CHAPTER IV

# DIFFERENTIAL GEOMETRY OF DIFFEOMORPHISM GROUPS

In 1963 E.N. Lorenz stated that a two-week forecast would be the theoretical bound for predicting the future state of the atmosphere using large-scale numerical models [Lor]. Modern meteorology has currently reached a good correlation of the observed versus predicted for roughly seven days in the northern hemisphere, whereas this period is shortened by half in the southern hemisphere and by two-thirds in the tropics for the same level of correlation [Kri]. These differences are due to a very simple factor: the available data density.

The mathematical reason for the differences and for the overall long-term unreliability of weather forecasts is the exponential scattering of ideal fluid (or atmospheric) flows with close initial conditions, which in turn is related to the negative curvatures of the corresponding groups of diffeomorphisms as Riemannian manifolds. We will see that the configuration space of an ideal incompressible fluid is highly "nonflat" and has very peculiar "interior" and "exterior" differential geometry.

"Interior" (or "intrinsic") characteristics of a Riemannian manifold are those persisting under any isometry of the manifold. For instance, one can bend (i.e., map isometrically) a sheet of paper into a cone or a cylinder but never (without stretching or cutting) into a piece of a sphere. The invariant that distinguishes Riemannian metrics is called Riemannian curvature. The Riemannian curvature of a plane is zero, and the curvature of a sphere of radius $R$ is equal to $R^{-2}$. If one Riemannian manifold can be isometrically mapped to another, then the Riemannian curvature at corresponding points is the same.

The Riemannian curvature of a manifold has a profound impact on the behavior of geodesics on it. If the Riemannian curvature of a manifold is positive (as for a sphere or for an ellipsoid), then nearby geodesics oscillate about one another in most cases, and if the curvature is negative (as on the surface of a one-sheet hyperboloid), geodesics rapidly diverge from one another.

It turns out that diffeomorphism groups equipped with a one-sided invariant metric look very much like negatively curved manifolds. In Lagrangian mechanics a motion of a natural mechanical system is a geodesic line on a manifold-configuration space in the metric given by the difference of kinetic and potential energy. In the case at hand the geodesics are motions of an ideal fluid. Therefore, calculation of the curvature of the diffeomorphism group provides a great deal of information on instability of ideal fluid flows.

In this chapter we discuss in detail curvatures and metric properties of the groups of volume-preserving and symplectic diffeomorphisms, and present the applications of the curvature calculations to reliability estimates for weather forecasts.

## §1. The Lobachevsky plane and preliminaries in differential geometry

**1.A. The Lobachevsky plane of affine transformations.** We start with an oversimplified model for a diffeomorphism group: the (two-dimensional) group $G$ of all affine transformations $x \mapsto a + bx$ of a real line (or, more generally, consider the $(n+1)$-dimensional group $G$ of all dilations and translations of the $n$-dimensional space $\mathbb{R}^n : x, a \in \mathbb{R}^n,\ b \in \mathbb{R}_+$).

Regard elements of the group $G$ as pairs $(a, b)$ with positive $b$ or as points of the upper half-plane (half-space, respectively). The composition of affine transformations of the line defines the group multiplication of the corresponding pairs:

$$(a_2,\, b_2) \circ (a_1,\, b_1) = (a_2 + a_1 b_2,\, b_1 b_2).$$

To define a one-sided (say, left) invariant metric on $G$ (and the corresponding Euler equation of the geodesic flow on $G$; see Chapter I), one needs to specify a quadratic form on the tangent space of $G$ at the identity.

Fix the quadratic form $da^2 + db^2$ on the tangent space to $G$ at the identity $(a, b) = (0, 1)$. Extend it to the tangent spaces at other points of $G$ by left translations.

PROPOSITION 1.1. *The left-invariant metric on $G$ obtained by the procedure above has the form*

$$ds^2 = \frac{da^2 + db^2}{b^2}.$$

PROOF. The left shift by an element $(a, b)$ on $G$ has the Jacobian matrix $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. Hence, the quadratic form $da^2 + db^2$ on the tangent space $T_{(0,1)}G$ at the identity is the pullback of the quadratic form $(da^2 + db^2)/b^2$ on $T_{(a,b)}G$ at the point $(a, b)$. $\square$

Note that starting with *any* positive definite quadratic form, one obtains an isometric manifold (up to a scalar factor in the metric).

