CHAPTER V

KINEMATIC FAST DYNAMO PROBLEMS

Stars and planets possess magnetic fields that permanently change. Earth, for instance, mysteriously interchanges its north and south magnetic poles, so that the time pattern of the switches forms a Cantor-type set on the time scale (see [AnS]). The mechanism of generation of magnetic fields in astrophysical objects (or in electrically conducting fluids) constitutes the subject of *dynamo theory.* Kinematic dynamo theory studies what kind of fluid motion can induce exponential growth of a magnetic field for small magnetic diffusivity. Avoiding analytical and numerical results (though crucial for this field), we address below the topological side of the theory.

§1. Dynamo and particle stretching

1.A. Fast and slow kinematic dynamos.

DEFINITION 1.1. The *kinematic dynamo equation* is the equation

$$\begin{cases} \frac{\partial B}{\partial t} = -\{v, B\} + \eta \Delta B, \\ \operatorname{div} B = 0 \end{cases} \tag{1.1}$$

(for a suitable choice of units).

It assumes that the velocity field v of an incompressible fluid filling a certain domain M is known. The unknown magnetic field $B(t)$ is stretched by the fluid flow, while a low diffusion dissipates the magnetic energy. Here η is a small dimensionless parameter (representing *magnetic diffusivity*), which is reciprocal to the so-called *magnetic Reynolds number* $R_m = 1/\eta$. The bracket $\{v, B\}$ is the Poisson bracket of two vector fields (for divergence-free fields v and B in Euclidean 3-space, the latter expression can be rewritten as $-\{v, B\} = \operatorname{curl}(v \times B)$). The vector field v is supposed to be tangent to the boundary of the domain M at any time. The boundary conditions for B are different in various physical situations. For instance, the magnetic field of the Sun extends out into space, forming loops based on the

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Sun's surface and seen as protuberances. This magnetic field is not tangent to the boundary.

Alternatively, one can suppose that the boundary conditions are periodic (the "star" or "planet" is being replaced by the three-dimensional torus $\mathbb{R}^3/(2\pi\mathbb{Z})^3$) or, more generally, that M is an arbitrary Riemannian manifold of finite volume and Δ is the Laplace–Beltrami operator on M.

The linear dynamo equation is obtained from the full nonlinear system of magnetohydrodynamics by neglecting the feedback action of the magnetic field on the velocity field due to the Lorentz force. This is physically motivated when the magnetic field is small. The latter corresponds to the initial stage of the amplification of a "seed" magnetic field by the differential rotation.

The following question has been formulated by Ya.B. Zeldovich and A.D. Sakharov [Zel2, Sakh]:

PROBLEM 1.2. *Does there exist a divergence-free velocity field v in a domain M such that the energy $E(t) = \|B(t)\|^2_{L^2(M)}$ of the magnetic field $B(t)$ grows exponentially in time for some initial field $B(0) = B_0$ and for arbitrarily low diffusivity?*

Consider solutions of the dynamo equation (1.1) of the form $B = e^{\lambda t} B_0(x)$. Such a field B_0 must be an eigenfunction for the (non-self-adjoint) operator $L_{v,\eta} : B_0 \mapsto -\{v, B_0\} + \eta \Delta B_0$ with eigenvalue $\lambda^{\mathbb{C}} = \lambda^{\mathbb{C}}(v, \eta)$. The eigenparameter $\lambda^{\mathbb{C}}$ is the *complex growth rate* of the magnetic field.

DEFINITION 1.3. A field v is called a *kinematic dynamo* if the increment $\lambda(\eta) := \operatorname{Re} \lambda^{\mathbb{C}}(\eta)$ of the magnetic energy of the field $B(t)$ is positive for all sufficiently large *magnetic Reynolds numbers* $R_m = 1/\eta$. The dynamo is *fast* if there exists a positive constant λ_0 such that $\lambda(\eta) > \lambda_0 > 0$ for all sufficiently large Reynolds numbers. A dynamo that is not fast is called *slow.*

There exist many possibilities for the dynamo effect in some "windows" in the range of the Reynolds numbers. In our formalized terminology, we shall *not* call such vector fields dynamos.

REMARK 1.4. The existence of an exponentially growing mode of B is a property of the operator $L_{v,\eta}$, and this is why we call the velocity field v, rather than the pair $(v,\ B)$, a dynamo. Kinematic dynamo theory neglects the reciprocal influence of the magnetic field B on the conducting fluid itself (i.e., the velocity field v is supposed to be unaffected by B). This assumption is justified when the magnetic field is small. The theory describes the generation of a considerable magnetic field from a very small "seed" field. Whenever the growing field gets large, one should take into account the feedback that is described by a complete system of MHD equations involving the Lorentz forces and the hydrodynamical viscosity.

The above question is reformulated now as the following

PROBLEM 1.2$'$. *Does there exist a divergence-free field on a manifold M that is a fast kinematic dynamo?*

Our main interest is related to *stationary velocity fields* v in 2- and 3-dimensional domains M. There are several (mostly simplifying) modifications of the problem at hand. We shall split the consideration of the *dissipative* (realistic, $\eta \to +0$) and nondissipative (idealized, or perfect, $\eta = 0$) cases. In the idealized *nondissipative* case the magnetic field is frozen into the fluid flow, and we are concerned with the exponential growth of its energy.

In a *discrete* (in time) version of the question, one keeps track of the magnetic energy at moments $t = 1, 2, \dots$. Instead of the transport by a flow and the continuous diffusion of the magnetic field, one has a composition of the corresponding two discrete processes at each step. Namely, given a (volume-preserving) diffeomorphism $g : M \to M$ and the Laplace–Beltrami operator $\eta\Delta$ on a Riemannian manifold M, the magnetic field B is first transported by the diffeomorphism to $B' := g_* B$, and then it dissipates as a solution of the diffusion equation $\partial B'/\partial t = \eta \Delta B'$:

$$B' \mapsto B'' := \exp(\eta\Delta) B'.$$

PROBLEM 1.5. *Does there exist a discrete fast kinematic dynamo, i.e., does there exist a volume-preserving diffeomorphism $g : M \to M$ such that the energy of the magnetic field B grows exponentially with the number n of iterations of the map*

$$B \mapsto \exp(\eta\Delta)(g_* B),$$

as $n \to \infty$ (provided that η is close enough to 0)? The question is whether the energy of the n^{th} iteration of B is minorated by $\exp(\lambda n)$ with a certain $\lambda > 0$ independent of η within an interval $0 < \eta < \eta_0$ for some η_0?

Other modifications of interest include chaotic flows, "periodic" versions of the dynamo problem (in which the field v on a 2- or 3-dimensional manifold is supposed to be periodic in time rather than stationary), as well as flows with various space symmetries (see [Bra, Bay1,2, Chi2,3, AZRS2, Sow2, Gil1, PPS, Rob]). In the sequel, we describe in detail certain sample dynamo constructions and the principal antidynamo theorems, along with their natural higher-dimensional generalizations. We shall see that the topology of the underlying manifold M enters unavoidably into our considerations.

The following remark of Childress shows that the difference between fast and slow dynamos is rather academic. Suppose that the dynamo increment $\lambda(\eta)$ decays extremely slowly, say, at the rate of $1/(\ln|\ln\eta|)$, as the diffusivity η goes to zero. (This is the case for a steady flow with saddle stagnation points, considered in [Sow1].) Though theoretically this provides the existence of only a slow dynamo, in practice, the dynamo is definitely fast: For instance, for $\eta = 1/(e^{e^3}) < 10^{-8}$ the increment $\lambda(\eta)$ is of order $1/3$, noticeably above zero.

REMARK 1.6. A more general (and much less developed) dynamo setting is the so-called *fully self-consistent theory.* It seeks to determine both the magnetic field B and the (time-dependent) velocity field v from the complete system of the magnetohydrodynamics equations:

$$\begin{cases} \frac{\partial B}{\partial t} = -\{v, B\} + \eta \Delta B, \\ \frac{\partial v}{\partial t} = -(v, \nabla)\, v + (\text{curl } B) \times B + \nu \Delta v - \nabla p, \\ \text{div } B = \text{div } v = 0, \end{cases}$$

for the fields B and v in a Euclidean domain (with standard necessary changes of symbols $\nabla, \Delta, \times$, and curl for a three-dimensional Riemannian manifold). We refer to Section I.10 and to [HMRW] for a group-theoretical treatment of magnetohydrodynamics, and to the interesting and substantial reviews [R-S, Chi2] for recent developments in both the kinematic and the fully self-consistent theories. Here we are solely concerned with the topological side of the fast kinematic dynamo mechanism.

1.B. Nondissipative dynamos on arbitrary manifolds. Unlike the dissipative ("realistic") dynamo problem, which is still unsolved in full generality, nondissipative ($\eta = 0$) dynamos are easy to construct on any manifold. First look at the case of a two-dimensional disk.

At first sight, *a nondissipative continuous-time fast dynamo on a disk* (or on a simply connected two-dimensional manifold) *is impossible.*

PSEUDO-PROOF. Every area-preserving velocity field v on a simply connected two-dimensional manifold is Hamiltonian and can be described by the corresponding Hamiltonian function. All the orbits of the field v that are noncritical level curves of such a function are closed (Fig.58).

Consider the linearized Poincaré map along every closed orbit. The derivative g_*^T of the flow map g^T at a point of an orbit of period T is generically a Jordan 2×2 block with units on the diagonal. Indeed, the tangent vector to the orbit is mapped to itself under the Poincaré map, and hence it is eigen with eigenvalue 1.

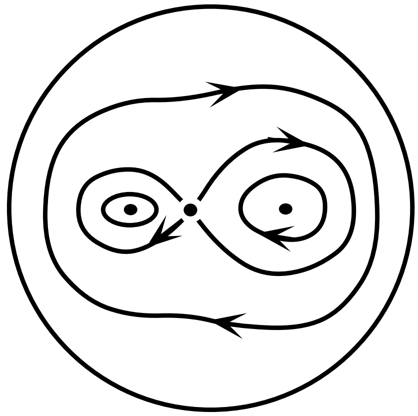

FIGURE 58. A typical Hamiltonian velocity field on a disk. Almost all orbits of the field are closed.

Then the Jordan block structure immediately follows from the incompressibility of the flow v, provided that it has a nondegenerate shear along the orbit (the orbit periods change with the value of the Hamiltonian).

Such a Jordan operator stretches the transported vectors of a magnetic field B linearly with the number of iterations of the Poincaré map (see Section II.5). The linear growth of the norm of B on a set of full measure implies the existence of a certain linear majorant for the increase of the energetic norm $\sqrt{E}$ over time. □

However, one cannot neglect the contribution of the singular level sets to the magnetic energy. The following statement is folklore that directly or indirectly is assumed in any study on dynamos (see [VshM, Gil2, Koz1]).

THEOREM 1.7. *On an arbitrary n-dimensional manifold any divergence-free vector field having a stagnation point with a unique positive eigenvalue (of the linearized field at the stagnation point) is a nondissipative dynamo.*

PROOF. The main point of the proof is that the energy of the evolved magnetic field *inside* a small neighborhood of the stagnation point is already growing exponentially in time.

Consider the following special case: The manifold is a two-dimensional plane $M = \mathbb{R}^2$ with coordinates (x, y), while the velocity v on M is the *standard linear hyperbolic field* $v(x, y) = (-\lambda x, \lambda y)$ with $\lambda > 0$. Specify the magnetic field B to be the vertical constant field $B = (0, b)$ with support in a rectangle $R := \{|x| \leq p/2, |y| \leq q/2\}$, see Fig.59.

At the initial moment the magnetic energy, i.e., the square of the L^2-norm of the field B, is

$$E_2(B) = \int_R B^2 \ \mu = pq \cdot b^2.$$

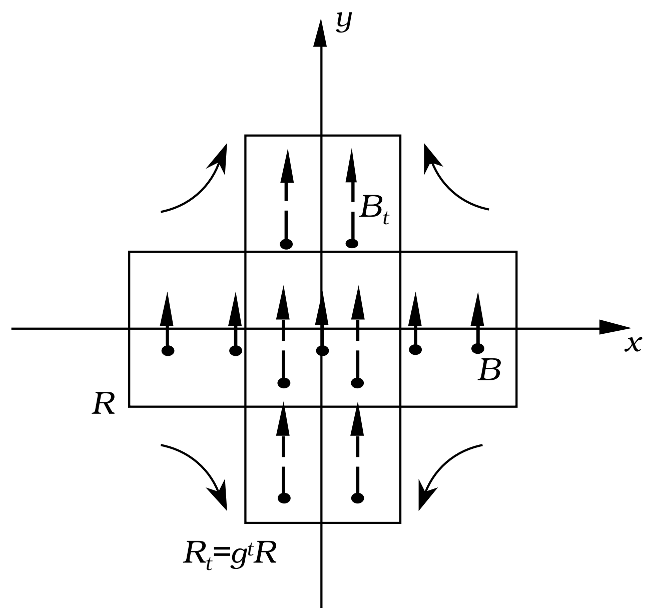

FIGURE 59. A nondissipative dynamo arising from a hyperbolic stagnation point.

After a time period t, the image R_t of the rectangle R is squeezed in the horizontal direction by the factor $e^{\lambda t}$ and is stretched along the vertical by the same factor (as is the field B as well). Then the magnetic energy of the field $B_t := g_*^t B$ is minorated by the field restriction to the initial rectangle:

$$E_2(B_t) = \int\limits_{R_t} B_t^2 \ \mu > \int\limits_{R_t \cap R} B_t^2 \ \mu$$
$$=(\text{area of } R_t \cap R) \cdot (e^{\lambda t} \ b)^2 = (pqe^{-\lambda t}) \cdot (e^{\lambda t} \ b)^2 = e^{\lambda t} \cdot E_2(B).$$

In turn, the latter expression $e^{\lambda t} \cdot E_2(B)$ grows exponentially with time.

The same argument applies to an arbitrary manifold M and an arbitrary velocity field v having a stagnation point with only one positive eigenvalue. One can always direct the initial magnetic field along the stretching eigenvector in some neighborhood of the stagnation point. In a cylindrical neighborhood of the stagnation point one obtains

$$E_2(B_t) = \|B_t\|^2_{L_2(M)} \geq \|B_t\|^2_{L_2(R)} \geq e^{\lambda t} \cdot \|B\|^2_{L_2(R)}$$
$$\geq C \cdot e^{\lambda t} \cdot \|B\|^2_{L_2(M)} = C \cdot e^{\lambda t} \cdot E_2(B),$$

where $B_t := g_*^t B$ is the image of the field B under the phase flow of the vector field v, and C is some positive constant. □

REMARK 1.8. This gives the exponential growth of B in any L^d-norm with $d > 1$. An exponential stretching of particles (being the key idea of the above construction) will be observed in all dynamo variations below. The result is still true if the stagnation point has several positive eigenvalues, say, for a point with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$, provided that $d \cdot \lambda_1 + \lambda_{k+1} + \cdots + \lambda_n > 0$, or even if the same inequality holds for the real parts of complex eigenvalues.

Even for the L^1-norm, one can provide such growth of the E_1-magnetic energy if the number of connected components of the intersection $R_t \cap R$ increases exponentially with time t. We shall observe it in the next section for the Anosov diffeomorphism of the two-torus and for any map with a Smale horseshoe.

