) is equivalent to the following Hamiltonian equation on the dual space $S\operatorname{Vect}(M)^*=\Omega^1(M)/d\Omega^0(M)$ to the Lie algebra of divergence-free vector fields $S\operatorname{Vect}(M)$:*

$$\frac{\partial[u]}{\partial t}=-L_v[u+\alpha]. \tag{2.4}$$

Here the 1-form u is related to the vector field v by means of the metric inertia operator, $[u]\in\Omega^1(M)/d\Omega^0(M)$ is the coset of the 1-form u, and α is a 1-form whose differential $d\alpha$ obeys the identity $d\alpha=-i_{\mathbf{B}}\mu$.

PROOF. The proof follows just as in the ideal incompressible case considered in Chapter I (see Equation (I.7.11)). The form α defined by $d\alpha=-i_{\mathbf{B}}\mu$ (up to the differential of a function) is precisely chosen to fit the term $v \times \mathbf{B}$ with the cross product in (2.3).

The infinite conductivity Equation (2.3) is Hamiltonian, with the Hamiltonian function being (minus) the quadratic energy form shifted away from the origin of $S\,\mathrm{Vect}(M)^*$:

$$-H([u]) = -\frac{1}{2}\int_M (u+\alpha, u+\alpha)\ \mu.$$

The Euler equation corresponding to the latter function has the form

$$\frac{\partial [u+\alpha]}{\partial t} = -L_v[u+\alpha],$$

which is equivalent to (2.4). Indeed, the field $\mathbf{B}$ is constant in time, and hence

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial \alpha}{\partial t} = 0.$$

□

COROLLARY 2.13. *The infinite conductivity Equation (2.3) has either at least one or infinitely many first integrals, according to the parity of* $n = \dim(M)$. *The integrals are given by* $I(v)$ *and* $I_f(v)$ *in formula (I.9.2) with* u *replaced by* $u+\alpha$ *and where the 1-form* α *is as defined above.*

REMARK 2.14. The equation of infinite conductivity (and its generalization to an n-dimensional manifold M) can be regarded as the Euler equation on the central extension of the Lie algebra of divergence-free vector fields on M [Rog, Ze2]. The corresponding two-cocycle, extending the Lie algebra of divergence-free vector fields $S\,\mathrm{Vect}(M)$, is the Lichnerowicz 2-cocycle [Lich]: For any closed 2-form β on M,

$$c_\beta(v,w) = \int_M (i_w i_v \beta)\mu,$$

cf. Remark I.11.6 on the extension of the sine-algebra and the algebra of Hamiltonian vector fields on a two-dimensional torus.

§3. Kähler geometry and dynamical systems on the space of knots

Infinite-dimensional spaces of curves appear in the hydrodynamical setting as certain special "low-dimensional" coadjoint orbits of the diffeomorphism group of $\mathbb{R}^3$. This point of view connects many seemingly unrelated symplectic and Poisson varieties and dynamical systems on them.

3.A. Geometric structures on the set of embedded curves. Consider the *space $\mathcal{C}$ of smooth embedded nonparametrized oriented closed curves* (or the *space of knots*) in Euclidean three-dimensional space $\mathbb{R}^3$. It can be thought of as the set of all smooth maps $\gamma : S^1 \to \mathbb{R}^3$ of the circle into $\mathbb{R}^3$ such that γ is an immersion ($\gamma'(x) \neq 0$ for all $x \in S^1$), γ has no double points, and where any two maps with the same image are indistinguishable:

$$\mathcal{C} = \{\gamma : S^1 \to \mathbb{R}^3 \mid \gamma'(x) \neq 0 \ \forall x \in S^1,\ \gamma(x) = \gamma(y) \text{ iff } x = y\}/\gamma \sim (\gamma \circ \phi).$$

Here ϕ runs over all diffeomorphisms of the circle S^1.

Connected components of $\mathcal{C}$ are the classes of equivalent (oriented) knots. We will call two knots *equivalent* if there is an isotopy of the ambient space $\mathbb{R}^3$ sending one of the knots into the other. Locally constant functions on $\mathcal{C}$ are called the *knot invariants*.

The space of knots $\mathcal{C}$ can be equipped with a natural symplectic structure. Consider an embedded curve $\gamma = \gamma(S^1) \subset \mathbb{R}^3$. A tangent vector v to $\mathcal{C}$ at γ is an infinitesimal variation of the curve γ, that is, a normal vector field attached to $\gamma(S^1)$. In parametrized form the vector $v(x)$ is orthogonal to $\gamma'(x)$ in $\mathbb{R}^3$ for all $x \in S^1$.

Definition 3.1. The *(Marsden–Weinstein) symplectic structure* on the space of knots is the 2-form β on $\mathcal{C}$ whose value on the pair of elements $u, v \in T_\gamma\mathcal{C}$ is the oriented volume of the following collar along the curve γ. At every point $\gamma(x)$ the vectors $u(x)$ and $v(x)$ span a parallelogram, and the collar is the union of these parallelograms along $\gamma \subset \mathbb{R}^3$; see Fig.74.

For a chosen parameter $x \in S^1$ one has

$$\beta(v, w) = \int_{S^1} \operatorname{vol}(\gamma'(x),\ v(x),\ w(x))\, dx,$$

where vol is the volume of the parallelepiped spanned by the three vectors. The integral clearly does not depend on the parametrization.

Note that we do not need the Euclidean structure but only the volume form in $\mathbb{R}^3$. The definition can be easily generalized to an arbitrary three-dimensional manifold with a volume form. Moreover, for manifolds of any dimension $n \geq 2$ the same definition gives the symplectic structure on the space of submanifolds of codimension 2 (i.e., of dimension $n - 2$).

The symplectic structure described above has a hydrodynamical meaning. It is based on the fact that every connected component of the space $\mathcal{C}$ (i.e., every

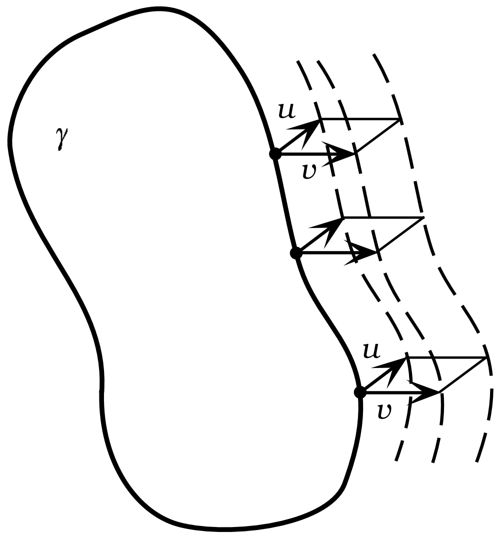

FIGURE 74. The value of the symplectic structure on two variations of a knot is the volume of the collar spanned by the variations.

isotopy class of knots) can be viewed as a special coadjoint orbit of the group of volume-preserving diffeomorphisms of $\mathbb{R}^3$.