DEFINITION 1.2. The Riemannian manifold $G$ equipped with the metric $ds^2 = (da^2 + db^2)/b^2$ is called the *Lobachevsky plane* $\Lambda^2$ (respectively, the *Lobachevsky space* $\Lambda^{n+1}$ for $x, a \in \mathbb{R}^n$).

PROPOSITION 1.3. *Geodesic lines (i.e., extremals of the length functional) are the vertical half lines ($a =$ const, $b > 0$) and the semicircles orthogonal to the $a$-axis (or $a$-hyperplane, respectively).*

Here vertical half lines can be viewed as semicircles of infinite radius; see Fig.46.

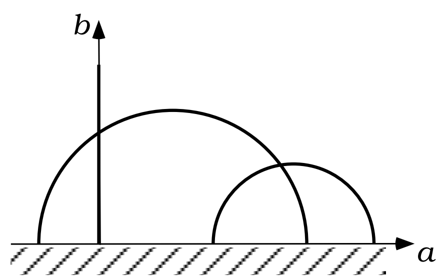



FIGURE 46. The Lobachevsky plane $\Lambda^2$ and geodesics on it.

PROOF. Reflection of $\Lambda^2$ in a vertical line or in a circle centered at the $a$-axis is an isometry. $\square$

Note that two geodesics on $\Lambda^2$ with close initial conditions diverge exponentially from each other (in the Lobachevsky metric). On a path only few units long, a deviation in initial conditions grows 100 times larger. The reason for practical indeterminacy of geodesics is the negative curvature of the Lobachevsky plane (it is constant and equal to $-1$ in the metric above). The curvature of the sphere of radius $R$ is equal to $R^{-2}$. The Lobachevsky plane might be regarded as a sphere of imaginary radius $R = \sqrt{-1}$.

PROBLEM 1.4 (B.Ya. Zeldovich). Prove that medians of every geodesic triangle in the Lobachevsky plane meet at one point. (Hint: Prove it for the sphere; then regard the Lobachevsky plane as the analytic continuation of the sphere to the imaginary values of the radius.)

**1.B. Curvature and parallel translation.** This section recalls some basic notions of differential geometry necessary in the sequel. For more extended treatment see [Mil3, Arn16, DFN].

Let $M$ be a Riemannian manifold (one can keep in mind the Euclidean space $\mathbb{R}^n$, the sphere $S^n$, or the Lobachevsky space $\Lambda^n$ of the previous section as an example); let $x \in M$ be a point of $M$, and $\xi \in T_x M$ a vector tangent to $M$ at $x$.

Denote by any of $\{\gamma(x, \xi, t)\} = \{\gamma(\xi, t)\} = \{\gamma(t)\} = \gamma$ the *geodesic line* on $M$, with the initial velocity vector $\xi = \dot\gamma(0)$ at the point $x = \gamma(0)$. (Geodesic lines can be defined as extremals of the action functional: $\delta \int \dot\gamma^2 dt = 0$. It is called the "principle of least action.")

*Parallel translation along a geodesic segment* is a special isometry mapping the tangent space at the initial point onto the tangent space at the final point, depending smoothly on the geodesic segment, and obeying the following properties.

(1) Translation along two consecutive segments coincides with translation along the first segment composed with translation along the second.
(2) Parallel translation along a segment of length zero is the identity map.
(3) The unit tangent vector of the geodesic line at the initial point is taken to the unit tangent vector of the geodesic at the final point.

EXAMPLE 1.5. The usual parallel translation in Euclidean space satisfies the properties (1)–(3).

The isometry property, along with the property (3), implies that the angle formed by the transported vector with the geodesic is preserved under translation. This observation alone determines parallel translation in the two-dimensional case, i.e., on surfaces, see Fig.47.

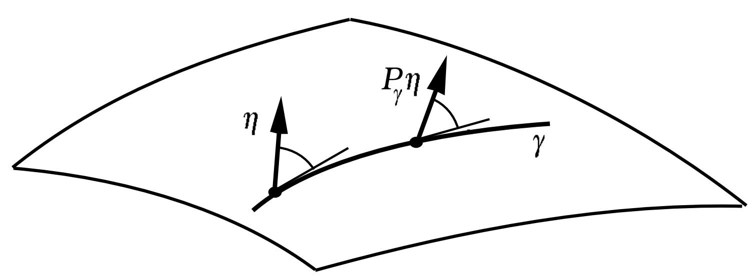



FIGURE 47. Parallel translation along a geodesic line $\gamma$.