§2. Discrete dynamos in two dimensions

2.A. Dynamo from the cat map on a torus. The main features of diversified dynamo schemes can be traced back to the following simple example (see [Arn8, AZRS1]).

Let the underlying manifold M be a two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ endowed with the standard Euclidean metric. Define a linear map $A : \mathbb{T}^2 \to \mathbb{T}^2$ to be the *cat map*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mod 1.$$

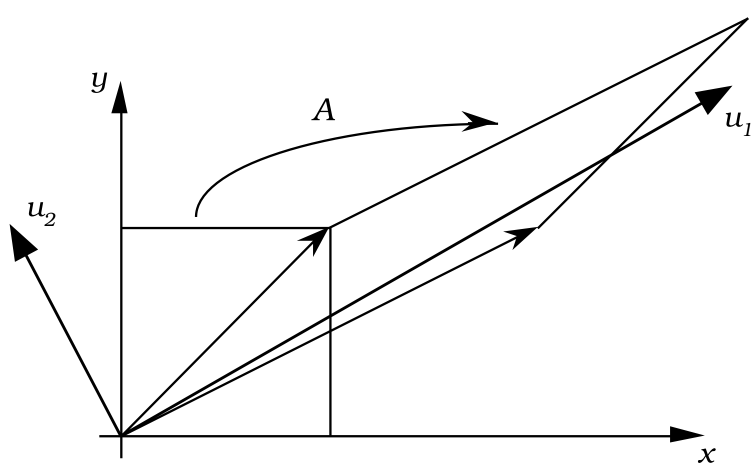

FIGURE 60. The cat map.

The stretching (respectively, contracting) directions at all points of the torus are given by the eigenvector $u_1 \in \mathbb{R}^2$ (respectively, $u_2 \in \mathbb{R}^2$) of A, corresponding to the eigenvalue $\chi_1 = (3+\sqrt{5})/2 > 1$ (respectively, $\chi_2 = (3-\sqrt{5})/2 < 1$; see Fig.60).

The constant magnetic field B_0, assuming the value $u_1 = \begin{pmatrix} 1+\sqrt{5} \\ 2 \end{pmatrix}$ at every point of $\mathbb{T}^2$, is stretched by the factor χ_1 with every iteration of A.

A diffeomorphism $A: M \to M$ of a compact manifold M is called an *Anosov map* if M carries two invariant continuous fields of planes of complementary dimensions such that the first one is uniformly stretched and the second one is uniformly contracted. The cat map is a basic example of an Anosov map.

REMARK 2.1. Taking the magnetic diffusion into account does not spoil the example of the cat dynamo. The iterations $B_{n+1} = \exp(\eta\Delta)\,[A\,(B_n)]$ with $B_0 = B$ (and $\eta \neq 0$) give the same exponential growth in spite of the diffusion. Indeed, the field B is constant, and hence the diffusion does not change the field or its iterations:

$$\|B_n\|_{L_2} = \chi_1^n \|B_0\|_{L_2}.$$

Furthermore, one can pass from a linear automorphism of the two-torus to an arbitrary smooth diffeomorphism $g: \mathbb{T}^2 \to \mathbb{T}^2$ ([Ose3]; see Section 2.C below).

REMARK 2.2. The cat map $A: \mathbb{T}^2 \to \mathbb{T}^2$ provides an example of a nondissipative L^1-dynamo. It provides the exponential growth of the number of connected components in the intersection $R \bigcap A^n(R)$ of the rectangle R (from Theorem 1.7) with its iterations.

The cat map on the two-torus can be adjusted to produce a nondissipative dynamo action on a two-dimensional disk. The idea is the use of a ramified two-sheet covering $\mathbb{T}^2 \to S^2$, along with an Anosov automorphism of $\mathbb{T}^2$; see Fig.61. The central symmetry of the plane $\mathbb{R}^2$ provides an involution on the torus, and its orbit space is homeomorphic to the sphere S^2. The automorphism

$$A^3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

of $\mathbb{R}^2$ has four fixed points on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the points with integral and semi-integral coordinates on $\mathbb{R}^2$, and therefore it descends to the quotient space $\mathbb{T}^2/\mathbb{Z}_2 = S^2$.

This idea was explored as early as in 1918 by Lattes [Lat], and is rather popular now in models of ergodic theory and holomorphic dynamics [Lyub, Kat1].

In the context of dynamo theory, constructions exploiting the maps on the (nonsmooth) quotient $\mathbb{T}^2/\mathbb{Z}_2$ appeared in [Gil2], along with results of numerical simulations. A substantial analysis given there shows that for the Lattes map of the disk any magnetic field after several iterations has a fine structure in which oppositely oriented vectors appear arbitrarily close to each other (Fig.62). In the presence of

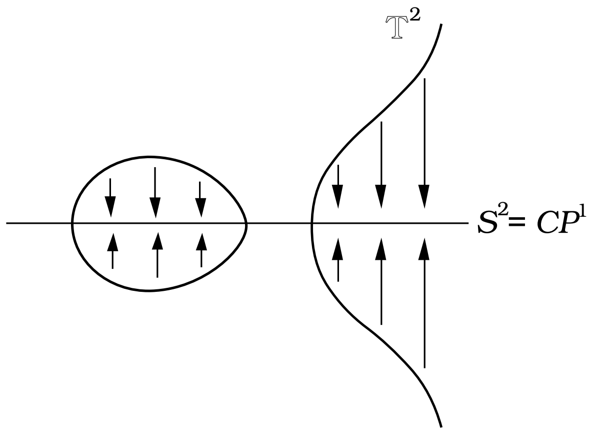

FIGURE 61. The covering of the sphere by a torus ramified at four points. The torus $w = \sqrt{z^3 - z}$ in $\mathbb{C}P^2$ maps to the sphere $S^2 = \mathbb{C}P^1$ by the projection $(z, w) \mapsto z$.

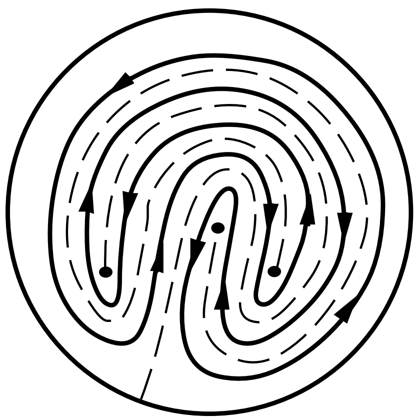

FIGURE 62. Cancellations in the magnetic field under iterations (from [Gil2]).

diffusion the dissipation action, large at these places, inevitably prevents the rapid growth of magnetic energy.

The trick to overcoming this difficulty in three-dimensional dynamo models is to include a nontrivial shear of "different pieces" of the manifold into an iteration procedure such that diffusion averaging mostly affects the parts with the same direction of the magnetic field (see [Gil2, B-C, ChG]).

There remains a possibility that a dissipative fast dynamo action in domains in $\mathbb{R}^3$ can be produced analytically, starting with the construction, known in ergodic theory, of a Bernoulli diffeomorphism on the disk.

DEFINITION 2.3. The *Lyapunov exponent* of a map g at a point x in the direction of a tangent vector B is the growth rate of the image length of B under the iterations of g measured by

$$\chi(x, B) := \liminf_{n\to\infty} \frac{\ln \|g_*^n B\|}{n}.$$

The Lyapunov exponents of the Lattes type diffeomorphism of the two-dimensional disk D^2 can be made positive almost everywhere (see [Kat2]). The fields of stretching directions are, in general, nonsmooth. The diffusion term of a dissipative dynamo should correspond to "random jumps of particles," in addition to the smooth evolution along the flow of v (in the spirit of [K-Y]).

2.B. Horseshoes and multiple foldings in dynamo constructions.

DEFINITION 2.4. A phase point of a (discrete or continuous) dynamical system is said to be *homoclinic* if its trajectory has as its limits as $t \to \pm\infty$ one and the same stationary point of the system (Fig.63).

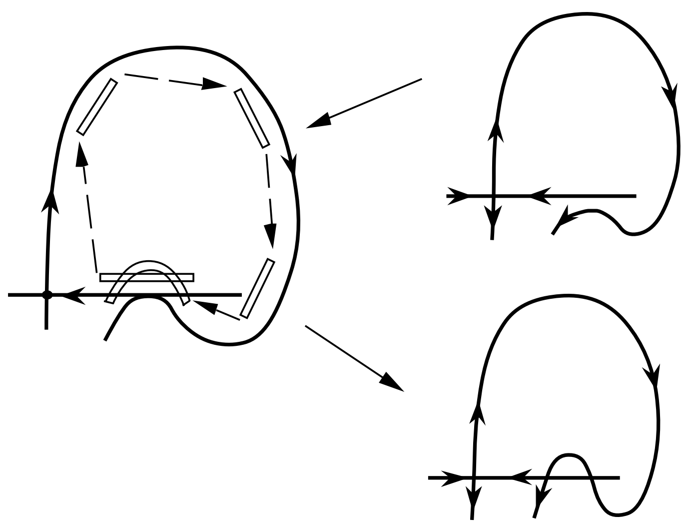

FIGURE 63. A homoclinic point and its bifurcation.

PROPOSITION 2.5 [Koz1]. *Any area-preserving map of a surface having a homoclinic point can serve as a nondissipative two-dimensional L^1-dynamo.*

PROOF. Assume that $g : D^2 \to D^2$ is a (volume-preserving) map of a two-dimensional disk to itself having a *Smale horseshoe*. This means that there is a rectangle $R \subset D^2$ on which the map g is a composition of the following two steps. First, the rectangle is squeezed in the horizontal direction by the factor e^λ and stretched in the vertical direction by the same factor, keeping its area the same (Fig.64).

Then the rectangle obtained is bent in such a way that it intersects the original rectangle twice (see Fig.64).

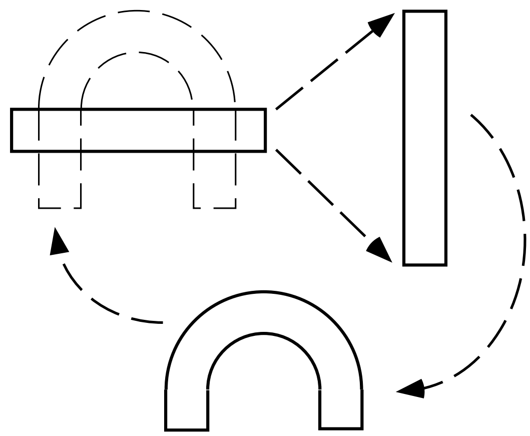

FIGURE 64. Smale's horseshoe.

Under the iterations of the procedure described, the number of connected components of the intersections $(g^n R) \cap R$ grows as 2^n, where n is the number of iterations. The argument of the preceding theorem now applies to the L^1-norm of the magnetic field B. Hence, $\|B_n\|_{L^1} \geq C \cdot 2^n \|B_0\|_{L^1}$.

In a neighborhood of a homoclinic point a generic map admits a Smale horseshoe. The L^1-norm of the restriction of the field to this horseshoe grows exponentially. This completes the proof. □

REMARK 2.6. The dynamics of points in the invariant set of the horseshoe is described by means of *Bernoulli sequences* of two symbols. We put the label 0 or 1 at position n if the point $g^n x$ belongs, respectively, to the left or to the right leg of the Smale horseshoe. The invariant sets of all C^2-horseshoes in a disk have measure zero [BoR]. The condition on smoothness is essential here: There is an example of a C^1-horseshoe of positive measure (see [Bow]).

We have here the same difficulty that is well known in the theory of stochastization of analytical Hamiltonian dynamical systems in a neighborhood of a periodic orbit that is the limit of the trajectory of a homoclinic point. Bifurcations of nontransversal intersections of stable and unstable manifolds of such a periodic orbit leads to the appearance of the so-called invariant set of nonwandering points. (A point a of a dynamical system g^t is called *wandering* if there exists a neighborhood $U(a)$ such that $U(a) \cap g^t U(a) = \emptyset$ for all sufficiently large t.) Though the existence of Bernoulli-type chaos on this set has been known since the classical work of Alekseev [Al], it is still unknown whether the corresponding invariant set of the phase space has positive or zero measure. The "multiple folding" occurring in such a system is basically of the same nature as the folding in nondissipative dynamo models.

We observed such a folding of the evolved magnetic field in both the horseshoe and Lattes constructions. The following theorem shows that it is unavoidable in all dynamo constructions on the disk.

PROPOSITION 2.7 [Koz1]. *Let $g : D^2 \to D^2$ be a smooth volume-preserving diffeomorphism of the two-dimensional disk with the following properties. There exist an open subset $U \subset D^2$ invariant for g and a continuous* oriented *line field that is defined on U and invariant for g. Then the Lyapunov exponents of g vanish almost everywhere on U.*

Notice that the Lattes map allows one to construct a diffeomorphism of the disk such that the invariant set U is this disk with 3 small disks removed and the Lyapunov exponents are positive (and equal to $\ln \chi_1 = \ln(3+\sqrt{5})/2 > 0$) on U. However, the field of the stretching directions is not oriented. This is the major obstacle to constructing a realistic dynamo on a disk: A nonzero diffusion mixes up the vectors of the magnetic field B that are oppositely oriented and hence prevents exponential growth of the field energy.

PROOF OF PROPOSITION. Assume the contrary, i.e., that the Lyapunov exponents do not vanish on a set U_1 that has positive measure. It is shown in [Kat2] that periodic points of g with homoclinic intersections of their stable and unstable manifolds are dense in the closure of U_1. Consider such a point x_0 and orient upwards the unstable direction at this point (Fig.65). Then all lines defined on the unstable manifold W^u of x_0 are tangent to it and have a compatible orientation. However, if the unstable manifold W^u meets the stable manifold W^s in one direction, then it intersects W^s roughly in the opposite direction the next time, by virtue of the simple-connectedness of the disk. (On the other hand, for instance on the torus, the unstable manifold *can* intersect the stable manifold at two consecutive points in the same direction.) Thus, the orientation of the lines oscillates and cannot be extended continuously to the point x_0. □

In order to take into account this "mixing up" effect in the nondissipative case ($R_m = \infty$), we introduce the following definition.

DEFINITION 2.8. A volume-preserving diffeomorphism $g : M \to M$ of a manifold M is called a nondissipative *mean dynamo* if there exist a divergence-free vector field B and a 1-form ω such that the integral of the contraction of the form ω with the field $g_*^n B$ grows exponentially as n tends to infinity. Denote by λ_m the maximal increment of the growth:

$$\lambda_m = \sup_{\omega, B} \limsup_{n\to\infty} \frac{1}{n} \ln \left| \int_M \omega(g_*^n B)\, \mu \right| .$$

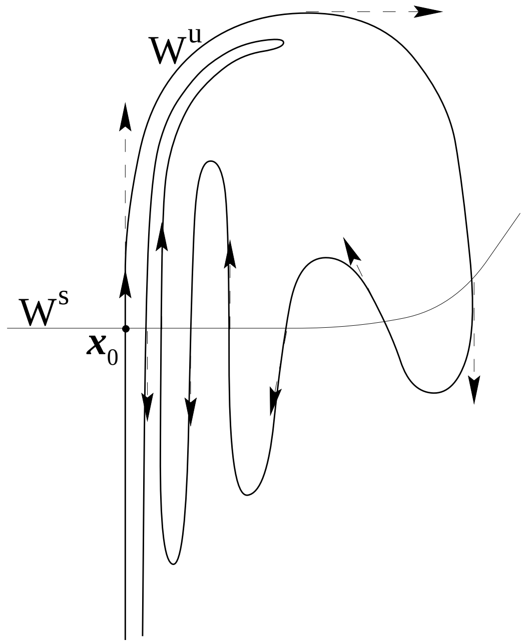

FIGURE 65. Oriented linear elements on the unstable manifold.