DEFINITION 3.2. Let γ be a knot in $\mathbb{R}^3$. Then it defines the *functional* ℓ_γ on divergence-free vector fields in the space: The value of ℓ_γ on a field v is the flux of the field v across any oriented surface in $\mathbb{R}^3$ bounded by the contour γ (such an embedded surface σ is called a Seifert surface).

PROPOSITION 3.3. *The knot functional ℓ_γ is well-defined on divergence-free vector fields; i.e., its value does not depend on the choice of the surface σ such that $\partial\sigma = \gamma$.*

PROOF. The difference between the fluxes of a field v through two surfaces with the same boundary γ is the flux of v across a closed surface. The latter vanishes by virtue of the divergence-free property of the field v. □

REMARK 3.4. We now relate the functional $\ell_\gamma \in S\,\mathrm{Vect}(\mathbb{R}^3)^*$ to another description of the dual space as the quotient $S\,\mathrm{Vect}(\mathbb{R}^3)^* = \Omega^1(\mathbb{R}^3)/d\Omega^0(\mathbb{R}^3) = Z^2(\mathbb{R}^3)$ of all 1-forms on $\mathbb{R}^3$ modulo exact 1-forms, or as the space of all closed 2-forms. The exterior derivative d takes a coset of 1-forms (an element of $\Omega^1/d\Omega^0$) to a closed 2-form (an element of Z^2) without any loss of information, since $H^1(\mathbb{R}^3) = 0$; see Corollary I.7.9.

The curve γ is identified with a singular 2-form ω_γ in $\mathbb{R}^3$ supported on γ. It is a δ-type form whose integrals over any piece of a two-dimensional surface vanish, unless the piece intersects the curve. In the latter case, the integral equals the algebraic number of the intersection points, where the points are counted according to orientation determined by the orientation of the curve γ and the orientation of the piece at every point of intersection.

The 2-form ω_γ is closed, which corresponds to the closedness of the curve γ itself. Thus ω_γ belongs to $Z^2(\mathbb{R}^3)$ (more precisely, it is a so-called *De Rham current*, and it belongs to a certain closure of the space of smooth closed 2-forms; see [DeR]). To represent the closed (and hence, exact) 2-form ω_γ on $\mathbb{R}^3$ as an element of the quotient $\Omega^1/d\Omega^0$, we have to take d^{-1} of it. A 1-form $d^{-1}\omega_\gamma$ is not uniquely defined, and it can be thought of as the δ-type 1-form supported on a Seifert surface σ of the curve γ (Fig.75). The coset of such a 1-form u_σ belongs to (a certain closure of) the space $\Omega^1/d\Omega^0$.

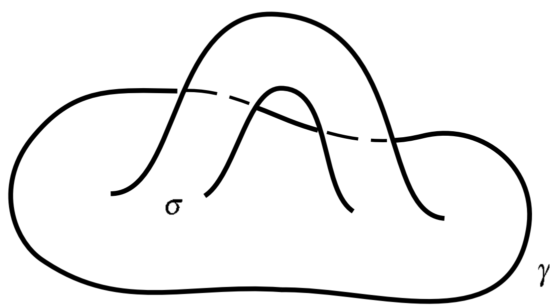

FIGURE 75. The 2-form ω_γ is supported on the curve γ. The 1-form $d^{-1}\omega_\gamma$ is supported on a Seifert surface σ.

PROPOSITION 3.5. *The pairing of the 1-form u_σ with a divergence-free vector field v, according to the rules of Chapter I (see formula (I.7.7)), coincides with the flux of the field v across the surface σ.*

PROOF. Let μ be a volume form in the space $\mathbb{R}^3$. Then the pairing of the (coset of the) 1-form u_σ and a divergence-free field v is

$$\langle [u_\sigma],\ v\rangle = \int_{\mathbb{R}^3} (i_v u_\sigma)\ \mu = \int_{\mathbb{R}^3} u_\sigma \wedge i_v\mu = \int_\sigma i_v\mu.$$

The last integral is a coordinate-free expression for the flux of v across σ. □

We have identified knots with certain points in the dual space $S\,\mathrm{Vect}(\mathbb{R}^3)^*$. The coadjoint action of volume-preserving diffeomorphisms on knots is geometric, and hence all knots isotopic to a given one constitute a coadjoint orbit. The same consideration is applicable to links as well. Thus, the classification of knot invariants, though difficult enough by itself, becomes part of a much more complicated question on classification of all Casimir functions for the group of volume-preserving diffeomorphisms of the space $\mathbb{R}^3$ (or of the three-dimensional sphere S^3).

PROPOSITION 3.6 [M-W]. *Identify the set of isotopic knots with a coadjoint orbit of the group of volume-preserving diffeomorphisms of* $\mathbb{R}^3$. *Then the Kirillov–Kostant symplectic structure on this set coincides with the Marsden–Weinstein symplectic structure.*

PROOF. Assume that two fields v and w in $\mathbb{R}^3$ define two variations of a curve γ. Then the Kirillov–Kostant symplectic structure on the coadjoint orbit at the point ω_γ associates to these variations the number

$$\begin{aligned}\langle \omega_\gamma,\ [v,w]\rangle &:= \langle d^{-1}\omega_\gamma,\ [v,w]\rangle = \langle u_\sigma,\ -\{v,w\}\rangle \\ &= -\int_{\mathbb{R}^3} (i_{\{v,w\}} u_\sigma)\mu = -\int_{\mathbb{R}^3} u_\sigma \wedge (i_{\{v,w\}}\mu).\end{aligned}$$

Here we have used the fact that the commutator in the Lie algebra of vector fields is equal to minus their Poisson bracket. Since $\{v,w\} = -\operatorname{curl}(v\times w)$, we have, according to the definition of $\operatorname{curl}(v\times w)$,

$$-i_{\{v,w\}}\mu = i_{\operatorname{curl}(v\times w)}\mu = d\alpha.$$

Here α is the 1-form related to the vector field $(v\times w)$ by means of the Riemannian metric: $\alpha(\xi) = (v\times w,\ \xi)$ for any vector field ξ. Then

$$\langle \omega_\gamma,\ [v,w]\rangle = \int_{\mathbb{R}^3} u_\sigma \wedge d\alpha = \int_{\mathbb{R}^3} du_\sigma \wedge \alpha = \int_{\mathbb{R}^3} \omega_\gamma \wedge \alpha.$$

The last integral, by definition of the 1-form α, is the circulation of the field $v\times w$ along γ, or, equivalently, the volume of the collar spanned by the variations v and w of the curve γ:

$$\int_{\mathbb{R}^3} \omega_\gamma \wedge \alpha = \int_{S^1} \operatorname{vol}(\gamma',\ v,\ w)\, dx = \beta(v,w).$$

This is the symplectic structure given in Definition 3.1. ☐

REMARK 3.7. Note that the coadjoint orbits corresponding to knots satisfy a kind of "quantization" condition. Associate to every knot a narrow current supported in a tubular neighborhood of the knot and whose flux across any transverse to the neighborhood is equal to 1. The flux of this current across a Seifert surface of any other knot is an integer.