In the higher-dimensional case, parallel translation is not determined uniquely by the condition of preserving the angle: One has to specify the plane containing the transported vector.

DEFINITION 1.6. *(Riemannian) parallel translation* along a geodesic is a family of isometries obeying properties (1)–(3) above, and for which the translation of a vector $\eta$ along a short segment of length $t$ remains tangent (modulo $O(t^2)$-small correction as $t \to 0$) to the following two-dimensional surface. This surface is formed by the geodesics issuing from the initial point of the segment with the velocities spanned by the vector $\eta$ and by the velocity of the initial geodesic.

REMARK 1.7. A physical description of parallel translation on a Riemannian manifold can be given using the adiabatic (slow) transportation of a pendulum along a path on the manifold (Radon, see [Kl]). The plane of oscillations is parallelly translated.

A similar phenomenon in optics is called the inertia of the polarization plane along a curved ray (see [Ryt, Vld]). It is also related to the additional rotation of a gyroscope transported along a closed path on the surface (or in the oceans) of the Earth, proportional to the swept area [Ish]. A modern version of these relations between adiabatic processes and connections explains, among other things, the Aharonov–Bohm effect in quantum mechanics. This version is also called the Berry phase (see [Berr, Arn23]).

DEFINITION 1.8. The *covariant derivative* $\nabla_\xi \bar\eta$ of a tangent vector field $\bar\eta$ in a direction $\xi \in T_x M$ is the rate of change of the vector of the field $\bar\eta$ that is parallel-transported to the point $x \in M$ along the geodesic line $\gamma$ having at this point the velocity $\xi$ (the vector observed at $x$ at time $t$ must be transported from the point $\gamma(\xi, t)$).

Note that the vector field $\dot\gamma$ along a geodesic line $\gamma$ obeys the equation $\nabla_{\dot\gamma}\dot\gamma = 0$.

REMARK 1.9. For calculations of parallel translations in the sequel we need the following explicit formulas. Let $\gamma(x, \xi, t)$ be a geodesic line in a manifold $M$, and let $P_{\gamma(t)} : T_{\gamma(0)}M \to T_{\gamma(t)}M$ be the map that sends any $\eta \in T_{\gamma(0)}M$ to the vector

$$P_{\gamma(\xi,t)}\eta := \frac{1}{t}\frac{d}{d\tau}\,|_{\tau=0}\; \gamma(x, \xi + \eta\tau, t) \in T_{\gamma(t)}M. \tag{1.1}$$

The mapping $P_{\gamma(t)}$ approximates parallel translation along the curve $\gamma$ in the following sense. The *covariant derivative* $\nabla_\xi \bar\eta$ of the field $\bar\eta$ in the direction of the vector $\xi \in T_x M$ is equal to

$$\nabla_\xi \bar\eta := \frac{d}{dt}\,|_{t=0}\; P^{-1}_{\gamma(t)}\bar\eta(\gamma(\xi, t)) \in T_x M. \tag{1.2}$$

REMARK 1.10. The following properties uniquely determine the covariant derivative on a Riemannian manifold and can be taken as its axiomatic definition (see, e.g., [Mil3, K-N]):

1) $\nabla_\xi v$ is a bilinear function of the vector $\xi$ and the field $v$;

2) $\nabla_\xi fv = (L_\xi f)v + f(\nabla_\xi v)$, where $f$ is a smooth function and $L_\xi f$ is the derivative of $f$ in the direction of the vector $\xi$ in $T_xM$;
3) $L_\xi\langle v, w\rangle = \langle \nabla_\xi v, w(x)\rangle + \langle v(x), \nabla_\xi w\rangle$; and
4) $\nabla_{v(x)}w - \nabla_{w(x)}v = \{v, w\}(x)$.