A similar definition can be introduced in the case of a vector field in place of the diffeomorphism g. The notion of a mean dynamo is stronger than that of a nondissipative L^d-dynamo ($d \geq 1$): Any mean dynamo is a nondissipative L^d-dynamo. Another important distinction between these two concepts is the following. A sufficient condition for a nondissipative dynamo is provided by the special behavior of the diffeomorphism g in a neighborhood of a fixed point (Theorem 1.7). The situation in cases of a mean dynamo or dissipative dynamo is different. Knowing only the local behavior of g is not enough to determine whether g is a mean or a dissipative fast dynamo.

If the dimension of the manifold equals 2, a diffeomorphism g is a fast dissipative dynamo if and only if it is a mean nondissipative dynamo. In this case ($\dim M = 2$) the growth rate λ_m is determined by the operator $g_{*1} : H_1(M) \to H_1(M)$, the action of g on the first homology group of the surface M, just as in the case of the dynamo increment.

THEOREM 2.9 [Koz1]. *An area-preserving diffeomorphism g of a surface M is a mean nondissipative dynamo if and only if the linear operator g_{*1} has the eigenvalue χ with $|\chi| > 1$. The mean dynamo increment λ_m is equal to* $\ln |\chi|$.

2.C. Dissipative dynamos on surfaces. Now suppose that there is a nonzero dissipation in the system. In the case of a torus, an arbitrary diffeomorphism g can be described as $g(x) = \Phi x + \psi(x)$, $(\ x \mod 1)$, the sum of a linear transformation $\Phi \in SL(2, \mathbb{Z})$ and a doubly periodic function ψ. In [Ose3] it is shown that for a dissipative dynamo, as $\eta \to 0$, the energy growth of a magnetic field on $\mathbb{T}^2$ is controlled solely by the matrix Φ. This matrix represents the action of g on the homology group $H_1(\mathbb{T}^2, \mathbb{R})$.

THEOREM 2.10 [Ose3]. *Let $g(x) = \Phi x + \psi(x)$ be a diffeomorphism (not necessarily area-preserving) of the two-dimensional torus $\mathbb{T}^2$. Then g is a fast dissipative dynamo as $\eta \to 0$ if and only if the matrix Φ has the eigenvalue χ with $|\chi| > 1$. The dynamo increment $\lambda_0 = \lim_{\eta\to 0} \lambda_\eta$ is equal to the eigenvalue $\ln|\chi|$:*

$$\lambda(\eta) = \lim_{n\to\infty} \frac{\ln\|B_n\|}{n} \to \ln|\chi| \quad \textit{as} \quad \eta \to 0,$$

for almost every initial vector field B_0.

Here

$$B_{n+1} = \exp(\eta\Delta)\,[g_* B_n]\,,\ n = 0, 1, \ldots,$$

in the area-preserving case, and

$$B_{n+1} = \exp(\eta\Delta)\left[(g_* B_n)/|\frac{\partial g}{\partial x}|\right],$$

where $|\frac{\partial g}{\partial x}|$ is the Jacobian of the map g in the non-area-preserving case. The norm $\|.\|$ is the L^2-norm of a vector field.

It turns out that the dynamo increment is determined exclusively by the action of g on the first homology group in the much more general situation of an arbitrary two-dimensional manifold M. For any M, each diffeomorphism $g : M \to M$ induces the linear operator g_{*i} in every vector space $H_i(M, \mathbb{R})$, the i^{th} homology group of M, $i = 0, \ldots, \dim M$. The following statement generalizes Theorem 2.10 (and is similar to the discrete dynamos considered in Theorem 3.20).

THEOREM 2.11 [Koz1]. *Let $g : M \to M$ be an area-preserving diffeomorphism of the two-dimensional compact Riemannian manifold M. Then g is a dissipative fast dynamo if and only if the linear operator g_{*1} has an eigenvalue χ with $|\chi| > 1$. The dynamo increment $\lambda(\eta)$ is equal to $\ln|\chi|$ and hence is independent of η:*

$$\lim_{n\to\infty} \frac{\ln\|B_n\|}{n} = \ln|\chi|$$

for almost every initial vector field B_0. ($B_{n+1} = \exp(\eta\Delta)\,[g_* B_n]$, $n = 0, 1, \ldots,$ and Δ is the Laplace–Beltrami operator on M.)

REMARK 2.12. An eigenvalue χ with $|\chi| > 1$ exists for "most" of the diffeomorphisms of the surfaces different from the 2-sphere. Indeed, the determinant of $g_{*1} : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$ is equal to 1, since g is a diffeomorphism.

PROOF OF THEOREM 2.11. First show that

$$\lim_{n\to\infty} \frac{\ln \|B_n\|}{n} \leq \ln |\chi|.$$

Indeed, consider the operator $A^* = g^* \circ \exp(\eta\Delta)$ in the space of 1-forms that is L^2-conjugate to the operator $A = \exp(\eta\Delta) \circ g_*$. Let ω be its (complex) eigenvector, i.e., $A^*\omega = \kappa\ \omega$ with $\ln|\kappa| = \lambda(\eta)$. Such an ω exists because the norm of the conjugate operator A^* equals the norm of the operator A, and A^* is a compact operator. Note that $|\kappa| \geq 1$, since $\det|g_{*1}| = 1$. Assume that $|\kappa| > 1$ (otherwise the statement is evident).

The exterior derivative operator d commutes with g^* and with Δ. Therefore, $g^* \exp(\eta\Delta) d\omega = \kappa\, d\omega$, where g^* and Δ now act in the space of 2-forms. The pullback operator $g^* : \Omega^2(M, \mathbb{C}) \to \Omega^2(M, \mathbb{C})$ preserves the L^2-norm, while the Laplace–Beltrami operator Δ does not increase it. Hence, if $|\kappa| > 1$, it follows that the form ω is closed, $d\omega = 0$ (cf. Theorem 3.6 below).

Furthermore, the Laplace–Beltrami operator Δ does not affect the cohomology class $[\omega]$ of the closed form ω, so $g^{*1}[\omega] = \kappa[\omega]$, where g^{*1} is an action of g on the first cohomology group $H^1(M, \mathbb{C})$ containing $[\omega]$.

Therefore, either $[\omega] \neq 0$ and hence $|\kappa| \leq |\chi|$ (i.e., $\lambda(\eta) \leq \ln|\chi|$), or $[\omega] = 0$. In the latter case there is a function α such that $d\alpha = \omega$ and $g^* \exp(\eta\Delta)\alpha = \kappa\,\alpha$. The same argument as before shows that $\alpha = 0$, which contradicts the assumption that ω is an eigenvector. Thus, there remains only the possibility that $\lambda(\eta) \leq \ln|\chi|$.

To show that $\lambda(\eta) \geq \ln|\kappa|$, we consider a cohomological class that is an eigenvector of g^{*1} with eigenvalue χ. Such a class is invariant under A^* and there is an eigenvector of A^*, with eigenvalue χ, so $\lambda(\eta) = \ln|\chi|$. □

Theorem 2.11 holds also if g is not area preserving

$$B_{n+1} = \exp(\eta\Delta) \left[(g_* B_n) / |\frac{\partial g}{\partial x}| \right].$$

It is easy to see that the conjugate operator has the same form as before: $A^* = g^* \exp(\eta\Delta)$.

2.D. Asymptotic Lefschetz number. The dynamo increment $\lambda(\eta)$ can also be viewed as an asymptotic version of the Lefschetz number of the diffeomorphism g (see [Ose3]).

DEFINITIONS 2.13. Let $g : M \to M$ be a generic diffeomorphism of an oriented compact connected manifold M. The *Lefschetz number* $L(g)$ of the diffeomorphism g is the following sum over all fixed points $\{x_i\}$ of g:

$$L(g) = \sum_{x_i} \operatorname{sign} \det \left[\frac{\partial g}{\partial x}(x_i) - \mathrm{Id} \right],$$

where $\frac{\partial g}{\partial x}$ is the Jacobi matrix of the diffeomorphism at a fixed point and Id is the identity matrix. The *asymptotic Lefschetz number* $L_{\rm as}(g)$ is

$$L_{\rm as}(g) = \limsup_{n\to\infty} \frac{1}{n}\ln\,|L(g^n)|$$

(in our example the lim sup is simply lim, as we shall see).

The *Lefschetz formula* relates the contribution of fixed points of the diffeomorphism g to its action on the homology groups:

$$L(g) = \sum_i (-1)^i \text{ Trace } (g_{*i}),$$

where the linear operators g_{*i} in the vector spaces $H_i(M,\mathbb{R})$, the $i^{\rm th}$ homology group of M, are induced by the diffeomorphism $g : M \to M$.

Now the visualization of the dynamo increment $\ln|\chi|$ as the asymptotic Lefschetz number $L_{\rm as}(g)$ for $g : \mathbb{T}^2 \to \mathbb{T}^2$ (and more generally, for any $g : M \to M$) is an immediate consequence of the following rewriting of the Lefschetz formula:

$$\begin{aligned} L(g^n) &= \sum_i (-1)^i \text{ Trace } ((g^n)_{*i}) = 1 - \text{ Trace } (\Phi^n) + 1 \\ &= (1-\chi^n)(1-\chi^{-n}) = -\chi^n + O(1) \quad \text{for } |\chi| > 1, \;\; n \to \infty. \end{aligned}$$

Here we used that for $i = 0, 2$ the maps g_{*i} act identically on $H_i(M,\mathbb{R}) = \mathbb{R}$. The automorphism $g_{*1} : H_1(M,\mathbb{R}) \to H_1(M,\mathbb{R})$ can be nontrivial, and it is given by the matrix Φ in the case of a torus $M = \mathbb{T}^2$.

§3. Main antidynamo theorems

3.A. Cowling's and Zeldovich's theorems. Traditionally, necessary conditions on the mechanism of a dynamo are formulated in the form of *antidynamo theorems.* These theorems specify (usually, geometrical) conditions on the manifold M and on the velocity vector field v under which exponential growth of the L^2-norm of a magnetic vector field (or, more generally, of any tensor field) on the manifold is impossible. In this section, the magnetic diffusivity η is assumed to be nonzero.

This direction of dynamo theory began with the following theorem of Cowling [Cow]: A steady magnetic field in $\mathbb{R}^3$ that is symmetric with respect to rotations about a given axis cannot be maintained by a steady velocity field that is also symmetric with respect to rotations about the same axis. This theorem stimulated numerous generalizations (see [Zel1, K-R, R-S]). These works show that the symmetry properties of the velocity field are irrelevant. The symmetry of the magnetic field alone prevents its growth:

THEOREM 3.1. *A translationally, helically, or axially symmetric magnetic field in* $\mathbb{R}^3$ *cannot be maintained by a dissipative dynamo action.*

In what follows we shall be concerned mostly with a somewhat dual problem, in which one studies restrictions on the geometry of *velocity fields* that cannot produce exponential growth of any magnetic field.

Consider a domain in three-dimensional Euclidean space that is invariant under translations along some axis (say, the vertical z-axis). A *two-dimensional motion* in this three-dimensional domain is a (divergence-free) horizontal vector field ($v_z = 0$) invariant under translations along the vertical axis.

Ya.B. Zeldovich considered the case where the projection of the domain to the horizontal (x, y)-plane along the vertical z-axis is bounded and simply connected.

THEOREM 3.2 [Zel1]. *Suppose that the initial magnetic field has finite energy. Then, under the action of the transport in a two-dimensional motion and of the magnetic diffusion, such a field decays as* $t \to \infty$.

In short, "there is no fast kinematic dynamo in two dimensions."

We put this consideration into a general framework of the transport–diffusion equation for tensor densities on a (possibly non-simply connected) manifold.

3.B. Antidynamo theorems for tensor densities. Here we discuss to what extent the antidynamo theorems can be transferred to a multiconnected situation. It happens that in the nonsimply connected case, instead of the decay of the magnetic field, one observes the approach of a stationary (in time) regime.

The assumption that the medium is incompressible turns out to be superfluous. In the compressible case we need merely consider the evolution of tensor densities instead of that of vector fields. The condition on the evolving velocity field v to be divergence-free can be omitted as well: We shall see that the evolution automatically leads, in the end, to a solenoidal density for an arbitrary initial condition. What really matters is the dimension of the underlying manifold.

Throughout this section we follow the paper [Arn10], to which we refer for further details.

Now we deal with an evolution of differential k-forms on a compact n-dimensional connected Riemannian manifold M without boundary. A differential k-form ω on

M evolves under transport by the flow with velocity field v and under diffusion with coefficient $\eta > 0$ according to the law

$$\frac{\partial \omega}{\partial t} + L_v \omega = \eta \Delta \omega. \tag{3.1}$$

The Lie derivative operator L_v is defined by the condition that the form is frozen into the medium. In other words, draw vectors on the particles of the medium and on their images as the particles move with the velocity field v to a new place. Then the value of the form carried over by the action (3.1) with $\eta = 0$ does not change with time when the form is evaluated on the vectors drawn.

The linear operator L_v is expressed in terms of the operator i_v (substitution of the field v into a form as the first argument) and the external derivative operator d via the *homotopy formula* $L_v = i_v \circ d + d \circ i_v$. The Laplace–Beltrami operator Δ on k-forms is defined by the formula $\Delta = d\delta + \delta d$, where $\delta = *d*$ is the operator conjugate to d by means of the Riemannian metric on M. The metric operator $* : \Omega^k \to \Omega^{n-k}$ (pointwise) identifies the k-forms on the n-dimensional Riemannian manifold with $(n-k)$-forms.

In the case of a manifold M with boundary, one usually needs specification of vanishing boundary conditions for the forms and fields.

EXAMPLES 3.3. A) Suppose $M = \mathbb{E}^3$, Euclidean space with the metric $ds^2 = dx^2 + dy^2 + dz^2$. Specify a 2-form $\omega = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$ by choosing the vector field B with components P, Q, R; i.e., $\omega = i_B \mu$, where $\mu = dx \wedge dy \wedge dz$ is the volume element. For solenoidal fields v and B, Equation (3.1) on ω results in Equation (1.1) on the evolution of the magnetic field B.

B) For functions on $M = \mathbb{R}^3$ (the case of $(k = 0)$-forms), Equation (3.1) becomes the heat equation with transport:

$$\frac{\partial f}{\partial t} = -(v, \nabla) f + \eta \Delta f. \tag{3.2}$$

C) For a scalar density g (i.e., for $k = n$ and $\omega = g \cdot \mu$, where μ is the volume element on a Riemannian n-dimensional manifold), Equation (3.1) has the form

$$\frac{\partial g}{\partial t} = -\text{div } (g \cdot v) + \eta \Delta g, \tag{3.3}$$

where the relation $d(i_\xi \mu) = (\text{div } \xi) \cdot \mu$ is used.