If m is a point of such a "quantized" orbit, then the orbit of the point λm ($\lambda \in \mathbb{R}$) for a nonintegral λ does not correspond, in general, to a (nonparametrized) knot. It follows from the description of the coadjoint orbits as of the cosets $[\alpha]$ of 1-forms modulo differentials: The form $\lambda\cdot\alpha$ corresponds to the orbit λm. These knot-type orbits of the coadjoint representation depend on the form period $\oint \alpha$ as a parameter. The orbits of nonparametrized knots correspond to the 1-forms of period 1 in the description via cosets.

Consider the coadjoint orbit of one such link or knot. The considerations above imply that this "manifold" has the following peculiar property: The values of the "coordinates" of its points, equal to the linking numbers with other knots, are always integers, except for those knots that intersect the given one. In this sense the orbit is similar to a polyhedron whose faces are parallel to the coordinate subspaces and have integer-valued projections along these subspaces. The simplest example of a polyhedron of this type is a broken line on a chessboard consisting of parts of the square boundaries.

In general, one can think of these elements as a certain subset of the dual space, somewhat similar to the set of those points in a vector space with at least one coordinate being an integer. Presumably, replacing the integral coefficients of the knots forming the links by the rational ones, one obtains a set that is dense in some reasonable topology.

Remark 3.8. The space $\mathcal{C}$ can also be endowed with a natural almost complex structure: a continuous operator field $J_\gamma : T_\gamma\mathcal{C} \to T_\gamma\mathcal{C}$ such that $J_\gamma^2 = -1$ for all $\gamma \in \mathcal{C}$. This operator has a simple geometric meaning: Every variation field v along the curve γ is sent to another field Jv along γ whose vectors $Jv(x)$ at each point are obtained from $v(x)$ through a rotation by $\pi/2$ in the positive direction in the plane normal to $\gamma'(x)$ (see Fig.76).

It turns out that the curvature tensor of this structure vanishes [PeS, Bry1]. In finite dimensions, this condition would be enough to introduce complex coordinates on the manifold (using the Newlander–Nirenberg theorem, [N-N]). However, the construction of complex coordinates does not carry over without restrictions to arbitrary infinite-dimensional manifolds. Here we deal with the infinite-dimensional manifold of all C^∞-curves, and one can show that it does not admit a complex structure [Lem, Wan]. According to V. Drinfeld and C. LeBrun, the situation is different in the category of analytic knots in an analytic manifold, see the discussion in [Bry1]. (We refer to [PeS] for a discussion of geometric quantization for vortex configurations.)

Note also that the moduli space of isometric maps of a circle into Euclidean space $\mathbb{R}^3$ (modulo orientation-preserving Euclidean motions) admits a complex structure [MiZ]. Another example of an infinite-dimensional complex manifold is given by a typical Virasoro coadjoint orbit; see [Ki3].

Remark 3.9. The above structures, as well as most of the dynamical systems

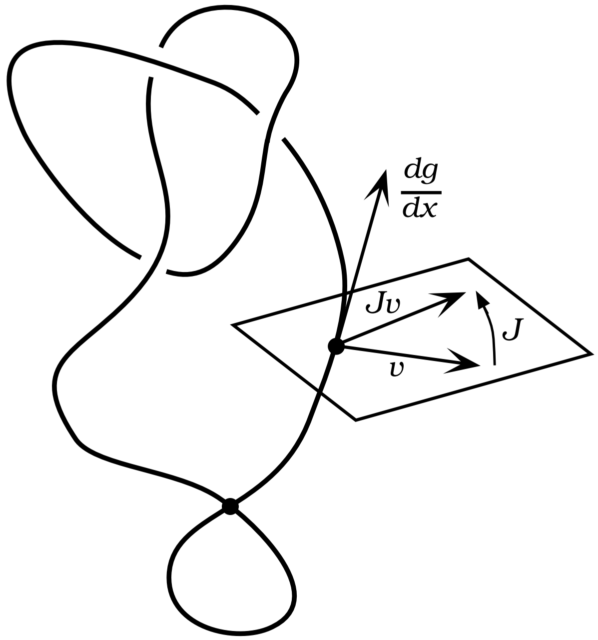

FIGURE 76. A singular knot and the almost complex structure in the space of knots.

we discuss below, can be defined on a bigger set $\bar{\mathcal{C}}$ of *immersed knots*, which has a nicer topology; see [Bry1]. The latter set is obtained by allowing the immersions γ to have self-intersections in a finite number of points and of finite multiplicity (Fig.76). The extension of the invariants of knots from the set $\mathcal{C}$ to the space of immersed knots $\bar{\mathcal{C}}$ is a cornerstone of the Vassiliev theory of knot invariants of finite order [VasV].

At first glance it seems that the space of singular knots with one self-intersection has (infinite) dimension that is one less than that of the symplectic space (the coadjoint orbit) of regular knots in the space of all circle immersions, and hence, it cannot carry a symplectic structure. This is, however, not the case. The corresponding coadjoint orbit has dimension two less than that of the regular knot: The singular knots with one double point (of a given topological type) form a two-parameter family of orbits (since the integral of the corresponding 1-form along each of the two loops is an invariant), while the regular knot orbits of a given topological type form a one-parameter family (the invariant being the integral along the whole knot).

3.B. Filament-, Nonlinear Schrödinger-, and Heisenberg chain equations. To define a dynamical system on the (symplectic) space of nonparametrized (immersed) oriented knots $\bar{\mathcal{C}}$ (the closure of the set of embedded curves $\mathcal{C}$), we need a Hamiltonian function. A natural choice of the function H is the *length functional* on curves:

$$H(\gamma) := \text{length of } \gamma = \int_{S^1} \sqrt{(\gamma'(x),\ \gamma'(x))}\ dx.$$

Note that just as in ideal hydrodynamics, to define the motion we need to specify the Riemannian metric, in addition to the volume form on the manifold.