Here $\langle\ ,\ \rangle$ is the inner product defined by the Riemannian metric on $M$, and $\{v, w\}$ is the Poisson bracket of two vector fields $v$ and $w$. In local coordinates $(x_1, \dots, x_n)$ on $M$ the Poisson bracket is given by the formula

$$\{v, w\}_j = \sum_{i=1}^{n} \left( v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right).$$

Parallel translation along any curve is defined as the limit of parallel translations along broken geodesic lines approximating this curve. The increment of a vector after the translation along the boundary of a small region on a smooth surface is (in the first approximation) proportional to the area of the region.

DEFINITION 1.11. The (Riemannian) *curvature tensor* $\Omega$ describes the infinitesimal transformation in a tangent space obtained by parallel translation around an infinitely small parallelogram. Given vectors $\xi, \eta \in T_xM$, consider a curvilinear parallelogram on $M$ "spanned" by $\xi$ and $\eta$. The main (bilinear in $\xi, \eta$) part of the increment of any vector in the tangent space $T_xM$ after parallel translation around this parallelogram is given by a linear operator $\Omega(\xi, \eta) : T_xM \to T_xM$. The action of $\Omega(\xi, \eta)$ on a vector $\zeta \in T_xM$ can be expressed in terms of covariant differentiation as follows:

$$\Omega(\xi, \eta)\zeta = (-\nabla_{\bar\xi}\nabla_{\bar\eta}\bar\zeta + \nabla_{\bar\eta}\nabla_{\bar\xi}\bar\zeta + \nabla_{\{\bar\xi,\bar\eta\}}\bar\zeta)|_{x=x_0}, \tag{1.3}$$

where $\bar\xi, \bar\eta, \bar\zeta$ are any vector fields whose values at the point $x$ are $\xi, \eta$, and $\zeta$. The value of the right-hand side does not depend on the extensions $\bar\xi, \bar\eta, \bar\zeta$ of the vectors $\xi, \eta$, and $\zeta$.

The *sectional curvature* of $M$ in the direction of the 2–plane spanned by two orthogonal unit vectors $\xi, \eta \in T_xM$ in the tangent space to $M$ at the point $x$ is the value

$$C_{\xi\eta} = \langle \Omega(\xi, \eta)\xi, \eta\rangle. \tag{1.4}$$

For a pair of arbitrary (not necessarily orthonormal) vectors the sectional curvature $C_{\xi\eta}$ is

$$C_{\xi\eta} = \frac{\langle \Omega(\xi, \eta)\xi, \eta\rangle}{\langle \xi, \xi\rangle\langle \eta, \eta\rangle - \langle \xi, \eta\rangle^2}. \tag{1.5}$$

EXAMPLE 1.12. The sectional curvature at every point of the Lobachevsky plane (Section 1) is equal to $-1$. (Hint: use the explicit description of the geodesics in the plane. See also Section 2 for general formulas for sectional curvatures on Lie groups.)

DEFINITIONS 1.13. The *normalized Ricci curvature* (at a point $x$) in the direction of a unit vector $\xi$ is the average of the sectional curvatures of all tangential 2-planes containing $\xi$. It is equal to $r(\xi)/(n-1)$, where the *Ricci curvature* $r(\xi)$ is the value $r(\xi) = \sum_i C_{\xi e_i} = \sum_i \langle \Omega(\xi, e_i)\xi, e_i\rangle$, calculated in any orthonormal basis $e_1, \dots, e_n$ of the tangent space $T_xM$.

The *normalized scalar curvature* at a point $x$ is the average of all sectional curvatures at the point. It is equal to $\rho/n(n-1)$, the *scalar curvature* $\rho$ being the sum $\rho = r(e_i) + \cdots + r(e_n) = 2\sum_{i<j} C_{e_ie_j}$ (see, e.g., [Mil4]).