DEFINITION 3.4. A closed k-form ω on M is called *stationary* if it obeys the equation

$$-L_v \omega + \eta \Delta \omega = 0. \tag{3.4}$$

THEOREM 3.5 [Arn10]. *The number of linearly independent stationary k-forms is not less than the k^{th} Betti number b_k of the manifold M.*

Recall, that the k^{th} Betti number of M is $b_k = \dim H_k(M, \mathbb{Z})$. Examples in which the number of stationary forms is strictly larger than b_k are given below.

THEOREM 3.6 [Arn10]. *If the diffusion coefficient η is large enough, then the number of linearly independent stationary k-forms is equal to the k^{th} Betti number, and*

a) *In each cohomology class of closed k-forms there is a stationary form.*
b) *There is exactly one such form.*
c) *Any closed k-form evolved according to Equation (3.1) tends as $t \to \infty$ to a stationary form belonging to the same cohomology class, i.e., to a stationary form with the same integrals over every k-dimensional cycle.*
d) *The evolution defined by Equation (3.1) with any initial conditions leads in the limit to a closed form.*
e) *All solutions of Equation (3.4) are closed forms.*

REMARK 3.7. Examples below show that items b) and c) are no longer true if the viscosity is sufficiently low (except for the cases $k = 0, 1$ or n). Exponentially growing solutions are observed for the case $k = 2, n = 3$ (which is most interesting physically, see, e.g., [AKo]) on a Riemannian manifold M, where for small diffusivity η the dimension of the space of stationary solutions is at least $2 > b_2(M) = 1$. The general Theorem 3.6 admits the following special cases:

THEOREM 3.8 ($k = 0$). *For the heat equation (3.2) with transport for scalars at every positive value of the diffusion coefficient η: a) every stationary solution is constant and b) the solution with any initial condition tends to a constant as $t \to \infty$.*

THEOREM 3.9 ($k = n$). *For the heat equation (3.3) with transport for scalar densities at every positive value of η:*

a) *The dimension of the space of stationary solutions of Equation (3.3) is equal to 1.*
b) *There exists a unique stationary solution with any value of the integral over the entire manifold.*
c) *The solution with any initial conditions tends as $t \to \infty$ to a stationary solution with the same integral.*
d) *In particular, the solution with initial conditions $g = \operatorname{div} B$ converges to 0 as $t \to \infty$ regardless of the field B.*

REMARK 3.10. The dynamo problem for scalar densities retains many features of the vector dynamo problem. The discussion and numerical evidence in [Bay2] show that the eigenfunctions develop a singular structure as diffusivity tends to zero.

On the other hand, the study of scalar densities (or more generally, of differential k-forms) and of their asymptotic eigenvalues allows one to prove the Morse inequalities and their generalizations by means of the method of short-wave ("quasi-classical") asymptotics [Wit1].

THEOREM 3.11 ($k = 1$). *For any positive value of η in Equation (3.1) for closed 1-forms:*

a) *The dimension of the space of stationary solutions is equal to $b_1(M)$, the one-dimensional Betti number of the manifold.*
b) *There exists a unique stationary solution with any given values of the integrals over independent 1-cycles.*
c) *The solution with any initial conditions tends as $t \to \infty$ to a stationary solution with the same integrals.*

The dynamo problem for a magnetic vector field on a compact n-dimensional Riemannian manifold is described by Equation (3.3) for an $(n-1)$-form ω. The corresponding evolution of the vector density B where $\omega = i_B\mu$ is given by the law

$$\frac{\partial B}{\partial t} = -\{v, B\} - B \text{ div } v + \eta \Delta B.$$

THEOREM 3.12 ($k = n-1$). *The divergence of the evolved density B tends to zero for every value of the diffusion coefficient $\eta > 0$. In particular, every stationary solution of Equation (3.3) for $(n-1)$-forms is closed.*

COROLLARY 3.13. *Every solution of Equation (3.1) for 1-forms on a compact two-dimensional manifold tends to a stationary closed 1-form as $t \to \infty$. For a simply connected two-dimensional manifold, every solution of Equation (3.1) tends to zero (cf. Theorem 3.2).*

3.C. Digression on the Fokker–Planck equation. A problem of large-time asymptotics for scalar density transport with diffusion is already interesting in the one-dimensional case, and it arises in the study of the *Fokker–Planck equation*

$$u_t + (uv)_x = \eta u_{xx}.$$

It describes the transport of a density form $u(x)dx$ by the flow of a vector field $v(x)\partial/\partial x$ accompanied by small diffusion with diffusion coefficient η.

Suppose, for instance, that the system is periodic in x and that the velocity field v is potential. Introduce the potential U for which $v = -\text{grad}\, U$ (the attractors of v are then the minima of U).

The stationary *Gibbs solution* of the equation has the form

$$\bar{u}(x) = \exp(-U(x)/\eta),$$

and is sketched in Fig.66. It means that if the diffusion coefficient η is small, the density distribution is concentrated near the minima of the potential. These minima are the attractors of the velocity field v. The mass is (asymptotically) concentrated in the vicinity of the attractor, corresponding to the lowest level of the potential. (Note that the total mass is preserved by the equation: $\int u(x,t)\, dx = \text{const.}$) In the sequel, we suppose that the potential is generic and has only one global minimum.

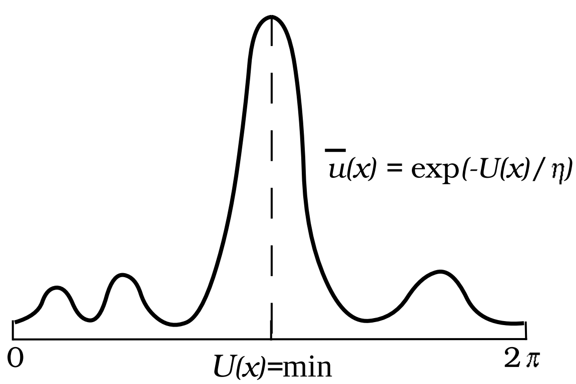

FIGURE 66. The stationary solution of the Fokker-Planck equation.

Suppose we start with a uniformly distributed density, say, $u = 1$ everywhere. The evolution will immediately make it nonuniform, and we shall see Gauss-type maxima near all the attractors of v.

At the beginning the attractor that produces the most pronounced maximum will be the one for which the contraction coefficient (the modulus of the eigenvalue of the derivative of v at its zero point) assumes the maximal value.

Later, however, after some finite time (independent of η), the distribution will be similar to a finite set of point masses at the attractors. At this stage the most pronounced attractor will be the one collecting the largest mass. This mass, at the beginning, will be the initial mass in the basin of the attractor. Hence, in general, this attractor will be different from the one that appeared first.

The next step will consist in (slow) competition between different attractors for the masses of particles kept in their neighborhoods. This competition is (asymptotically) described by a system $\dot{m} = Am$ of linear ordinary differential equations with constant coefficients. The elements of the corresponding matrix A are the so-called *tunneling coefficients.* They are exponentially small in η, and hence the tunneling phase of the relaxation process is exponentially long ($t \sim \exp(\text{const }/\eta)$). In practice, this means that in most numerical simulations one observes, instead of the limiting (Gibbs) distribution (where almost all the mass is concentrated in one place), an intermediate distribution (concentrated at several points). This intermediate distribution evolves so slowly that one does not observe this evolution in numerical simulations.

At the end ($t \to \infty$), one of the attractors will win and attract almost all the mass. This attractor is given by the Gibbs solution, and it is somewhat unexpected: It is neither the one with the maximal initial growth of density, nor the one containing initially the most mass. In Russian, it was called the "general attractor," or the "Attractor General" (since it is as difficult to predict as it was to predict who would become the next "Secretary General").

Consider an evolution of the density (i.e., of a differential n-form) $u\mu$ on a connected compact n-dimensional Riemannian manifold with Riemannian volume element μ. The evolution under the action of a gradient velocity field $v = -\text{grad } U$ and of the diffusion is described by the equation

$$u_t + \text{div } (uv) = \eta \Delta u.$$

(Note that $(\text{div } (uv))\mu = d(i_v(u\mu)) = L_v(u\mu)$ and $(\Delta u)\mu = (\text{div grad } u)\mu = d\delta(u\mu)$, since $u\mu$ is a differential n-form, and hence it is closed.)

The spectrum of the evolution operator $u \mapsto -\text{div } (uv) + \eta\Delta u$ consists of a point 0 (corresponding to the Gibbs distribution), accompanied by a finite set of eigenvalues very close to 0 as $\eta \to 0$. The number of such eigenvalues is equal to the number of attracting basins of the field v, and it is defined by the Morse complex of the potential U. There is a "spectral gap" between these "topologically necessary" eigenvalues and the rest of the spectrum (which remains at a finite distance to the left of the origin as $\eta \to 0$).

The tunneling linear ordinary differential equation is the asymptotic ($\eta \to 0$) description of what is happening in the finite-dimensional space spanned by the eigenvectors corresponding to the eigenvalues close to 0. The eigenvalues are of order at most $\exp(-\text{const}/\eta)$ as $\eta \to 0$, while the characteristic tunneling time is of order $\exp(\text{const}/\eta)$. This explains the slow decay of the modes corresponding to the nongeneral attractors.

REMARK 3.14. In spite of the evident importance of the problem, the description of the events given above does not seem to be presented in the literature (cf., e.g., [F-W]). The above description is based on an unpublished paper by V.V. Fock [Fock] and on the work of Witten [Wit1] and Helffer [Helf].[1] Fock also observed that the asymptotics of the density at a generic point of the border between the basins of two competing attractors involve a universal (erf) function in the transversal direction to the boundary hypersurface. Here, time is supposed to be large but fixed while $\eta \to 0$. The density is asymptotically given by an almost eigenfunction (quasimode) concentrated in one basin from one side of the boundary, and by the quasimode corresponding to the other basin from the other side. The transition from one asymptotics to the other at the boundary is, according to Fock, described by the step-like "erf."

The preceding theory has an extension to the case of k-forms, where the small eigenvalues correspond to the critical points of the potential of index k (see [Wit1]).

Remark 3.15 (C. King). In the potential (one- and higher-dimensional) case, the operator $L_v - \eta\Delta$ is conjugate to a nonnegative self-adjoint operator. It shows that the spectrum is real and nonnegative.

Namely, the change of variables $\tilde{u}(x) = e^{U(x)/2\eta}u(x)$ sends the one-dimensional operator $L_v - \eta\Delta$ to the operator $\eta D_\eta^* D_\eta$, where

$$D_\eta := \frac{d}{dx} + \frac{v(x)}{2\eta} = e^{U(x)/2\eta}\frac{d}{dx}e^{-U(x)/2\eta}.$$

The latter is known as the Witten deformation of the gradient (see its spectral properties in [Helf]).

Semiclassical asymptotics of spectra of a very general type of elliptic self-adjoint operators are treated in [Sh1] (see also [Sh2]) .

Remark 3.16. The case where the velocity field is locally (but not globally) gradient is very interesting. This may already happen on the circle. In that case, the Gibbs formula $\bar{u}(x) = \exp(-U(x)/\eta)$ is meaningful only on the covering line. The potential function is no longer a periodic function, but a pseudoperiodic one (the sum of linear and periodic functions).

For every local minimum of the potential, we define the *threshold* as the minimal height one has to overcome to escape out of the well to infinity, Fig.67. The general attractor is the one for which the threshold is maximal.

Many facts described above admit generalizations to the case of a pseudoperiodic potential in higher dimensions. In particular, the number of decaying eigenvalues is equal to the number of the field's critical points of the corresponding index [Fock].

[1] We thank M.A. Shubin and C. King for the adaptation of the general theory to our situation.

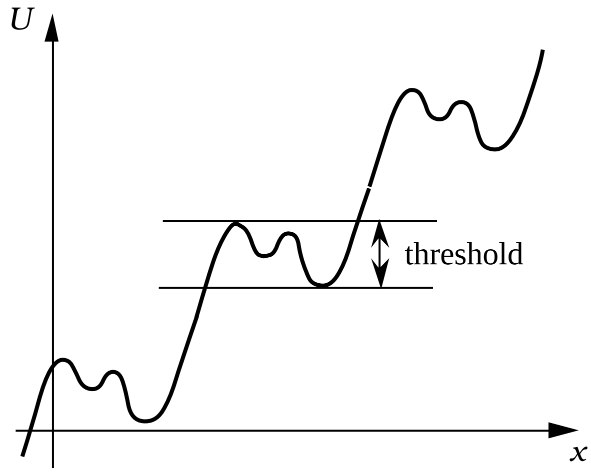

FIGURE 67. The threshold for a local minimum of the potential is the minimal height one has to overcome to escape to infinity.

Note that the description of the topology of pseudoperiodic functions is a rich and interesting question by itself already in two dimensions (see [Arn19, Nov3, SiK, GZ, Zor, Dyn, Pan]), where much remains to be done.

REMARK 3.17. B. Fiedler and C. Rocha developed in [F-R] an interesting topological theory of the attractors of nonlinear PDEs of the type

$$u_t = f(x, u, u_x) + \eta a(x) u_{xx}.$$

They computed the Morse complex defined by the heteroclinic connections between stationary solutions in terms of some permutations and meanders. A meander is formed by a plane curve and a straight line. The corresponding permutation transforms the order of the intersection points along the straight line into their order along the curve.

3.D. Proofs of the antidynamo theorems.

PROOF OF THEOREM 3.5 (according to E.I. Korkina). The operator $A = -L_v + \eta\Delta$ acts on the space H of closed k-forms on M. Denote by Ker A the set of solutions of the homogeneous equation $A\omega = 0$, and by Im A the image of A in the space H. The index ind $A =$ dim Ker $A -$ dim Coker A, where Coker $A = H/\text{Im } A$. The index of the Laplace operator Δ is zero and so is the index of A (which differs from $\eta\Delta$ only in lower-order terms: L_v is of the first order). This means that dim Ker $A =$ dim Coker A. But Im $A \subset$ Im d (since $A\omega = d(-i_v\omega + \eta \cdot \delta\omega)$ if $d\omega = 0$). It follows that

$$\dim\, (H/AH) \geq \dim\, (H/\{d\omega^{k-1}\}) = b_k$$

(De Rham's theorem). □

PROOF OF THEOREM 3.6. *(i)* The evolution defined by Equation (3.1) does not affect the cohomology class of the closed form ω, since $A\omega = d(-i_v\omega + \eta \cdot \delta\omega)$ is an exact form.

(ii) For forms ω from the orthogonal complement to the subspace of harmonic forms in the space of closed forms H the following relations hold:

$$(\omega, \omega) \leq \alpha(\delta\omega, \delta\omega), \tag{3.5a}$$

$$|(\omega, L_v\omega)| \leq \beta(\delta\omega, \delta\omega), \tag{3.5b}$$

where α and β are positive constants independent of ω.