DEFINITION 3.10. The evolution equation for the length Hamiltonian function H, with respect to the symplectic Marsden–Weinstein structure, is called the *filament equation.*

The time evolution $\gamma(x,t)$ of the curve $\gamma(x,0)$, $x \in S^1$, according to the filament equation, is

$$\frac{\partial \gamma}{\partial t} = k(x,t)\frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2}, \tag{3.1}$$

where $k(x,t)$ is the curvature of the curve at the point x at time t.

Indeed, the variational derivative $\frac{\delta H}{\delta \gamma}$ is

$$-(\text{length of } \gamma)^{-1} \frac{d^2\gamma}{dx^2}$$

for (a multiple of) arc-length parametrization x. Then, the corresponding Hamiltonian field can be found, say, by using the almost complex structure J_γ:

$$\begin{aligned} \operatorname{sgrad} H &= -(\text{length of } \gamma)^{-1}\ J_\gamma \left(\frac{\delta H}{\delta \gamma}\right) = (\text{length of } \gamma)^{-2}\ J_\gamma \left(\frac{d^2\gamma}{dx^2}\right) \\ &= (\text{length of } \gamma)^{-2} \left(\frac{d\gamma}{dx} \times \frac{d^2\gamma}{dx^2}\right). \end{aligned} \tag{3.1$'$}$$

The return from the arc-parametrization to an arbitrary one results in the curvature factor $k(x,t)$ in Equation (3.1).

REMARK 3.11. Hasimoto noticed in [Has] that the filament equation (3.1) is equivalent to the *nonlinear Schrödinger equation*

$$-i\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}|\psi|^2\psi \tag{3.2}$$

for a complex-valued wave function $\psi : S^1 \to \mathbb{C}$. This equation is known to be a completely integrable infinite-dimensional system and to possess soliton solutions (see, e.g., [DKN]).

The transformation reducing one of the equations to the other is called the *Hasimoto transformation*:

$$\psi(x,t) = k(x,t) \exp\left(i \cdot \int_0^x \tau(u,t)\ du\right),$$

where $\tau(u,t)$ is the torsion of the curve γ at the point u and time t.

The paper [LaP] shows that the Hasimoto transformation respects the Hamiltonian property of the equations: It sends the Marsden–Weinstein structure on curves to a certain (nonconstant) Poisson structure on wave functions.

REMARK 3.12. Another equivalent form of the filament equation is the *equation of gas dynamics* we dealt with in Section 2. Rewriting Equation (3.1) in the Frenet frame of γ, one obtains the evolution equations on the curvature $k(x,t)$ and the torsion $\tau(x,t)$, which in the coordinates $\rho := k^2$ ("energy density" of the curve) and τ are

$$\begin{cases} \partial_t \rho + \partial_x(\rho\tau) = 0, \\ \partial_t \tau + \tau\partial_x\tau = \partial_x\left(\frac{1}{4}\rho + \frac{1}{2}\rho^{-1/2}\partial_x^2\rho^{1/2}\right), \end{cases}$$

where $\partial_x := \partial/\partial x$ and $\partial_t := \partial/\partial t$; see [Tur]. Thus ρ and τ become the velocity and density fields for a one-dimensional fluid.

REMARK 3.13. The *Heisenberg magnetic chain* provides one more version of the filament equation. This equation describes the dynamics of the vector function $L(x) \in \mathbb{R}^3$, $x \in S^1$:

$$\frac{\partial L}{\partial t} = L \times \frac{\partial^2 L}{\partial x^2}. \tag{3.3}$$

One immediately obtains this equation from (3.1) by using the arc-parametrization x along the curve γ. Indeed, the filament equation (3.1-3.1′) assumes the form

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2},$$

equivalent to Equation (3.3) for the corresponding tangent vector $L := \partial\gamma/\partial x$.

The vector $L(x) \in \mathbb{R}^3$ can also be regarded as an element of the three-dimensional Lie algebra $\mathfrak{so}(3)$. From this point of view Equation (3.3) is a particular case of the *Landau–Lifschitz equation*, which can be associated to an arbitrary finite-dimensional Lie group, or rather to the corresponding gauge group.

REMARK 3.14. The filament equation can be regarded as an "approximation" of the Euler–Helmholtz equation for the vorticity concentrated on a curve if one considers the contribution of the local terms only; cf. Section I.11.C. The integrable dynamics in this case is a consequence of the approximation. The inclusion of the next (nonlocal) term into the picture makes the dynamics much more complicated; see [KlM].

3.C. Loop groups and the general Landau–Lifschitz equation. Let G be a finite-dimensional matrix group with a nondegenerate Killing form $\langle A, B\rangle = tr(AB)$ for $A, B \in G$, i.e., a *reductive group* (one can think of $SO(3)$ in the example above, or the group of all nondegenerate matrices $GL(n)$), and let $\mathfrak{g}$ be the corresponding Lie algebra.

DEFINITION 3.15. The *loop group* $\tilde{G}$ (or the *gauge group*) is the group of G-valued functions on the circle $\tilde{G} = C^\infty(S^1,\ G)$ with pointwise multiplication. The corresponding *loop Lie algebra* $\tilde{\mathfrak{g}}$ is the Lie algebra of $\mathfrak{g}$-valued functions on the circle with pointwise commutator.

DEFINITION 3.16. The *Landau–Lifschitz equation* is the evolution equation

$$\partial_t L = L \times \partial_x^2 L$$

for a vector-valued function $x \mapsto L(x) \in \mathbb{R}^3$ on the circle $x \in S^1$, and $\partial_x^2 := \frac{\partial^2}{\partial x^2}$.

More generally, the *Landau–Lifschitz equation* associated to a Lie algebra $\mathfrak{g}$ is the following evolution equation:

$$\partial_t m = [m\ ,\ \partial_x^2 m], \tag{3.4}$$

where m is a $\mathfrak{g}$-valued function on S^1.

According to the latter definition, the classical Landau–Lifschitz equation (3.3) is associated to the Lie group $\mathfrak{so}(3)$ upon the identification of the vectors in $\mathbb{R}^3$ with angular velocities, the elements in $\mathfrak{so}(3)$:

$$L = (v_1, v_2, v_3) \mapsto \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix}.$$

THEOREM 3.17. *The Landau–Lifschitz equation associated to a Lie algebra $\mathfrak{g}$ is the Euler equation corresponding to the loop group $\tilde{G}$ with the quadratic Hamiltonian function*

$$H(m) = \frac{1}{2}\int_{S^1} tr\,(\partial_x m)^2\ dx$$

on the dual space $\tilde{\mathfrak{g}}^*$, where $\partial_x m$ is the $\mathfrak{g}$-valued function, and tr stands for the trace in the matrix algebra $\mathfrak{g}$.