**1.C. Behavior of geodesics on curved manifolds.** From the definition of curvature one easily deduces the following

PROPOSITION 1.14. *The distance $y(t)$ between two infinitely close geodesics on a surface satisfies the differential equation*

$$\ddot{y} + Cy = 0, \tag{1.6}$$

*where $C = C(t)$ is the Riemannian curvature of the surface along the geodesic.*

In order to describe in general how the curvature tensor affects the behavior of geodesics, we look at a variation $\gamma_\alpha(t)$ of a geodesic $\gamma = \gamma_0(t)$. For each $\alpha$ sufficiently close to 0, the curve $\gamma_\alpha$ is a geodesic whose initial condition is close to that of $\gamma$. The field $\xi(t) = \frac{d}{d\alpha}\,|_{\alpha=0}\,\gamma_\alpha(t)$ (defined along $\gamma$) is called the *vector field of geodesic variation.*

LEMMA–DEFINITION 1.15. *The vector field of geodesic variation satisfies the second-order linear differential equation, called the Jacobi equation,*

$$\nabla_{\dot\gamma}^2\xi + \Omega(\dot\gamma, \xi)\dot\gamma = 0. \tag{1.7}$$

PROOF. Define the vector field of geodesic variation $\xi(t,\alpha)$ for all geodesics of the family $\gamma_\alpha(t)$ (with small $\alpha$) as the derivative $\xi(t,\alpha) = \frac{d}{d\alpha}\gamma_\alpha(t)$. Then the fields $\xi$ and $\dot\gamma$ commute ($\{\xi, \dot\gamma\} \equiv 0$) as partial derivatives of the map $(t,\alpha) \mapsto \gamma_\alpha(t)$. Using the properties of covariant differentiation listed above and the definition of the curvature tensor, we get

$$\nabla_{\dot\gamma}^2\xi = \nabla_{\dot\gamma}\nabla_{\dot\gamma}\xi = \nabla_{\dot\gamma}\nabla_{\xi}\dot\gamma = -\Omega(\dot\gamma, \xi)\dot\gamma.$$

□

Assume for a moment that the curvature is positive in all two-dimensional directions containing the velocity vector of the geodesic. A closer analysis of the Jacobi equation (or its analogy with the standard pendulum; see Proposition 1.14) shows that the normal component of the vector field of geodesic variation oscillates with $t$. This means that geodesics with close initial velocities on a manifold of positive curvature oscillate around each other (or locally converge); see Fig.48a. On the other hand, negativity in sectional curvatures prompts analogy with the upside-down pendulum and implies the exponential divergence of nearby geodesics from the given one; see Fig.48b.

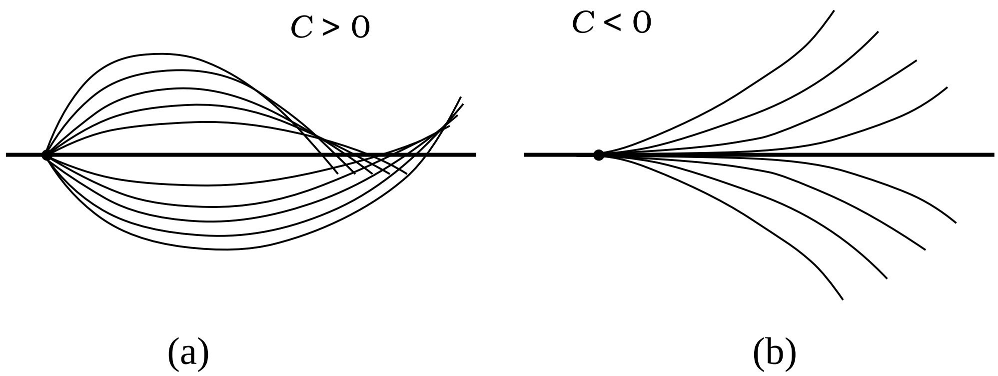



FIGURE 48. Geodesics on manifolds of (a) positive and (b) negative curvature.

REMARK 1.16. For numerical estimates of the instability, it is useful to define the *characteristic path length* as the average path length on which small errors in the initial conditions are increased by the factor of $e$. If the curvature of our manifold in all two-dimensional directions is bounded away from zero by the number $-b^2$, then the characteristic path length is not greater than $1/b$ (cf. Proposition 1.14). In view of the exponential character of the growth of error, the course of a geodesic line on a manifold of negative curvature is practically impossible to predict.

**1.D. Relation of the covariant and Lie derivatives.** Every vector field on a Riemannian manifold defines a differential 1-form: the pointwise inner product with vectors of the field. For a vector field $v$ we denote by $v^\flat$ the 1-form whose value on a tangent vector at a point $x$ is the inner product of the tangent vector with the vector $v(x)$.