Indeed, (3.5a) is the Poincaré inequality (or it can be viewed as the compactness of the inverse Laplace–Beltrami operator):

$$(\omega, \omega) \leq \alpha|(\Delta\omega, \omega)| = \alpha(\delta\omega, \delta\omega).$$

The inequality (3.5b) is a combination of the Schwarz and Poincaré inequalities. First note that $L_v\omega = d\,i_v\omega$ by virtue of the homotopy formula $L_v = i_v d + d\,i_v$ and since the form ω is closed. Then $(\omega, L_v\omega) = (\omega, d\,i_v\omega) = -(\delta\omega, i_v\omega)$, whence applying the Schwarz inequality to the latter inner product, we get

$$|(\delta\omega, i_v\omega)|^2 \leq (\delta\omega, \delta\omega)(i_v\omega, i_v\omega).$$

Now the required inequality (3.5b) follows from the above and (3.5a) in the form $(i_v\omega, i_v\omega) \leq \text{const} \cdot (\delta\omega, \delta\omega)$.

(iii) From (i) and (ii) it follows that in the space of exact forms the evolution defined by Equation (3.1) contracts everything to the origin if η is sufficiently large:

$$\frac{d}{dt}(\omega, \omega) = -2(\omega, L_v\omega) + 2\eta(\omega, d\delta\omega) \leq 2(\beta - \eta)(\delta\omega, \delta\omega) \leq -2\gamma(\omega, \omega),$$

if $\eta \geq \beta + \alpha\gamma$.

(iv) From (i) and (ii) it also follows that in an affine space of closed forms lying in one and the same cohomological class, Equation (3.1) defines the flow of *contracting* transformations (in the Hilbert metric of H), and hence, it has a fixed point. This proves assertions a)-c).

(v) Both L_v and Δ commute with d, and therefore $d\omega$ satisfies Equation (3.1) as well as ω. But the form $d\omega$ is exact, and therefore, in accordance with (iii), it tends exponentially to zero as $t \to \infty$. Thus the distance between $\omega(t)$ and the space of closed forms tends exponentially to zero as $t \to \infty$.

Moreover, the same contraction to zero is observed in H^1-type metrics that take into account the derivative, provided that the diffusion coefficient η is sufficiently large (it is proved similarly to (iii) by using inequalities of the type

$$(\Delta\omega, L_v\Delta\omega) \leq \beta(\Delta\omega, \Delta^2\omega)$$

for exact forms).

We now denote by $\omega = p + h + q$ the orthogonal decomposition of the initial form ω into exact, harmonic, and coexact (i.e., lying in the image of the operator δ) terms. Equation (3.1) assumes the form of the system

$$\dot{p} = A_1 p + A_2 h + A_3 q,\ \dot{h} = A_4 q,\ \dot{q} = A_5 q,$$

since for $q(0) = 0$ the form remains closed (i.e., $q(t) \equiv 0$), and a closed form retains its cohomology class (i.e., $\dot{h} = 0$ for $q = 0$).

Now, since $q(t) \to 0$ (in metrics with derivatives) exponentially, $h(t)$ tends to a finite limit (also in metrics with derivatives). But in accordance with (iii), the transformation $\exp(A_1 t)$ is contracting, and hence $p(t)$ also tends to a finite limit.

Therefore, $\omega(t)$ converges to a finite limit $p(\infty) + h(\infty)$, which is a closed form. This completes the proof of assertions d) and e). □

Proof of Theorem 3.8 (according to Yu.S. Ilyashenko and E.M. Landis). If the stationary solution were at any point larger than its minimum, it would immediately increase everywhere (since heat is propagated instantaneously) and would not be stationary (the so-called strengthened maximum principle). Consequently, it must be everywhere equal to its minimum, i.e., it must be constant.

The same reasoning shows that a time-periodic solution of Equation (3.2) must also be a constant. Hence, the operator $A = -L_v + \eta\Delta$ on functions has no pure imaginary eigenvalues and has a single eigenvector with eigenvalue zero (by the maximum principle); this means that zero is an eigenvalue of multiplicity one and all other eigenvalues lie strictly in the left half-plane.

Since A is the sum of an elliptic operator $\eta\Delta$ and the operator $-L_v$ of lower order, we can derive by standard arguments (from the information we have obtained about the spectrum) the convergence of all solutions to constants (even in metrics with derivatives). □

Proof of Theorem 3.9. The operator $A = -L_v + \eta\Delta$ on the right-hand side of Equation (3.3), which sends a density g to $-\text{div}\,(g \cdot v) + \eta\Delta g$, is conjugate to the operator $A^* = L_v + \eta\Delta$ on functions.

The eigenvalues of the operators A and A^* coincide, and therefore the dimensions of the spaces of stationary solutions of Equations (3.3) and (3.2) are identical. These dimensions are equal to 1, by Theorem 3.8. Assertions b) and c) of Theorem 3.9 follow from the information on the spectrum of the operator A^* that we obtained in proving Theorem 3.8. Assertion d) follows from c) since $\int(\operatorname{div} B)\,\mu = 0$. □

PROOF OF THEOREM 3.11. The operator $-L_v+\eta\Delta$ commutes with d. It follows that the solution with initial conditions $\omega_0 = df_0$ evolves under Equation (3.1) in the same way as the derivative df of the solution f of Equation (3.2) with initial condition f_0. From Theorem 3.8, it follows that $f \to$ const (with derivatives). This means that $df \to 0$; i.e., the exact 1-form degenerates over time. Thus the sole stationary solution that is an exact form is zero. But by Theorem 3.5, the dimension of the space of solutions of the stationary equation is not less than the first Betti number b_1, i.e., than the codimension of the subspace of exact 1-forms in the space of closed forms. Since the space of stationary solutions intersects the subspace of exact forms only at zero, its dimension is exactly equal to the Betti number b_1, and its projection onto the space of cosets of closed forms modulo exact forms is an isomorphism. This proves assertions a) and b). Assertion c) follows from the fact that the exact 1-forms have vanishing integrals over all 1-cycles. □

PROOF OF THEOREM 3.12. Since d and $-L_v + \eta\Delta$ commute, the n-form $d\omega = g \cdot \mu$ evolves according to the law (3.3). By Theorem 3.9d), the density g tends to zero as $t \to \infty$ (the condition $d\omega = g \cdot \mu$ means that g is the divergence of the vector field ξ that specifies the $(n-1)$-form $\omega = i_\xi\mu$). □

PROOF OF COROLLARY 3.13. For $n = 2$, the 1-form ω is an $(n-1)$-form. By Theorem 3.12, it becomes closed ($d\omega \to 0$) as $t \to \infty$. (Here the convergence to zero is exponential even in a metric with derivatives.) Using the same reasoning as in the proof of Theorem 3.6(v), and using Theorem 3.11 to study the behavior of the exact forms, we arrive at the conclusion that the limit of ω as $t \to \infty$ exists and is closed. □

3.E. Discrete versions of antidynamo theorems. Suppose that $g : M \to M$ is a diffeomorphism of a compact Riemannian manifold, $g^\star = (g^*)^{-1}$ is its action on differential forms (by forward translation), and h_η is the evolution of forms during some fixed time η under the action of the diffusion equation:

$$h_\eta := \exp(\eta\Delta), \quad f_\eta := h_\eta \circ g^\star.$$

Denote by $G^\star$ the action of $g^\star$ in the cohomology groups, $G^\star : H^*(M, \mathbb{R}) \to H^*(M, \mathbb{R})$.

THEOREM 3.18 [Arn10]. *i) The cohomology class of the closed form $f_\eta \omega$ is obtained from the class of the closed form ω by the action of $G^\star$.*

ii) If t is chosen sufficiently large and $G^\star$ is the identity transformation, then

a) *for any closed form ω the limit $\lim_{n\to\infty} f_\eta^n \omega$ exists;*

b) *this limit is a unique closed form cohomological to ω, and it is fixed under the action of f_η;*

c) *if the form ω is exact, then $f_\eta^n \omega \to 0$ as $n \to \infty$;*

d) *for any form ω (not necessarily closed), the sequence of forms $f_\eta^n \omega$ is convergent as $n \to \infty$, and the limit is a closed form.*

THEOREM 3.19 [Arn10]. *Let M be a two-dimensional manifold and $G^\star = \mathrm{Id}$; then assertions a)-d) of Theorem 3.18 are true for all $\eta > 0$ (not only for sufficiently large values of η).*

These discrete versions of Theorem 3.12 and Corollary 3.13 (and counterparts of other theorems) are proved in the same way as the original statements themselves. Moreover, one can give up the identity condition on $G^\star$. To obtain the discrete analogues of Theorems 3.6-3.12 with $G^\star \neq \mathrm{Id}$, one should not confine oneself to the stationary forms but consider the eigenvectors of the map f_η with eigenvalues λ, $|\lambda| > 1$. Denote by $G_k^\star$ the action of a diffeomorphism g (by forward translation) on the cohomology group $H^k(M, \mathbb{R})$ and let χ be an eigenvalue of $G_k^\star$ of maximal magnitude.

THEOREM 3.20 [Koz1]. *For sufficiently large η,*

a) *and any exact form ω, the image under iterations $f_\eta^n \omega$ tends to zero as $n \to \infty$;*

b) *every eigenvector of f_η is a closed form;*

c) *and a closed k-form ω, the norm $\|f_\eta^n \omega\|$ grows with the same increment as $\|(G_k^\star)^n [\omega]\|$; i.e., for all k-forms from the same cohomology class, the growth rate coincides with the growth rate of this class under the action of $G_k^\star$;*

d) *if a cohomology class Ω is an eigenvector for the operator $G_k^\star$ with the eigenvalue χ, $G_k^\star \Omega = \chi\, \Omega$, then there is a form-representative $\omega \in \Omega$ such that $f_\eta \omega = \chi\, \omega$;*

e) *one has for any k-form ω*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|f_\eta^n \omega\| \leq \ln |\chi|,$$

while for a generic k-form ω the inequality becomes equality:

$$\lim_{n\to\infty} \frac{1}{n} \ln \|f_\eta^n \omega\| = \ln |\chi|.$$

PROOF. a) can be proved using the same estimates as in the proof of Theorem 3.6(ii). For b)-e), note that the operator f_η is compact, so for any value in its spectrum there is an eigenvector. Let λ be in the spectrum and $|\lambda| > 1$. Then there is an ω such that $f_\eta \omega = \lambda\, \omega$. The exterior derivative d commutes with f_η, so $f_\eta d\omega = \lambda\, d\omega$, and a) implies that $d\omega = 0$. The "diffusion" operator $\exp(\eta\Delta)$ does not change the cohomological class, so the condition $G_k^\star \Omega = \chi\, \Omega$ implies $f_\eta \Omega \in \Omega$. If there is a k-form $\omega \in \Omega$ such that $f_\eta \omega = \lambda\, \omega$, then either $\lambda = \chi$, or $[\omega] = 0$ and $|\lambda| = 1$. □

The same method can be used to prove, for example, that if $M = \mathbb{T}^2$ and $G^\star = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $f_\eta^n \omega$ increases no more rapidly than the first power of n.

REMARK 3.21. The case of $G^\star = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is used in [Arn8, AZRS1] to construct a fast kinematic dynamo on a three-dimensional compact Riemannian manifold; see the next section.

To the best of our knowledge, the preceding theory has not been settled for manifolds with boundary, though it certainly deserves to be.

§4. Three-dimensional dynamo models

4.A. "Rope dynamo" mechanism. The topological essence of contemporary dynamo constructions goes back to the following scheme proposed by Sakharov and Zeldovich (see [V-Z, ChG]) and depicted in Fig.68.

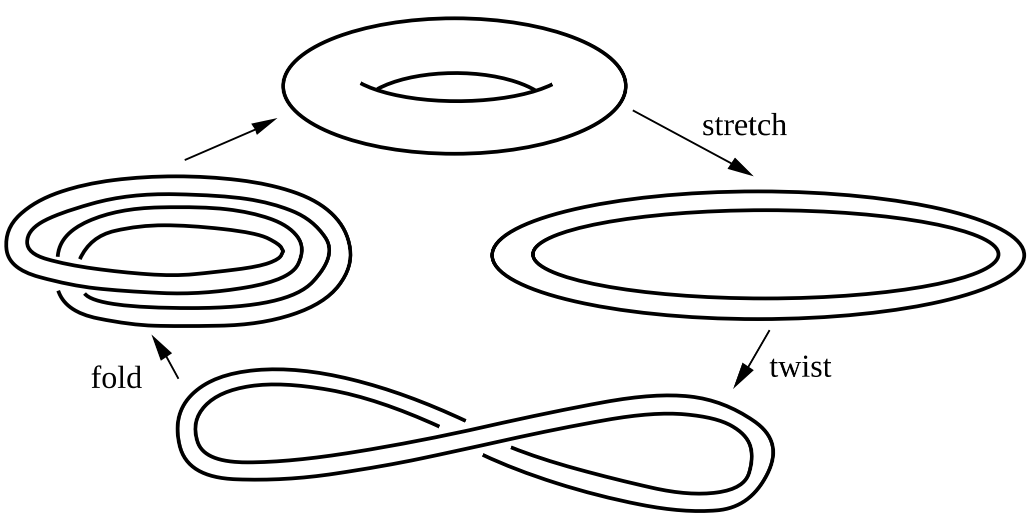

FIGURE 68. Rope dynamo: the stretch–twist–fold mechanism.

The rapid growth of magnetic energy is achieved by iterations of the three-step transformation of a solid torus: *stretch–twist–fold (STF).*

We start with a solid torus $S^1 \times D^2$ embedded in a three-ball. Take it out and stretch S^1 twice, while shrinking D^2 in such a way that the volume element remains preserved. Then we twist and fold the new solid torus in such a way as to obtain a twofold covering of the middle circle, and finally we put the resulting solitorus in its initial place (Fig.68).

The energy of the longitudinal field in the solid torus (directed along the S^1 component) grows exponentially under iterations of the construction above, since the field is stretched by a factor of 2 along with the longitudinal elongation of the magnetic lines.

Though this construction is not a diffeomorphism of the solid torus onto itself, one can make it smooth, sacrificing control over stretching in a small portion of the solid torus. The loss of information about stretching of the flow in a small part of the manifold, though irrelevant for an idealized nondissipative dynamo, is essential when viscosity is taken into account.

4.B. Numerical evidence of the dynamo effect. The presence of chaos in ABC flows (see Chapter II) makes them extremely attractive for dynamo modeling. We confine ourselves to mentioning only the extensive studies in this field. The numerical and scale evidence for fast dynamo action in ABC and, more generally, in chaotic steady flows, can be found in, e.g., [Hen, G-F, AKo, Chi3, Bay1, Gil1, PPS] (see also [Zhel] for analogues of ABC flows in a three-dimensional ball).

The most extensive studies on ABC flows dealt with the case $A = B = C$ with the velocity field

$$\mathbf{v} = (\cos y + \sin z)\frac{\partial}{\partial x} + (\cos z + \sin x)\frac{\partial}{\partial y} + (\cos x + \sin y)\frac{\partial}{\partial z}.$$

One of the main problems in such a modeling is to estimate the increment $\lambda(\eta)$ of the fastest growing mode of the magnetic field B as a function of the magnetic diffusivity η, or of the magnetic Reynolds number $R_m = 1/\eta$. In other words, one is looking for the eigenvalue of the operator $L_{R_m} : B \mapsto -R_m\{\mathbf{v}, B\} + \Delta B$ with the largest real part. The first computations of E.I. Korkina (see [AKo]), by means of Galerkin's approximations, covered the segment of Reynolds numbers $R_m \leq 19$.