PROOF. The inverse inertia operator $A^{-1} : \tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}$ corresponding to this Hamiltonian sends a ($\mathfrak{g}^*$-valued) function m to the ($\mathfrak{g}$-valued) function $-\partial_x^2 m$. Then the Euler equation assumes the form

$$\partial_t m = \mathrm{ad}^*_{A^{-1}m}\, m = -[\partial_x^2 m\ ,\ m],$$

equivalent to (3.4). □

COROLLARY 3.18 (SEE, E.G.,[A-L]). *The classical Landau–Lifschitz equation (3.3) is the Hamiltonian equation on the dual space* $\tilde{\mathfrak{so}}(3)^*$ *with the Hamiltonian function*

$$H(L) = -\int_{S^1} (\partial_x v_1)^2 + (\partial_x v_2)^2 + (\partial_x v_3)^2 \; dx$$

(here $L(x) = (v_1(x), v_2(x), v_3(x)) \in \mathbb{R}^3 = \mathfrak{so}(3)^*$*).*

The paper [A-L] also contains the calculation of the sectional curvatures of the loop group $\widetilde{SO}(3)$ with respect to the right-invariant Riemannian metric induced by the Hamiltonian function $H(L)$.

§4. Sobolev's equation

Studying fluid oscillations in a fast rotating tank, and starting with the corresponding approximating equation

$$\frac{\partial v}{\partial t} - k(v \times e_z) + \operatorname{grad} p = F, \quad \operatorname{div} v = 0 \tag{4.1}$$

(with unknowns v and p), S.L. Sobolev obtained an equation of unusual type, now named after him.

DEFINITION 4.1. The *Sobolev equation* is the equation

$$\frac{\partial^2 \Delta u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{4.2}$$

for the unknown function u.

REMARK 4.2. Equation (4.1) is the the linearization of the Navier–Stokes equation in a rotating domain. Typical examples are atmospheres of planets and fuel tanks of rotating projectiles. Poincaré [Poi2] reduced the linear system (4.1) to one equation (4.2). The latter equation was named after Sobolev, who rediscovered it in the forties and studied the corresponding boundary problems.

Sobolev's work was declassified and published in [Sob2]. This paper was in fact written in Kazan', perhaps in 1942. Sobolev's neighbor was Pontriagin and they discussed many relevant problems in functional analysis. In particular, they considered the "pseudo-Hilbert" spaces with one (studied by Sobolev) or a finite number (studied by Pontriagin) of negative squares in the metric. These spaces are now called Pontriagin spaces. Very few people know that the theory of these spaces originated in the classified hydrodynamical work of Sobolev.

The work of Poincaré and of Sobolev was continued by Babin, Mahalov, and Nicolaenko, who extended the equation to the case of fast rotation and shallow domains, and considered nonlinear dynamics of the Navier–Stokes equation. Many features of Sobolev's study of the linear problem, such as the small denominators and the Diophantine incommensurability conditions on the domains' geometrical parameters, reappear in [BMN]. It is shown in [BMN] that solutions of the 3D Euler and Navier–Stokes equations of uniformly rotating fluids can be decomposed into the sum of the following terms: a solution of the 2D Euler (or Navier–Stokes) system with vertically averaged initial data, a vector field explicitly expressed in terms of the phases, and a small remainder.

REMARK 4.3. To derive the Sobolev equation from the system (4.1) with $F = 0$ (see [GaS] for details) we take the curl of both sides of the first equation. Since curl $(a \times b) = -\{a, b\}$, this gives

$$\frac{\partial \omega}{\partial t} + 2k\frac{\partial u}{\partial z} = 0, \qquad \text{where} \qquad \operatorname{curl} u = \omega. \tag{4.3}$$

Take the curl once more,

$$-\frac{\partial}{\partial t}\Delta u + 2k\frac{\partial \omega}{\partial z} = 0,$$

and differentiate it in t to get

$$-\frac{\partial^2}{\partial t^2}\Delta u + 2k\frac{\partial}{\partial z}\frac{\partial \omega}{\partial t} = 0.$$

Finally, substitute $\partial\omega/\partial t = -2k\ \partial u/\partial z$ from Equation (4.3) to obtain the Sobolev equation

$$-\frac{\partial^2 \Delta u}{\partial t^2} - 4k^2\frac{\partial^2 u}{\partial z^2} = 0.$$

A study of the spectral problems for the linear Sobolev equation showed a strong dependence of the eigen oscillations on the tank shape. Namely, after some transformations, Sobolev found it necessary to investigate the two-dimensional spectral problem

$$\frac{\partial^2 u}{\partial x^2} - \lambda\frac{\partial^2 u}{\partial y^2} = 0, \quad u|_\Gamma = g$$

(as well as the corresponding homogeneous problem $u|_\Gamma = 0$) in a plane domain bounded by a curve Γ.

For a given value of the spectral parameter λ, the equation above is the Dirichlet problem for the one-dimensional wave equation. Solving it by the method of characteristics, one immediately encounters the strong dependence of the results on the domain shape.

As we shall see below, on the boundary of the domain there appears a dynamical system. The ergodic properties of this system have a strong impact on the oscillation character.

Consider the case of a convex domain. Two families of characteristic lines cover the domain. Each of these two families defines the diffeomorphism of the boundary Γ into itself that is the involution exchanging the points of intersection of each characteristic with the boundary. The above-mentioned dynamical system on the boundary curve is the diffeomorphism of the curve Γ that is the composition of two involutions corresponding to the two families of characteristics.

In terms of this diffeomorphism $T : \Gamma \to \Gamma$, the solution of the above Dirichlet problem (for a fixed λ) is constructed as follows. First, by a linear change of variables, we transform the characteristics into the straight lines $x = \text{const}$ and $y = \text{const}$. Our problem assumes the form

$$\frac{\partial^2 u}{\partial x\, \partial y} = 0, \quad u|_\Gamma = g.$$

The solution u is the sum of two functions $f(x)+h(y)$, one of which depends only on x and the other only on y. To look for these functions, we fix some boundary point A and choose the value of one of the functions at this point (e.g., $f(A)$) arbitrarily. Then the value of the second function is determined by the boundary condition (i.e., $h(A) = g(A) - f(A)$).