Every vector field also defines a flow, which transports differential forms. For instance, one might transport the 1-form corresponding to some vector field by means of the flow of this field and get a new differential 1-form. Infinitesimally this transport is described by the Lie derivative of the 1-form (corresponding to the field) along the field itself, and the result is again a 1-form. This natural derivative of a 1-form is related to the covariant derivative of the corresponding vector field along itself by the following remarkable formula.

THEOREM 1.17. *The Lie derivative of the one-form corresponding to a vector field on a Riemannian manifold differs from the one-form corresponding to the covariant derivative of the field along itself by a complete differential:*

$$L_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2} d\langle v, v\rangle. \tag{1.8}$$

*Here $\langle v, v\rangle$ is the function on the manifold equal at each point $x$ to the Riemannian square of the vector $v(x)$.*

Note that this statement does not require the vector field to be divergence free.

PROOF. Let $w$ be a vector field commuting with the field $v$ (i.e., $\{v, w\} = 0$).

First, since parallel translation is an isometry,

$$L_a\langle b, c\rangle = \langle \nabla_a b, c\rangle + \langle b, \nabla_a c\rangle, \tag{1.9}$$

for any vector fields $a, b$, and $c$. Applying this to the fields $a = w,\ \ b = c = v$, gives $L_w\langle v, v\rangle = \langle \nabla_w v, v\rangle + \langle v, \nabla_w v\rangle$. From this we find that

$$\langle \nabla_w v, v\rangle = \frac{1}{2}(d\langle v, v\rangle)(w). \tag{1.10}$$

Applying the isometry property (1.9) once more to $a = c = v,\ \ b = w$, we get

$$L_v\langle w, v\rangle = \langle \nabla_v w, v\rangle + \langle w, \nabla_v v\rangle. \tag{1.11}$$

On the other hand, for commuting fields $v$ and $w$, property 4) of Remark 1.10 implies

$$\langle \nabla_v w, v\rangle = \langle \nabla_w v, v\rangle. \tag{1.12}$$

Substituting this into (1.11) we obtain that

$$L_v\langle w, v\rangle = \langle \nabla_w v, v\rangle + \langle w, \nabla_v v\rangle.$$

Using (1.10), we rewrite the above in the form

$$L_v\langle w, v\rangle = \langle \nabla_v v, w\rangle + \frac{1}{2}(d\langle v, v\rangle)(w). \tag{1.13}$$

Next, we use the identity

$$L_\xi(v^\flat(w)) = (L_\xi(v^\flat))(w) + v^\flat(L_\xi w), \tag{1.14}$$

which expresses the naturalness of the Lie derivative: The flow of $\xi$ transports the 1-form $v^\flat$, the vector $w$, and the value of the 1-form on this vector. (This is the reason why vector fields are transported in the opposite direction in the definition of the Lie derivative.)

Applying (1.14) to $\xi = v$, we obtain

$$L_v(v^\flat(w)) = (L_v(v^\flat))(w), \tag{1.15}$$

since $L_v w = -\{v, w\} = 0$. (Here we use the commutativity of $v$ and $w$ for the second time.)

Note that by the definition of the map $v \mapsto v^\flat$ one has

$$L_v(v^\flat(w)) = L_v\langle v, w\rangle.$$

Combining this with (1.15) and (1.13), we find that for any vector of a field $w$ commuting with the field $v$,

$$(L_v(v^\flat))(w) = (\nabla_v v)^\flat(w) + \frac{1}{2}(d\langle v, v\rangle)(w). \tag{1.16}$$

At every point $x$ where the vector field $v$ is nonzero, it is easy to construct a field $w$ commuting with $v$ and having at this point any value. Hence the identity (1.16) implies

$$L_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2}d\langle v, v\rangle,$$

which proves the theorem.

At the singular points $v(x) = 0$ there is nothing to prove, since both sides of the relation (1.8) are equal to zero. □

REMARK 1.18. One can take an arbitrary field $w$ instead of the one commuting with $v$, and then the formulas are slightly longer. Two commutator terms have to be introduced at the two places where commutativity was used: The additional term $\langle\{w, v\}, v\rangle$ on the right-hand side of (1.12) cancels with the extra term $\langle v, -\{v, w\}\rangle$ on the right-hand side of (1.15).