For small Reynolds numbers (i.e., for a large diffusivity η), every solution of the dynamo Equation (1.1) tends to a stationary field that is determined by the cohomology class of the initial field B_0; see Theorem 3.6. Hence, for such Reynolds numbers the eigenvalue of L_{R_m} is zero independent of R_m.

When confined to the case of the fields B_0 with zero average, the largest eigenvalue of the operator L_{R_m} becomes -1 for all numbers R_m less than the critical value $R_m \approx 2.3$. The reason for this phenomenon is that $\Delta \mathbf{v} = -\mathbf{v}$ (and of course, $\{\mathbf{v}, \mathbf{v}\} = 0$), and therefore the field $\mathbf{v}$ is eigen for L_{R_m} with eigenvalue -1.

As the Reynolds number grows, there appears a pair of complex conjugate eigenvalues with $\operatorname{Re} \lambda = -1$. The pair of eigenvalues moves to the right and crosses the "dynamo border" $\operatorname{Re} \lambda = 0$ at $R_m \approx 9.0$. The increment $\operatorname{Re} \lambda$ stays in the right half-plane until $R_m \approx 17.5$, when it becomes negative again.

Thus the field $\mathbf{v}$ is the dynamo for $9 < R_m < 17.5$. D.J. Galloway and U. Frisch [G-F] have discovered the dynamo in this problem for $30 < R_m < 100$. It is still unknown whether this field is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as $R_m \to \infty$.

Numerically, the kinematic fast dynamo problem is the first eigenvalue problem for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8. The corresponding matrices are (and will remain) beyond the reach of any computer.

Symmetry reasoning (involving, in particular, representation theory of the group of all rotations of the cube) allows one to speed up the computations significantly. In particular, the first harmonic of some actual eigenfield for any Reynolds number can be found explicitly [Arn13]. This mode is the fastest growing for $R_m \leq 19$ as numerical experiments show.

Computer computations also suggest that the growing mode is confined to a small neighborhood of the invariant manifolds of the stagnation points, at least for $A = B = C$. There still exists a hope that this observation might lead to some rigorous asymptotic results. The asymptotic solution constructed in [DoM] manifests concentration near the separatrices for a very long time, but not forever. No mode with such concentration was found as $t \to \infty$!

4.C. A dissipative dynamo model on a three-dimensional Riemannian manifold. In this section we consider an artificial example of a flow with exponential stretching of particles that provides the fast dynamo effect in spite of a nonzero diffusion (see [Arn8, AZRS1,2]). In this example, everything can be computed explicitly. Its disadvantage, however, is the unrealistic uniformity of stretching and the absence of places where the directions of the growing field are opposite.

The construction is based upon the cat map of a torus, discussed above, and can be thought of as a simplified version of the model of Section II.5. In that section we considered exponential stretching of particles according to the same equations.

However, unlike the magnetic field evolution in the kinematic dynamo problem, in ideal hydrodynamics the transported (vorticity) field is functionally dependent on the velocity field that is evolving it.

The domain of the flow is a three-dimensional compact manifold M that in Cartesian coordinates can be constructed as the product $\mathbb{T}^2 \times [0,1]$ of the two-dimensional torus $\mathbb{T}^2$ with the segment $0 \leq z \leq 1$, for which the end-tori are identified by means of the transformation

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$$

(i.e., according to the law $(x,y,0) = (2x+y, x+y, 1)$, or equivalently, $(x,y,1) = (x-y, 2y-x, 0)$ with $x \mod 1,\ y \mod 1$).

To introduce a Riemannian metric on this manifold we first pass from the Cartesian coordinates x, y, z to the Cartesian coordinates p, q, z, where p has the direction of the eigenvector of A with the eigenvalue $\chi_1 = (3+\sqrt{5})/2 > 1$, and q is directed along the eigenvector with the eigenvalue $\chi_2 = (3-\sqrt{5})/2 < 1$. Then the metric given by the line element

$$ds^2 = e^{-2\lambda z} dp^2 + e^{2\lambda z} dq^2 + dz^2,\ \lambda = \ln \chi_1 \approx 0.75$$

is invariant with respect to the transformation A, and therefore defines an analytic Riemannian structure on the compact three-dimensional manifold M. We choose the eigenvector directions in such a way that the (p,q,z)- and (x,y,z)-orientations of $\mathbb{R}^3$ coincide.

Further, on this manifold we consider a flow with the stationary velocity field $\mathbf{v} = (0,0,v)$ in (p,q,z)-coordinates, where v =const, so that $\operatorname{div} \mathbf{v} = 0$ and $\operatorname{curl} \mathbf{v} = 0$. Each fluid particle moving along this field is exponentially stretched in the q-direction and exponentially contracted along the p-axis when regarded as *a particle on M*. If the magnetic Reynolds number is small (the diffusivity is large), the magnetic field growth is damped by the magnetic diffusion, and there is no dynamo effect (cf. Theorem 3.6). For small magnetic diffusivity the situation is different.

THEOREM 4.1 [AZRS1]. *The vector field $\mathbf{v}$ defines a fast dynamo on the Riemannian manifold M for an arbitrarily small diffusivity η and in the limit $\eta \to 0$. For a given initial magnetic vector field, only its Fourier harmonic independent of p and q survives and grows exponentially as $t \to \infty$.*

PROOF. Consider the following three vector fields in $\mathbb{R}^3$:

$$\mathbf{e}_p = e^{\lambda z} \partial/\partial p,\ \mathbf{e}_q = e^{-\lambda z} \partial/\partial q,\ \mathbf{e}_z = \partial/\partial z.$$

These fields are A-invariant, and hence they descend to three vector fields on M^3, for which we shall keep the same notations $\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_z$. Those fields are orthogonal at every point in the sense of the above metric. Let f be a function on M, i.e., it is a function $f : \mathbb{R}^3 \to \mathbb{R}$, 1-periodic in x and y, and satisfying $f(x, y, z+1) = f(x-y, 2y-x, z)$. Similarly, suppose that $\mathbf{B} = B_p\mathbf{e}_p + B_q\mathbf{e}_q + B_z\mathbf{e}_z$ is a (magnetic) vector field on M. Direct calculation leads to the following

PROPOSITION 4.2. *The vector calculus formulas on M are*

$$\nabla f = (e^{\lambda z}\frac{\partial f}{\partial p})\ \mathbf{e}_p + (e^{-\lambda z}\frac{\partial f}{\partial q})\ \mathbf{e}_q + (\frac{\partial f}{\partial z})\ \mathbf{e}_z,$$

$$div\ (B_p\mathbf{e}_p + B_q\mathbf{e}_q + B_z\mathbf{e}_z) = e^{\lambda z}\frac{\partial B_p}{\partial p} + e^{-\lambda z}\frac{\partial B_q}{\partial q} + \frac{\partial B_z}{\partial z}$$

$$(in\ particular,\ div\ \mathbf{e}_p =\ div\ \mathbf{e}_q =\ div\ \mathbf{e}_z = 0),$$

$$curl\ (B_p\mathbf{e}_p + B_q\mathbf{e}_q + B_z\mathbf{e}_z) = (curl_p\mathbf{B})\ \mathbf{e}_p + (curl_q\mathbf{B})\ \mathbf{e}_q + (curl_z\mathbf{B})\ \mathbf{e}_z,$$

$$where\quad curl\ _p\mathbf{B} = e^{-\lambda z}(\frac{\partial B_z}{\partial q} - \frac{\partial e^{\lambda z}B_q}{\partial z}),$$

$$curl\ _q\mathbf{B} = e^{\lambda z}(\frac{\partial e^{-\lambda z}B_p}{\partial z} - \frac{\partial B_z}{\partial p}),$$

$$curl\ _z\mathbf{B} = e^{\lambda z}\frac{\partial B_q}{\partial p} - e^{-\lambda z}\frac{\partial B_p}{\partial q}$$

$$(in\ particular,\ curl\ \mathbf{e}_p = -\lambda\mathbf{e}_q,\quad curl\ \mathbf{e}_q = -\lambda\mathbf{e}_p,\quad curl\ \mathbf{e}_z = 0),$$

$$\Delta f = e^{2\lambda z}\frac{\partial^2 f}{\partial p^2} + e^{-2\lambda z}\frac{\partial^2 f}{\partial q^2} + \frac{\partial^2 f}{\partial z^2},$$

$$\Delta\mathbf{e}_p := -curl\ curl\ \mathbf{e}_p = -\lambda^2 e_p,\ \Delta\mathbf{e}_q = -\lambda^2\mathbf{e}_q,\ \Delta\mathbf{e}_z = 0,\quad and$$

$$\{\mathbf{e}_p, \mathbf{e}_q\} = 0,\ \{\mathbf{e}_z, \mathbf{e}_p\} = \lambda\mathbf{e}_p,\ \{\mathbf{e}_z, \mathbf{e}_q\} = -\lambda\mathbf{e}_q.$$

PROOF OF PROPOSITION. Denote by $\phi_p = e^{-\lambda z}dp,\ \phi_q = e^{\lambda z}dq,\ \phi_z = dz$ the dual 1-forms (in $\mathbb{R}^3$ and on M). Such a form is dual to the corresponding field, in the sense that, e.g., $\phi_p|_{\mathbf{e}_p} = 1,\ \phi_p|_{\mathbf{e}_q} = \phi_p|_{\mathbf{e}_z} = 0$, etc. Then, the expression for the differential

$$df = \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial q}dq + \frac{\partial f}{\partial z}dz = e^{\lambda z}\frac{\partial f}{\partial p}\ \phi_p + e^{-\lambda z}\frac{\partial f}{\partial q}\ \phi_q + \frac{\partial f}{\partial z}\ \phi_z$$

directly implies the gradient formula, etc. □

The evolution (1.1) of a magnetic field $\mathbf{B} = B_p\mathbf{e}_p + B_q\mathbf{e}_q + B_z\mathbf{e}_z$ on M along the velocity field $\mathbf{v} = v\partial/\partial z$ has the following description in components:

$$\frac{\partial B_p}{\partial t} + v\frac{\partial B_p}{\partial z} = -\lambda v B_p + \eta[(\Delta - \lambda^2)B_p - 2\lambda e^{\lambda z}\frac{\partial B_z}{\partial p}],$$
$$\frac{\partial B_q}{\partial t} + v\frac{\partial B_q}{\partial z} = \lambda v B_q + \eta[(\Delta - \lambda^2)B_q + 2\lambda e^{-\lambda z}\frac{\partial B_z}{\partial q}],$$
$$\frac{\partial B_z}{\partial t} + v\frac{\partial B_z}{\partial z} = \eta(\Delta - 2\lambda\frac{\partial}{\partial z})B_z.$$

The equation for the z-component of the field splits from the rest. Suppose that the function B_z has zero average. Then, asymptotically as $t \to \infty$, the B_z-component decays (cf. Zeldovich's antidynamo theorem, Section 3.A). Indeed, the latter is the heat equation in a moving liquid. It is easy to see that B_z diminishes, since each of its maxima tends to disappear (the maximum principle). Formally, one obtains

$$\frac{d}{dt}\int B_z^2\ \mu = \int B_z\frac{\partial B_z}{\partial t}\ \mu = \eta\int B_z(\Delta B_z)\ \mu = -\eta\int(\nabla B_z)^2\ \mu.$$

Based on this, we assume in the sequel that the component B_z is constant.

It suffices to consider only one component of the vector field B, since the equations for the p- and q-components differ only by the substitution $\lambda \to -\lambda$:

$$\frac{\partial B}{\partial t} + v\frac{\partial B}{\partial z} = \lambda v B + \eta(\Delta - \lambda^2)B, \tag{4.1}$$

where $B \equiv B_q$.

To specify the boundary conditions on B, we return to the (x, y, z)-coordinate system. Periodicity in x and y allows one to expand B into a Fourier series:

$$B(x, y, z, t) = \sum_{n,m} B_{n,m}(z, t)\exp\ [2\pi i(nx + my)]$$
$$= b(p, q, z, t) = \sum_{\alpha,\beta} b_{\alpha,\beta}(z, t)\exp\ [i(\alpha p + \beta q)],$$

where n and m are integers and α, β are related to $2\pi n, 2\pi m$ by a linear transformation corresponding to the passage from the coordinates x, y to p, q.

LEMMA 4.3. *The function $b_{0,0}(z, t)$ is periodic in z. The harmonics $b_{\alpha,\beta}(z, t)$ with $(\alpha, \beta) \neq (0, 0)$ decay exponentially in z for analytic functions B.*

PROOF. Restrictions on the Fourier amplitudes come from the symmetry with respect to a shift along the z-axis: $B(x, y, z, t) = B(2x + y, x + y, z + 1, t)$. This identity is equivalent to that on the Fourier coefficients that are acted upon by the operator conjugate to A:

$$B_{(n,m)}(z + 1, t) = B_{(n,m)A^*}(z, t). \tag{4.2}$$

Here A^* is the transpose of the matrix A, and in the case at hand $A^* = A$.

Thus the shift along the z-axis is equivalent to the transition from the Fourier amplitudes with indices (n, m) to the Fourier amplitudes with indices $(n, m)A$. Iterative applications of the matrix A shifts a typical vector (n, m) along a hyperbola in the (n, m) plane (see Fig.69).

The only exception is the case $n = m = 0$, when the magnetic field does not depend on x, y or p, q: $(0, 0)\,A = (0, 0)$. Here we use that the eigendirections of A do not contain integral points (n, m) (different from (0,0)), since the eigenvalues of A are irrational.

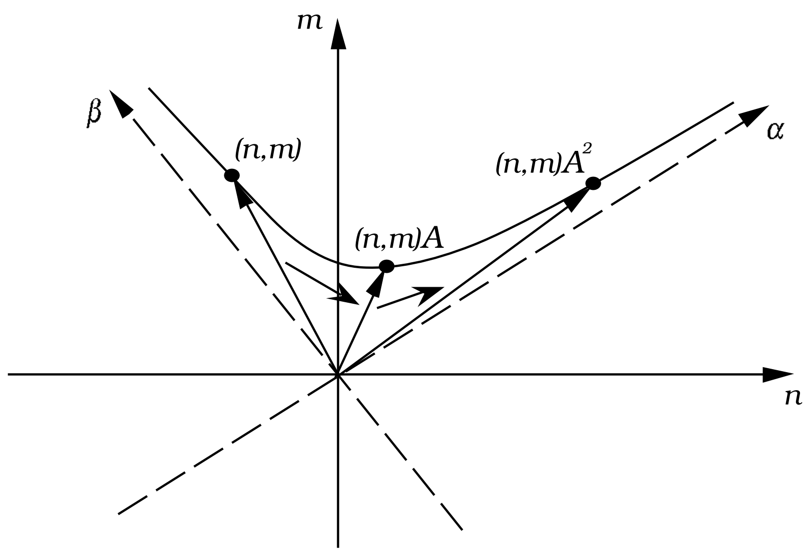

FIGURE 69. The invariant curves for the orbits $\{(n, m)A^k\}$ are hyperbolas in the (n, m)- (or (α, β)-) plane.