Let B be the intersection of the characteristic line of the first family ($x = \text{const}$) passing through A and the boundary Γ. At the point B we already know the value of the first function (it is the same as at the point A, i.e., $f(B) = f(A)$). Then the value of the second function h at B is determined by their sum $g(B)$ (namely, $h(B) = g(B) - f(B) = g(B) - f(A)$). Further, the characteristic of the second family ($y = \text{const}$) passing through B intersects the boundary Γ at the point $A' = TA$ (Fig.77). We already know the value of the second function along this line (which is the same as that at B: $h(A') = h(B)$). Given the sum $g(A')$, we find the value of the first function at A' (here $f(A') = g(A') - h(A') = g(A') - h(B) = g(A') - g(B) + f(A)$) and so on.

The infinite process of constructing the solution is described by a piecewise-linear trajectory. This trajectory is constituted by the intermittent segments of the characteristics joining the points $A^{(n)} = T^n A$. The solution is the sum of the initial value and the alternating sum of the boundary values at the vertices of the piecewise-linear trajectory.

The properties of the dynamical system $T : \Gamma \to \Gamma$ have the following impact on the solutions of our Dirichlet problem. Suppose that T has a periodic trajectory,

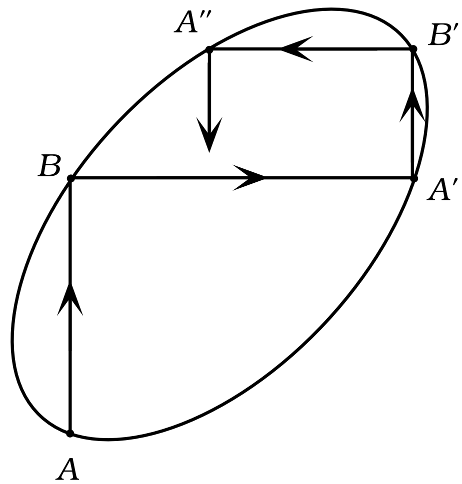

FIGURE 77. Constructing the solution of the Dirichlet problem for the wave equation from two families of characteristics.

$T^n A = A$. Then the alternating sum of values of the boundary function g at the vertices of the corresponding piecewise-linear trajectory $ABA'B' \ldots A$ must be equal to zero. Hence, each periodic trajectory of the map T corresponds to a solvability condition for the Dirichlet problem (and therefore, to a certain nontrivial "distributional solution" of the corresponding homogeneous equation, "supported near" this periodic trajectory).

There are more subtle properties of the dynamics of T that also affect the solvability of the Dirichlet problem (see details, e.g., in [Arn2]).

Consider first an elliptic domain. In this case, the diffeomorphism T becomes a rotation after an appropriate choice of the angle coordinate on the boundary. Indeed, an ellipse can be turned into a circle by an affine transformation of the plane. The characteristics of both families will turn into two families of parallel lines forming an angle α with each other. The map T will become the circle rotation by the angle 2α (by virtue of the "inscribed angle" theorem).

Depending on whether the angle α is commensurable with 2π or not, the orbits of the rotation T either consist each of a finite number of points (repeating periodically with the same period for all initial points) or are everywhere dense on the circle.

In the first ("resonance") case, the solution of the nonhomogeneous equation exists if and only if the function g satisfies an infinite number of independent conditions. The corresponding homogeneous problem has an infinite-dimensional space of solutions.

When the angle α is not commensurable with 2π, any T-orbit is everywhere dense (it is the second, "ergodic" case). Here the situation is more complicated. Formally, one can find the solution as a Fourier series. However, its convergence relies on the arithmetic Diophantine properties of the irrational number $\alpha/2\pi$ (as well as in what functional space the problem is considered). For almost all (in the sense of Lebesgue measure) irrational numbers $\alpha/2\pi$, the corresponding homogeneous problem has the unique solution $u = 0$. The nonhomogeneous problem has, in this case, a (smooth) solution for every sufficiently smooth right-hand side g (the necessary smoothness of g increases as the required smoothness of the solution increases; for an analytic solution the analyticity of the right-hand side is sufficient).

The case of an ellipse, discussed above, is not generic, since the dynamics of the corresponding diffeomorphism T is integrable. (According to Yurkin [Yur] a domain bounded by ellipses is the only type of cavity in a rotating symmetric top for which the study of oscillations described by the Sobolev equation can be reduced to a finite-dimensional problem.) For a typical boundary curve the diffeomorphism T cannot, in general, be reduced to a rotation, no matter what angle coordinate on the curve is chosen.

In the space of all diffeomorphisms (and hence, in the space of curves Γ), the *structurally stable* diffeomorphisms form an open and everywhere dense set. Such diffeomorphisms are of "resonance type" with a finite number of periodic orbits (all of which have the same period) and alternating attractors and repulsers.

People working in the axiomatic theory of dynamical systems usually assume that "generic" events are those occurring on an everywhere dense open set in the space of systems. From this viewpoint "generic" circle diffeomorphisms are the structurally stable ones.

However, from the physics point of view, these structurally stable systems are not the most widespread. Consider, for instance, a family of circle diffeomorphisms $x \to x + a + b\sin x \pmod{2\pi}$, where a and b are parameters. For most of the points (a, b) in the rectangle $0 \leq a \leq 2\pi$, $0 \leq b \leq \beta$ of a sufficiently small height β, the diffeomorphism does not have periodic points at all, and one can make it into a rotation by choosing an appropriate coordinate on the circle. (This will be the rotation by an angle incommensurable with 2π.) Every orbit of such a diffeomorphism is everywhere dense on the circle. For almost all values of the rotation angle, the solvability question for the Dirichlet problem, corresponding to such a diffeomorphism T, turns out to be the same as that for the integrable case of an elliptic boundary.

For instance, for the near-elliptic domains the "ergodic" situations are encountered in an overwhelming majority of cases, while the "resonance" ones are rare (but form an open and everywhere dense set) in the space of ellipse deformations;

see [Arn2].

We return now to the initial spectral problem with the parameter λ. For a typical boundary Γ, the two types of behavior of the dynamical system $T = T(\lambda)$ on the curve Γ alternate as λ changes. If Γ is a typical small perturbation of an ellipse, then the resonance values of the parameter λ (for which nontrivial eigenfunctions arise) form an infinite everywhere dense set (of small measure) on the axis λ. The ergodic values of λ (i.e., the values λ for which $T(\lambda)$ reduces to the circle rotation by a smooth coordinate change) form a Cantor-type set of almost full measure (for small perturbations of an ellipse).

As we can see, all topological subtleties of the nonlinear theory of dynamical systems (in particular, of their perturbation theory) appear in hydrodynamics in studying the spectrum of the *linear* problem of small oscillations of a fluid.