On the other hand, analyticity of $B(x, y, z, t)$ implies that its Fourier harmonics $b_{\alpha,\beta}$ must decay exponentially in α and β. It follows that the functions $b_{\alpha,\beta}(z, t)$ decrease rapidly for fixed $(\alpha, \beta) \neq (0, 0)$ as $|z| \to \infty$ due to the shift property above. Periodicity in z of the zero harmonic is evident. The Lemma is proved. □

To complete the proof of Theorem 4.1 we first fix $\eta = 0$. Equation (4.1) can be solved explicitly (due to the frozenness property):

$$b(p, q, z, t) = e^{\lambda v t} b(p, q, z - vt, 0) \tag{4.3}$$

(pass to the Lagrangian reference frame, solve the Cauchy problem, and return to the Eulerian coordinates).

Equation (4.1) may be written in the form $\partial b/\partial t = T_\eta b$, where the operator T_η (depending on the viscosity η) acts on the functions on M^3 (depending in our case on t as a parameter):

$$T_\eta b = \lambda v b + \eta(\Delta - \lambda^2) b - v\frac{\partial b}{\partial z}.$$

Consider first the nonviscous case $\eta = 0$. The nonviscous operator T_0 has a series of eigenfunctions $b_k = \exp(2\pi i k z)$, $k = 0, \pm 1, \pm 2, \dots$, with eigenvalues $\gamma_k = \lambda v - 2\pi i k v$.

Every solution of (4.3) that does not depend on p and q (i.e., that is constant on every 2-torus $z = \text{const}$, $t = \text{const}$) can be represented as a linear combination of the products $b_k \cdot \exp(\gamma_k t)$ (expand (4.3) into a Fourier series in z).

The operator T_0 has no other eigenfunctions. Indeed, suppose that $b : M^3 \to \mathbb{C}$ were an eigenfunction of T_0 with an eigenvalue γ. The function $b \cdot \exp(\gamma_k t)$ would then satisfy Equation (4.3). By choosing $t = 1/v$, we obtain from (4.3)

$$b(p, q, z) = e^{\lambda - \gamma} b(p, q, z - 1).$$

Using (4.2) we see that the Fourier coefficients $b_{\alpha,\beta}(z)$ along every hyperbola $\alpha = \lambda^n \alpha_0$, $\beta = \lambda^{-n}\beta_0$ form a geometric series. This contradicts the decay of the Fourier coefficients of the smooth function $b(\cdot, \cdot, z)$ on the 2-torus (unless $\alpha_0 = \beta_0 = 0$, in which case b does not depend on p and q).

The absence of eigenfunctions is explained by the continuity of the spectrum of T_0 (on the orthogonal complement to the space of functions, constant on the tori $z = \text{const}$).

Now turn to the general case $\eta \neq 0$. As before, Equation (4.1) has a sequence of solutions $b_k \cdot \exp(\gamma_k t)$, which are independent of p and q, with eigenvalues

$$\gamma_k = \lambda v + \eta(-4\pi^2 k^2 - \lambda^2) - 2\pi i k v, \ k = 0, \pm 1, \pm 2, \dots.$$

If η is small we find many ($\approx C\eta^{-1/2}$) growing modes. (If η is large, there is no growing mode at all, since $\operatorname{Re}\gamma_k < 0$.)

However, the behavior of the solutions whose initial field depends on p and q differs drastically from the behavior given by the frozenness condition (4.3). To explain this, consider the time evolution of b as consisting of two intermittent parts: the frozen-in stretching (4.3) ($\eta = 0$) and the pure diffusion action ($\mathbf{v} = 0$). If η is small, the stretching part might be long.

The long shift $z - vt \quad (vt \in \mathbb{Z})$ along the z-axis is equivalent to a translation (along the hyperbola) of the labels (α, β) of the harmonics $b_{\alpha,\beta}(z, t)$ for fixed z. Hence, any given harmonic will shift with time into the region of large wave numbers, where dissipation becomes important. Its amplitude will then decay in the diffusion part of the evolution. Asymptotically as $t \to \infty$, the evolving field will decay however small the viscosity η is.

Thus, we come to the conclusion that, asymptotically for $t \to \infty$, only the solution independent of p and q survives (see [AZRS1] for details on analysis of the solution asymptotics). Such a periodic in z solution in $\mathbb{R}^3$ grows exponentially in the metric ds^2 as $z \to \infty$. The increment of the corresponding exponent is bounded away from 0 by a positive constant independent of $\eta > 0$. Finally, due to the linear relation between shifts in the z and t directions, one obtains the same exponential growth of the solution as $t \to \infty$. □

4.D. Geodesic flows and differential operations on surfaces of constant negative curvature. Every compact Riemann surface can be equipped with a metric of constant curvature. This curvature is positive for a sphere, vanishes for a torus, and is negative for any surface with at least two handles (i.e., for any surface of genus ≥ 2).

In this section we show that the geodesic flow on every Riemann surface whose curvature is constant and negative provides an example of the fast (dissipative) kinematic dynamo. More precisely, let M^3 be the bundle of unit vectors over such a surface P : $M^3 = \{\xi \in TP \mid \|\xi\| = 1\}$. The geodesic flow defines a dynamical system on this three-dimensional manifold M^3 with exponential stretching of particles of M, similar to the example above. Avoiding repetition, we present here the basic formulas for the key differential operations on the bundle of unit vectors over P.

First of all, let us pass from the surface P to its universal covering $\tilde{P}$. Every such surface of constant negative curvature is covered by the Lobachevsky plane $\tilde{P} = \Lambda$, where the covering is locally isometric (that is, respecting the metrics on both spaces).

Remark 4.4. Sometimes it is convenient to think of the bundle of unit vectors $V^3 := T_1\Lambda$ over the Lobachevsky plane Λ as the group $SL(2, \mathbb{R})$. Then the space M^3 is the quotient of $SL(2, \mathbb{R})$ (or, more generally, of the universal covering $\widetilde{SL(2, \mathbb{R})}$) over a discrete uniform subgroup Γ:

$$M^3 = SL(2, \mathbb{R})/\Gamma.$$

We will deal with the following three "basic" flows on the Lobachevsky plane: the geodesic flow and two horocyclic flows. Introduce the natural coordinates (x, y, φ) in the space of line elements (or of unit vectors) $V^3 = T_1\Lambda^2$, where the Lobachevsky plane is the upper half-plane $\Lambda^2 = \{(x, y) \mid y > 0 \}$ equipped with the metric

$ds^2 = (dx^2 + dy^2)/y^2$, and $\varphi \in [0, 2\pi)$ is the angle of a line element with the vertical in Λ^2 (see Fig.70a). The x-axis is called the *absolute* of the Lobachevsky plane. Recall that the geodesics in Λ^2 are all semicircles and straight lines orthogonal to the absolute (Proposition IV.1.3).

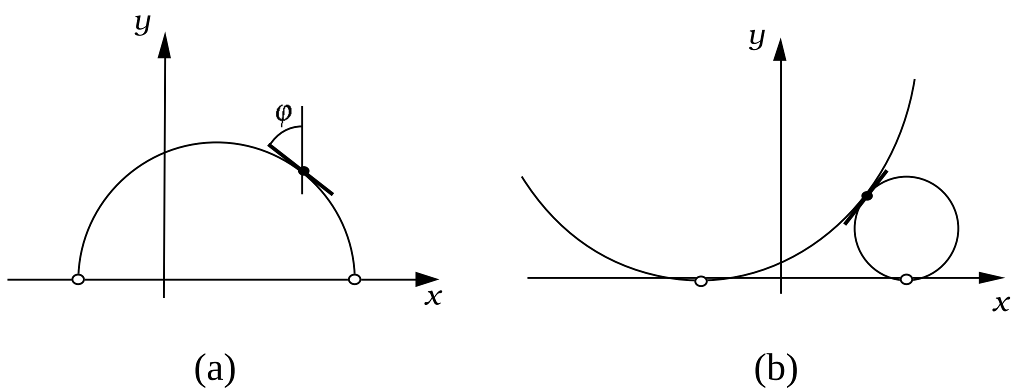

FIGURE 70. (a) Coordinates (x, y, φ) in the space of line elements along the geodesics in the Lobachevsky plane. (b) Two horocycles passing through one line element.

DEFINITION 4.5. The *geodesic flow* of the Lobachevsky plane is the flow in the space of unit line elements $V^3 = T_1\Lambda^2$ that sends, for time t, every element l into the line element on Λ^2 tangent to the same geodesics as l, but at the distance t (in the Lobachevsky metric) ahead of l.

The limit of a sequence of Euclidean circles tangent to each other at a given point and of increasing radius in the Lobachevsky plane is called a *horocycle*.

PROPOSITION 4.6 (SEE, E.G., [Arn15]). *The horocycles in the Lobachevsky plane are exactly the Euclidean circles tangent to the absolute and the straight lines parallel to it.*

Every line element (point with a specified direction) on Λ^2 belongs to two horocycles, "upper" and "lower"; see Fig.70b.

DEFINITION 4.5′. The *first (+)* and *second (−) horocyclic flows* on Λ^2 are the flows sending in a time t every line element on the Lobachevsky plane to the line element belonging to the same lower and upper horocycles respectively, and lying on the distance t ahead of it.

Explicitly, the flows are given by the following vector fields e (for the geodesic flow), h^- (for the "lower" horocyclic flow), and h^+ (for the "upper" horocyclic one) on V:

$$
\begin{aligned}
e &= -\,y \sin\varphi \frac{\partial}{\partial x} + y\cos\varphi \frac{\partial}{\partial y} + \sin\varphi \frac{\partial}{\partial \varphi},\\
h^- &= -\,y \cos\varphi \frac{\partial}{\partial x} - y\sin\varphi \frac{\partial}{\partial y} + (\cos\varphi - 1) \frac{\partial}{\partial \varphi},\\
h^+ &= -\,y \cos\varphi \frac{\partial}{\partial x} - y\sin\varphi \frac{\partial}{\partial y} + (\cos\varphi + 1) \frac{\partial}{\partial \varphi}.
\end{aligned}
$$

PROPOSITION 4.7. *The vector fields e, h^+, h^- generate the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.*

PROOF. $\{e, h^+\} = h^+$, $\{e, h^-\} = -h^-$, $\{h^+, h^-\} = 2e$, where $\{\ ,\ \}$ means the Poisson bracket of two vector fields: $L_{\{u,v\}} = L_u L_v - L_v L_u$. In coordinates it is $\{u, v\} = (u, \nabla)v - (v, \nabla)u$; see Section I.2. □

Notice that the difference of the horocyclic fields $f := \frac{1}{2}(h^- - h^+)$ is the rotation field $f = \frac{\partial}{\partial\varphi}$. Introduce also the sum field $\tilde{e} := \frac{1}{2}(h^+ + h^-)$. The Poisson brackets between the fields $e,\ \tilde{e},\ f$ are $\{f, e\} = \tilde{e},\ \{\tilde{e}, e\} = f,\ \{\tilde{e}, f\} = e$.

Now we define in $V^3 = T_1\Lambda^2$ a one-parameter family of metrics:

$$
d\ell^2 = \frac{dx^2 + dy^2}{y^2} + \lambda^2 \left(d\varphi + \frac{dx}{y} \right)^2. \tag{4.4}
$$

PROPOSITION 4.8. *The above metrics on the space of line elements $V^3 = T_1\Lambda^2$ are singled out by the following three conditions:*

1) Consider the planes defining the standard Riemannian connection in $T_1\Lambda^2$ related to the Lobachevsky metric on Λ^2. The condition on a metric in $T_1\Lambda^2$ is that the fibers of the projection $T_1\Lambda^2 \to \Lambda^2$ are orthogonal to those planes;

2) The above projection sends the planes to the tangent spaces to Λ^2 isometrically;

3) The metrics are invariant with respect to isometries of the Lobachevsky plane.

PROOF. From 2) one can see that

$$
d\ell^2 = \frac{dx^2 + dy^2}{y^2} + \lambda^2 (d\varphi + a\,dx + b\,dy)^2.
$$

Utilize condition 1) in the following form: The coefficients a and b obey the relation $d\varphi + a\,dx + b\,dy = 0$ along two curves in $V = T_1\Lambda^2$. One of the curves is the parallel transport of a line element along its geodesics (i.e., it is the orbit of e), and the other curve is obtained by the parallel transport of the same line element in the perpendicular direction (i.e., along the orbit of $\tilde{e}$). The calculation can be carried out at $x = 0,\ y = 1,\ \varphi = 0$, and extended by invariance due to 3). □

For each metric of the family (4.4), the basis $e,\ \tilde{e},\ f$ is orthogonal: $(e,\ \tilde{e}) = (e, f) = (\tilde{e}, f) = 0$, and moreover, $(e, e) = (\tilde{e},\ \tilde{e}) = 1$, $(f, f) = \lambda^2$. In this normalization the volume element spanned by the three fields is $\tau(e,\ \tilde{e},\ f) = \lambda$.

PROPOSITION 4.9. *All the fields $e,\ \tilde e,\ f,\ h^+,\ h^-$ are divergence-free and null-homologous. Their vorticities are as follows:*

$$\text{curl }\ e=-\frac{e}{\lambda},\ \text{curl }\ \tilde e=-\frac{\tilde e}{\lambda},\ \text{curl }\ f=\lambda f,\ \text{curl }\ h^{\pm}=-\frac{\tilde e}{\lambda}\mp\lambda f.$$

The helicities of both of the horocyclic flows are zero, while the helicity of the geodesic flow e in the compact manifold $M^3=T_1P^2$ is

$$\mathcal{H}(e)=-8\pi^2\lambda^2\left((\textit{genus of } P^2)-1\right).$$

PROOF. Introduce 1-forms $\alpha,\ \tilde\alpha,\ \beta$ dual to the fields $e,\ \tilde e,\ f$, respectively (i.e., $\alpha|_e=1,\ \alpha|_{\tilde e}=\alpha|_f=0$, etc). Now, the calculation of the vorticity and divergence is a straightforward application of the formula for the differential of a 1-form:

$$d\gamma(v_1,v_2)=\gamma(\{v_2,v_1\})+L_{v_1}\gamma(v_2)-L_{v_2}\gamma(v_1).$$

For instance, combining it with the formulas for Poisson brackets

$$\{e,f\}=-\tilde e,\ \{e,\tilde e\}=-f,\ \{\tilde e,f\}=e,$$

one obtains $d\alpha(e,\tilde e)=d\alpha(e,f)=0,\ d\alpha(\tilde e,f)=-1$. Therefore, $-\lambda d\alpha=i_e\tau$. By definition, this means that div $e=0$ and curl $e=-e/\lambda$ (see Section III.1 for more detail).

The helicity expression for the geodesic field e on $M^3=T_1P^2$ is

$$\mathcal{H}(e)=\int_M(e,\text{curl }^{-1}e)\tau=-\lambda\int_M\tau=-2\pi\lambda^2\cdot(\text{area of }P^2).$$

Here the volume of the bundle M is the product of the fiber length $2\pi\lambda$ and the area of the surface P^2. The Gauss-Bonnet theorem reduces the area of the surface P^2 (with constant curvature) to the number of handles:

$$\text{area of }P^2=4\pi(\text{genus of }P^2-1).$$

We leave to the reader the helicity calculations for horocyclic flows $h^{\pm}$ on M. □

Returning to hydrodynamics, we immediately obtain the following

COROLLARY 4.10. *The velocity fields $e,\ \tilde e,\ f$, as well as all linear combinations of e and $\tilde e$, are stationary solutions of the Euler equation on $M^3=T_1P^2$. They are* *also the stationary solutions of the corresponding Navier-Stokes equation on M for the vorticity field $\omega = \operatorname{curl}\ v$:*

$$\frac{\partial \omega}{\partial t} + \{\operatorname{curl}^{-1} \omega,\ \omega\} = -\eta \cdot \operatorname{curl}\ \operatorname{curl} \omega + R,$$

where R (the curl of the external force) is proportional to ω.

PROOF. $\{e, \operatorname{curl}^{-1} e\} = \{\tilde{e}, \operatorname{curl}^{-1} \tilde{e}\} = \{f, \operatorname{curl}^{-1} f\} = 0.$ □

The symmetry of this corollary and of the formulas of Proposition 4.9 under the interchange of e and $\tilde{e}$ is not surprising, since the flow of the field $f = \partial/\partial\varphi$ is an isometry of $V^3 = T_1\Lambda^2$, and it takes the field e to the field $\tilde{e}$ for the time $\pi/2$.

PROPOSITION 4.11. *Every steady solution $\omega_0 = Ae + B\tilde{e}$ of the Euler equation for vorticity is unstable in the linear approximation.*

PROOF. The linearized Navier-Stokes equation (cf. the linearized Euler equation (II.5.1)) for variations of velocity $v = v_0 + v_1$ and vorticity $\omega = \omega_0 + \omega_1$ is

$$\frac{\partial \omega_1}{\partial t} + \{v_0, \omega_1\} + \{v_1, \omega_0\} = -\eta \operatorname{curl}\ \operatorname{curl}\ \omega_1.$$

For the initial vorticity $\omega_0 = Ae + B\tilde{e}$ we have $v_0 = -\lambda\ \omega_0$, and hence

$$\frac{\partial \omega_1}{\partial t} = \{\omega_0, v_1 + \lambda\omega_1\} - \eta \operatorname{curl}\ \operatorname{curl}\ \omega_1, \tag{4.5}$$

where $v_1 = \operatorname{curl}^{-1}\omega_1$. Consider the three-dimensional space of special ("long-wave") perturbations

$$\omega_1 = a\, e + b\, \tilde{e} + c\, f, \qquad v_1 = -\lambda a\, e - \lambda b\, \tilde{e} + \frac{c}{\lambda} f.$$

The operator on the right-hand side of formula (4.5) maps this space (with the basis $e,\ \tilde{e},\ f$) into itself, and it is represented by the matrix

$$\begin{pmatrix} -\eta/\lambda^2 & 0 & B\xi \\ 0 & -\eta/\lambda^2 & -A\xi \\ 0 & 0 & -\eta\lambda^2 \end{pmatrix}, \qquad \text{where} \quad \xi = \lambda + \frac{1}{\lambda}.$$

Therefore, for nonzero viscosity η the eigenvalues are negative, and the corresponding modes decay. However, for $\eta = 0$ one has linear growth of the perturbations (in the direction perpendicular to ω_0 in the plane $(e, \tilde{e})$).

REMARK 4.12. It is natural to conjecture that our linearized equation for $\eta = 0$ has exponentially growing solutions, and even an infinite-dimensional space of those (it has not been proved). Indeed, at least for fast-oscillating solutions, one may neglect the second term $\{v_1, \omega_0\}$ and take into account only the first one $\{v_0, \omega_1\}$, since for such solutions $v_1 = \mathrm{curl}^{-1}\omega_1$ is small compared to ω_1. Then one obtains the equation of a frozen transported field, and it has exponentially growing solutions (directions h^- or h^+ for $\omega_0 = -e$ or $\omega_0 = e$, respectively).

One can also argue that for small positive η Equation (4.5) has many exponentially growing solutions.

In a similar way one can study the stationary solution $\omega_0 = f,\ v_0 = f/\lambda$. In this case the first term on the right-hand side of Equation (4.5) has the form $\{\omega_0, v_1 - \omega_1/\lambda\}$. The matrix of the evolution operator for the "long-wave" perturbations is

$$\begin{pmatrix} -\eta/\lambda^2 & \xi & 0 \\ -\xi & -\eta/\lambda^2 & 0 \\ 0 & 0 & -\eta\lambda^2 \end{pmatrix}.$$

Therefore, the eigenvalues in this case are always negative for $\eta > 0$, while for $\eta = 0$ the eigenvalues are purely imaginary, $\pm i\xi$, and 0. The "fluid motion" on M corresponding to the field f is "rigid" (i.e., an isometry) and apparently stable.

As usual, the problem simplifies as we pass to the dynamo equations, where the magnetic field is not related to the velocity. Consider, for instance, the velocity field $v = Ae + B\tilde{e} + Cf$, where A, B, C are constants. The three-dimensional space of "long-wave" magnetic fields $\mathbf{B} = ae + b\tilde{e} + cf$ is invariant with respect to stretching by the flow of v, as well as with respect to "diffusion." The evolution of a "long-wave" field is given by the matrix

$$\begin{pmatrix} -\eta/\lambda^2 & C & -B \\ -C & -\eta/\lambda^2 & A \\ -B & A & -\eta\lambda^2 \end{pmatrix}.$$

Let us confine ourselves to the case $B = C = 0$, where the particles are stretched by the geodesic flow. In this case one readily evaluates the eigenvalues of the matrix and obtains the following

COROLLARY 4.13. *For sufficiently small magnetic diffusion ($\eta < |A|$), the geodesic flow is a fast dynamo. The growing mode is a linear combination of the horocyclic flows (or of the flows $\tilde{e}$ and f). The growth rate (i.e., the increment of the growing mode) depends continuously on the magnetic viscosity η and tends to $|A|$ as $\eta \to 0$.*

The velocity field $\tilde{e}$ (corresponding to $A = C = 0$) shares the analogous dynamo properties.

Hypothetically, the number of exponentially growing modes in these cases increases without bound as the magnetic viscosity η tends to 0. On the other hand, for the "rigid" field f (i.e., for $A = B = 0$) one has the matrix

$$\begin{pmatrix} -\eta/\lambda^2 & C \\ -C & -\eta/\lambda^2 \end{pmatrix},$$

which indicates the absence of growth of the "long-wave" fields. Furthermore, for nonzero magnetic viscosity η the fields decay, since the matrix eigenvalues have negative real parts.

4.E. Energy balance and singularities of the Euler equation.

PROPOSITION 4.14. *If a vector field ω_0 is a solution of the following equation,*

$$\{\operatorname{curl}^{-1} \omega_0, \omega_0\} = \text{const} \cdot \omega_0, \tag{4.6}$$

then the constant is zero.

PROOF. We look for solutions of the Helmholtz equation $\dot{\omega} = -\{v, \omega\}$, $v = \operatorname{curl}^{-1} \omega$ in the form $\omega(t) = a(t)\omega_0$, where ω_0 satisfies the relation above, and $a(t)$ depends on t only. Then substitution gives the following equation in a : $\dot{a} = -a^2 \cdot \text{const}$. All nontrivial solutions of the latter equation go to infinity at finite time if the constant is nonzero. The unbounded growth of ω contradicts the energy conservation law $\dot{E} = 0$ for kinetic energy $E = \frac{1}{2} \int v^2 \ \mu$. □

Among the "long-wave" vector fields on $M = T_1 P^2$ studied above, only the fields $\omega_0 = ae + b\tilde{e}$ (which commute with $\operatorname{curl}^{-1}\omega_0$) satisfy Equation (4.6). Indeed, for $\omega_0 = ae + b\tilde{e} + cf$ one gets from the commutation relations discussed

$$\{\operatorname{curl}^{-1} \omega_0, \omega_0\} = ac\xi\tilde{e} - bc\xi e,$$

where $\xi = \lambda + 1/\lambda$. There is no f on the right-hand side, which implies that $c = 0$.

§5. Dynamo exponents in terms of topological entropy

5.A. Topological entropy of dynamical systems. We have seen in Section 1.B that the exponential growth of the L^2-magnetic energy (and more generally, of the L^q-energy for $q > 1$) can be easily achieved in a nondissipative dynamo model whose velocity field has a hyperbolic stagnation point or a hyperbolic limit cycle. However, the class of nondissipative dynamos providing the exponential growth of the L^1-energy of a magnetic field is much more subtle. To specify this class, as well as to formulate the conditions for realistic (dissipative) dynamos, we need the notion of entropy of a flow or of a diffeomorphism.

DEFINITION 5.1. Let dist be the metric on a compact metric space M, and let $g : M \to M$ be a continuous map. For each $n = 0, 1, 2, \ldots$, define a new metric $\mathrm{dist}_{g,n}$ on X by

$$\mathrm{dist}_{g,n}(x, y) = \max_{i=0,1,\ldots,n} \mathrm{dist}(g^i x, g^i y).$$

A set is said to be (n, ϵ)-*spanning* if in the $\mathrm{dist}_{g,n}$-metric, the ϵ-balls centered at the points of the set cover the space M. Let $N(n, \epsilon, g)$ be the cardinality of the minimal (n, ϵ)-spanning set. Then the *topological entropy of the map* g is defined by

$$h_{\mathrm{top}}(g) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln N(n, \epsilon, g).$$

The *topological entropy* $h_{\mathrm{top}}(v)$ *of a vector field* v is the topological entropy of the time 1 map of its flow.

One can give such a definition for an arbitrary compact topological space by replacing ϵ-balls with an open covering and maximizing over all coverings, see, e.g., [K-Y].

To visualize this notion, think of the trajectories of two points x and y as being indistinguishable if the images $g^i(x)$ and $g^i(y)$ are ϵ-close for each $i = 0, \ldots, n$. Then $N(n, \epsilon, g)$ measures the number of trajectories of length n for the diffeomorphism g that are pairwise distinguishable for given ϵ. Intuitively, positivity of entropy indicates that this number grows exponentially with n.

5.B. Bounds for the exponents in nondissipative dynamo models.

THEOREM 5.2 [Koz2, K-Y]. *Let v be a divergence-free C^∞-vector field on a compact Riemannian three-dimensional manifold M, and μ the Riemannian volume form on M. Assume that a magnetic field B_0 is transported by the flow g_v^t of the field v : $B(t) := g_{v*}^t B_0$. Then for every continuous field B_0 on M the increment of the L^1-growth rate is majorated by the topological entropy $h_{\mathrm{top}}(v)$ of the field v:*

$$\limsup_{t \to \infty} \frac{1}{t} \ln \int_M \|B(t)\| \ \mu \le h_{\mathrm{top}}(v).$$

Moreover, the increment is exactly equal to the topological entropy for any generic magnetic field B_0 (i.e., for a field from an open and dense subset in the space of vector fields).

The formulation of Theorem 5.2 allows one to regard it as a naive definition of topological entropy: Choose a vector field, act on it by the flow, and estimate the corresponding rate of change of the L^1-energy. To be sure that the chosen field B_0 is "generic," one can start with a pair of vector fields B_1 and B_2 that along with the field v form a basis in (almost) every tangent space of M. Then $h_{\text{top}}(v)$ is equal to the biggest rate of L^1-energy growth of these two vector fields under iterations:

$$h_{\text{top}}(v) = \max\{\lambda_1, \lambda_2\}, \qquad \text{where} \qquad \lambda_i = \limsup_{t\to\infty} \frac{1}{t} \ln \int_M \|B_i(t)\| \ \mu.$$

Furthermore, the topological entropy of a smooth map $g : M \to M$ is equal to the maximum of the L^1-growth rate under iterations of g of a generic differential form. More precisely,

$$h_{\text{top}}(v) = \lim_{n\to\infty} \frac{1}{n} \ln \int_M \|Dg^{n*}\| \ \mu,$$

where Dg^{n*} is a mapping between the full exterior algebras of the tangent spaces to M, and we integrate with respect to the Lebesgue measure μ. The measure μ is not supposed to be invariant under g. If g is measure-preserving, then the same statement holds for k-vector fields; see [Koz2]. The topological entropy also gives a lower bound for the growth of the magnetic field in any L^q-norm ($q \geq \dim(M) - 2$), even in the case of finite smoothness of the diffeomorphism; see [K-Y].

Theorem 5.2 provides a necessary and sufficient condition for the existence of the exponential growth in a nondissipative L^1-dynamo. In order to be a dynamo, the velocity field v has to have nonzero entropy, i.e., roughly speaking, to admit some chaos. Positive topological entropy is often related to the presence of horseshoes, and they essentially exhaust all the entropy for two-dimensional systems ([Kat2], see the discussion in [K-Y]). For other L^q-norms ($q > 1$), the topological entropy gives a lower bound for the growth rate of an appropriate magnetic field [Koz2, K-Y].

The spectrum of the nondissipative kinematic dynamo operator for a continuous velocity field on a compact Riemannian manifold without boundary is described in [CLMS] (see also [LL]).

5.C. Upper bounds for dissipative L^1-dynamos. Klapper and Young [K-Y] proved that the same necessary condition is valid for dissipative (realistic) dynamos: If the topological entropy of the field vanishes ($h_{\text{top}}(v) = 0$), then the field v cannot be a fast dynamo (in other words, the increment $\lambda(\eta)$ goes to zero as the magnetic diffusivity η tends to zero). Such a bound was proposed by Finn and Ott in 1988, and the proof was announced in 1992 by M.M. Vishik. The result and proof in [K-Y] is given in the more general form of finite smoothness of the magnetic and velocity fields:

THEOREM 5.3 [K-Y]. *Let v and B_0 be divergence-free vector fields supported on a compact domain $M \subset \mathbb{R}^n$. Assume that v is of class C^{k+1} and B_0 is of class C^k*

for some $k \geq 2$. Let $B_\eta(t)$ be the solution of the dynamo equation (1.1) with the initial condition $B_\eta(0) = B_0$. Then

$$\limsup_{\eta \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln \int_M \|B_\eta(n)\| \ \mu \leq h_{\text{top}}(v) + \frac{r(g)}{k},$$

where $g := g_v^1$ is the time 1 map of the flow defined by the field v, $B_\eta(n)$ is the value of $B_\eta(t)$ at the moments $t = n$, and

$$r(g) := \lim_{n \to \infty} \frac{1}{n} \ln \left(\max_{x \in M} \|\frac{\partial(g^n)}{\partial x}(x)\| \right).$$

(Here $\frac{\partial(g^n)}{\partial x}$ is the Jacobian matrix of the map g^n.) This upper bound is also valid for the vanishing magnetic diffusivity $\eta = 0$.

In the case of an idealized nondissipative dynamo $\eta = 0$ and a smooth vector field v $(k = \infty)$ this theorem reduces to Theorem 5.2.

A variety of questions related to the kinematic dynamo are discussed in the recent book [ChG], which deals particularly with the fast dynamo problem, as well as in the books and surveys [Mof3, K-R, R-S, Chi2, ZRS, Z-R].