After S.L. Sobolev, the spectral problem was studied by R.A. Alexandryan and his school (see [Ale]). We mention the series of papers by S.G. Ovsepjan [Ovs], in which the case of a nonconvex boundary was treated.

In the nonconvex case a new difficulty arises: A characteristic line intersects the boundary in more than just two points, so that the "dynamics" T turns out to be multivalued (or branching). The ergodic properties of this branching multivalued dynamics form an interesting, but an insufficiently explored, part of the theory of dynamical systems.

Consider, for example, a circle diffeomorphism that becomes a multivalued algebraic correspondence of an algebraic curve into itself when the diffeomorphism is extended into the complex domain. This means that the graph of the diffeomorphism is one of the components of a real algebraic curve on the Cartesian square of another algebraic curve. One believes that the number of attractors of such a diffeomorphism is bounded by a constant, depending only on the discrete invariants of the correspondence (the genera of the curves and the degree of the correspondence). However, it has been proved only for the correspondences univalued in one of the directions (say, for polynomial or rational maps of the Riemann sphere into itself), see [Jak, Herm].

Remark 4.4. The Dirichlet problem for the one-dimensional wave equation is encountered in many problems of different origins. For instance, J.-P. Dufour [Duf] treated in detail its local analogue for algebraic curves with singularities (say, $x^2 = y^3$). This problem arises in the study of the symmetry loss (for example, for the classification of Morse functions in a neighborhood of the fixed point of the line involution $x \to -x$, or for the classification of pairs of line involutions in a neighborhood of the common fixed point); see the survey of S. Voronin [Vor].

An analogous method was used in [Arn1] in the study of the representations of functions on trees by the sums of functions of the coordinates, which is related to the 13^{th} Hilbert problem. It is interesting that the main trick in all these problems is the composition of the alternating sums of values of a known function along a piecewise characteristic, and it is exactly the same as the one used in hydrodynamics in the study of spectral problems for the Sobolev equation.

§5. Elliptic coordinates from the hydrodynamical viewpoint

Imagine an electrically charged metallic ellipsoid. A theorem going back to Newton [NewI] and Ivory [Ivo] states that the potential (of the electrostatic field) induced by the charges is constant inside the ellipsoid, while outside of it the equipotential surfaces are the ellipsoids confocal to the initial one. As we shall see below (following [Arn12, ShV]), this fact, as well as its higher-dimensional generalizations, has a genuine hydrodynamical flavor: The electromagnetic fields of this type are generated by incompressible flows of electric charges along quadrics.

5.A. Charges on quadrics in three dimensions. We start with a quadric surface (say, ellipsoid) Q in three-dimensional space and include it first in the family of confocal quadrics.

DEFINITION 5.1. For a quadric Q defined by the equation

$$\frac{x^2}{a_1}+\frac{y^2}{a_2}+\frac{z^2}{a_3}=1,$$

the *confocal family of quadrics* $Q_{\text{cnf}}(\lambda)$ is the following family of surfaces:

$$Q_{\text{cnf}}(\lambda)=\left\{\frac{x^2}{a_1+\lambda}+\frac{y^2}{a_2+\lambda}+\frac{z^2}{a_3+\lambda}=1\right\}.$$

The quadrics of the family change signature at $\lambda=-a_1,\ -a_2$, or $-a_3$. For instance, for a hyperboloid of one sheet with $a_1>a_2>0>a_3$ the family consists of the hyperboloids of two sheets for $-a_1<\lambda<-a_2$, of the hyperboloids of one sheet for $-a_2<\lambda<a_3$, and of the ellipsoids for $a_3<\lambda$ (Fig.78).

We will also use another family of quadrics containing our initial surface Q: quadrics *homothetic* to Q with center at the origin. First let Q be an ellipsoid.

DEFINITION 5.2. A *homeoidal density* on the surface of an ellipsoid Q is the density of a layer between Q and an infinitely nearby ellipsoid homothetic to Q.

Now we can make mathematical sense of the "free distribution of electric charges" on the surface of an ellipsoid:

THEOREM 5.3 (IVORY THEOREM, SEE [Ivo, Arn12]). *A finite mass distributed on the surface of an ellipsoid with homeoidal density does not attract any internal point; it attracts every external point the same way as if the mass were distributed with homeoidal density on the surface of any smaller confocal ellipsoid.*

The attraction of the charges is defined by the Coulomb (or Newton) law: In $\mathbb{R}^n$ the force is proportional to r^{1-n} (as prescribed by the fundamental solution of the Laplace equation).

In the counterparts of Ivory's theorem for hyperboloids, one replaces the homeoidal densities on ellipsoids by harmonic forms of different degrees, and the Coulomb potential by the generalized potentials related to the Biot–Savart law.

In the simplest nontrivial case of a hyperboloid H of one sheet in three-dimensional Euclidean space, the result is as follows. Consider the intersection curves of the hyperboloid with other quadrics of the confocal family $H_{\mathrm{cnf}}(\lambda)$. We will be referring to the intersections with confocal ellipsoids (respectively, confocal hyperboloids of two sheets) as *parallels* (respectively, *meridians*) of H. (Notice that the parallels and meridians are orthogonal to one another at every point of the hyperboloid, Fig.78. This is the theorem on the existence of an orthogonal eigenbasis for a symmetric matrix, applied to the Legendre dual family of quadrics; see, e.g., [A-G].)

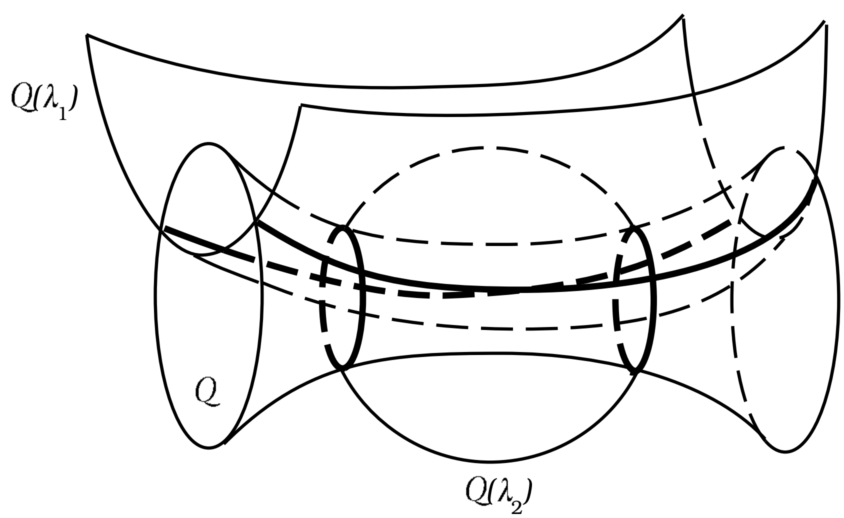

FIGURE 78. Quadrics of the confocal family intersect a hyperboloid along the orthogonal system of curves.

The hyperboloid H divides the space $\mathbb{R}^3$ into two parts, "internal" $I(H)$ and "external" $E(H)$, the latter being non-simply connected. The region inside the hyperboloidal tube is also smoothly fibered by meridians (orthogonal to the ellipsoids in the confocal family), while the annular region outside the hyperboloid is smoothly fibered by parallels (orthogonal to the hyperboloids of two sheets).

THEOREM 5.4 [Arn12]. *There exists a unique (modulo a constant factor) surface current flowing along the meridians of the hyperboloid that produces a magnetic field that vanishes in the inner domain and is directed along parallels in the exterior domain of the hyperboloid. Similarly, there exists a unique (modulo a constant factor) surface current flowing along the parallels of the hyperboloid that produces a magnetic field that vanishes in the exterior domain and is directed along meridians in the inner domain of the hyperboloid.*

The magnetic field in the inner domain for the hyperboloid, but outside of a charged ellipsoid from the same confocal family, coincides modulo sign with the electrostatic field of the ellipsoid. Furthermore, let us look at the electrostatic field produced by two charges of opposite signs, "equal in magnitude," and distributed with homeoidal density on the surfaces of a conducting hyperboloid of two sheets. This field between the surfaces coincides (modulo sign) with the magnetic field in the exterior domain of a confocal hyperboloid of one sheet. The explicit formulas are given below.

REMARK 5.5. The vector fields given by Theorem 5.4 are exact stationary solutions of the corresponding Euler equations of an incompressible fluid flowing, respectively, inside or outside of the hyperboloid in $\mathbb{R}^3$. The flow is potential in the inner domain of a triaxial hyperboloid, and it is vorticity-free in the exterior domain.

5.B. Charges on higher-dimensional quadrics. Let Q be a nondegenerate quadric centered at the origin of Euclidean n-dimensional space. Include it in the family of *confocal quadrics*

$$Q_{\mathrm{cnf}}(\lambda) := \left\{ \sum_{i=1}^{n} \frac{x_i^2}{a_i + \lambda} = 1 \right\},$$

as the hypersurface corresponding to $\lambda = 0$. Let us order the axes as follows: $a_n < \dots < a_1$.

DEFINITION 5.6. The *elliptic coordinates* of a point $x \in \mathbb{R}^n$ is the set of n values of λ (in increasing order) for which a quadric of the family $Q_{\mathrm{cnf}}(\lambda)$ passes through x. Note that it is an orthogonal coordinate system, since the confocal quadrics meet at right angles.

The results formulated above for the three-dimensional case have been extended by B. Shapiro and A. Vainshtein [ShV] to hyperboloids in Euclidean spaces of any number of dimensions. A nonsingular hyperboloid H in $\mathbb{R}^n$, diffeomorphic to $S^l \times \mathbb{R}^k$, divides the space into the exterior region $E(H)$ (diffeomorphic to the product of S^l with a half-space) and the interior $I(H)$.

Let ω be a differential form with distribution coefficients (see [DeR]). The form is said to be *harmonic off a hypersurface* Γ if it is continuous off this hypersurface, coclosed (i.e., $\delta\omega = 0$, where δ is the operator conjugate to the external derivative d, see Section V.3.B), and if its exterior derivative is a form (with distribution coefficients) supported on Γ.

THEOREM 5.7 [ShV]. *Given a hyperboloid H there exists a unique (modulo a constant factor) l-form harmonic off H, decomposable in elliptic coordinates, and vanishing in the interior region $I(H)$, and there exists a unique (modulo a constant factor) k-form harmonic off H, decomposable in elliptic coordinates, and vanishing in the exterior $E(H)$.*

These forms are induced by certain homeoidal densities on the *focal quadrics*, which are the limiting quadrics of the confocal family, when the shortest axis of the hyperboloids or ellipsoids shrinks to zero. We refer to [ShV] for explicit formulas and proofs.

For the hyperboloids of indices $(1, n-1)$ and $(n-1, 1)$, one can give the following magnetohydrodynamical meaning to those densities.

Let H^1 be a nondegenerate quadric with $a_n < \cdots < a_2 < 0 < a_1$ (and a quadric H^{n-1}, with $a_n < 0 < a_{n-1} < \cdots < a_1$, respectively). Similar to the above, the exterior region $E(H^{n-1})$ for the hyperboloid H^{n-1} is fibered by *parallels* (diffeomorphic to a circle) by fixing the values of all $n-1$ elliptic coordinates *positive* in $E(H^{n-1})$. The inner domain $I(H^1)$ for the quadric H^1 is fibered by *meridians* (diffeomorphic to a line) by fixing the values of all $n-1$ elliptic coordinates *negative* in $I(H^1)$.

THEOREM 5.7′ [ShV]. *There exists a unique (modulo a constant factor) potential flow of an incompressible fluid in the inner domain $I(H^1)$ whose trajectories coincide with the meridians. Similarly, there exists a unique (modulo a constant factor) flow of an incompressible fluid in the exterior region $E(H^{n-1})$ whose vorticity vanishes and the trajectories of which are the parallels.*

By construction, both of the flows are directed along the remaining elliptic coordinate. Say, in the 3-dimensional case, one has the following explicit formulas for the corresponding vector fields v_1 and v_2 in the regions $I(H^1)$ and $E(H^2)$, respectively, in the elliptic coordinates $\lambda_1 > \lambda_2 > \lambda_3$ (see [ShV]):

$$v_1 = \frac{\lambda_2 - \lambda_3}{\Phi(\lambda_2)\Phi(\lambda_3)} \frac{\partial}{\partial \lambda_1} \quad \text{and} \quad v_2 = \frac{\lambda_1 - \lambda_2}{\Phi(\lambda_1)\Phi(\lambda_2)} \frac{\partial}{\partial \lambda_3},$$

where

$$\Phi(\lambda_i) = \sqrt{(\lambda_i + a_1)(\lambda_i + a_2)(\lambda_i + a_3)}.$$

Noncomputational proofs of these geometric theorems are unknown, even in the three-dimensional case.

QUESTION 5.8. The presence of distinguished forms that are harmonic off hyperboloids suggests that one might try to find filtrations, analogous to those arising in the theory of mixed Hodge structures, in spaces of differential forms on noncompact real algebraic and semialgebraic varieties.