This theorem explains the form of the Euler equation of an incompressible fluid on an arbitrary Riemannian manifold $M$ presented in Sections I.5 and I.7:

COROLLARY 1.19. *The Euler equation*

$$\frac{\partial v}{\partial t} = -\nabla_v v - \nabla p$$

*on the Lie algebra* $\mathfrak{g} = S\,\mathrm{Vect}(M)$ *of divergence-free vector fields is mapped by the inertia operator* $A : \mathfrak{g} \to \mathfrak{g}^*$ *to the Euler equation*

$$\frac{\partial [u]}{\partial t} = -L_v[u] \tag{1.17}$$

*on the dual space* $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$ *of this algebra. Here the field* $v$ *and the 1-form* $u$ *are related by means of the Riemannian metric:* $u = v^\flat$, *and* $[u] \in \Omega^1/d\Omega^0$ *is the coset of the form* $u$.

PROOF. The inertia operator $A : S\,\mathrm{Vect}(M) \to \Omega^1/d\Omega^0$ sends a divergence-free field $v$ to the 1-form $u = v^\flat$ considered up to the differential of a function. By the above theorem, it also sends the covariant derivative $\nabla_v v$ to the Lie derivative $L_v u$ modulo the differential of another function. Hence the Euler equation for the 1-form $u$ assumes the form

$$\frac{\partial u}{\partial t} = -L_v u - df,$$

with the function $f = p - \frac{1}{2}\langle v, v\rangle$. It is equivalent to Equation (1.17) for the coset $[u]$. $\square$

## §2. Sectional curvatures of Lie groups equipped with a one-sided invariant metric

Let $G$ be a Lie group whose left-invariant metric is given by a scalar product $\langle\ ,\ \rangle$ in the Lie algebra. The sectional curvature of the group $G$ at any point is determined by the curvature at the identity (since by definition, left translations map the group to itself isometrically). Hence, it suffices to describe the curvatures for the two-dimensional planes lying in the Lie algebra $\mathfrak{g} = T_eG$.

THEOREM 2.1 [Arn4]. *The curvature of a Lie group* $G$ *in the direction determined by an orthonormal pair of vectors* $\xi, \eta$ *in the Lie algebra* $\mathfrak{g}$ *is given by the formula*

$$C_{\xi\eta} = \langle\delta, \delta\rangle + 2\langle\alpha, \beta\rangle - 3\langle\alpha, \alpha\rangle - 4\langle B_\xi, B_\eta\rangle, \tag{2.1}$$

*where*

$$2\delta = B(\xi,\eta) + B(\eta,\xi), \qquad 2\beta = B(\xi,\eta) - B(\eta,\xi)$$
$$2\alpha = [\xi,\eta], \qquad 2B_\xi = B(\xi,\xi), \qquad 2B_\eta = B(\eta,\eta), \tag{2.2}$$

*and where* $[\ ,\ ]$ *is the commutator in* $\mathfrak{g}$, *and* $B$ *is the bilinear operation on* $\mathfrak{g}$ *defined by the relation* $\langle B(u,v), w\rangle = \langle u, [v,w]\rangle$ *(see Chapter I).*

REMARK 2.2. In the case of a two-sided invariant metric, the formula for the curvature has the particularly simple form

$$C_{\xi\eta} = \frac{1}{4}\langle[\xi,\eta],[\xi,\eta]\rangle. \tag{2.3}$$

In particular, in this case the sectional curvatures are nonnegative in all two-dimensional directions.

REMARK 2.3. The formula for the curvature of a group with a right-invariant Riemannian metric coincides with the formula in the left-invariant case. In fact, a right-invariant metric on a group is a left-invariant metric on the group with the reverse multiplication law ($g_1 \star g_2 = g_2 g_1$). Passage to the reverse group changes the signs of both the commutator and the operation $B$ in the algebra. But in every term of the curvature formula there is a product of two operations changing the sign. Therefore, the formula for curvature is the same in the right-invariant case. The right-hand side of the Euler equation changes sign under passage to the right-invariant case.

The mapping of the group $G$ to itself, sending each element $g$ to the inverse element $g^{-1}$, is an involution preserving the identity element. It sends any left-invariant metric on the group to the corresponding right-invariant metric (defined on the Lie algebra by the same quadratic form). Hence the group with the left-invariant metric is isometric to the same group with the corresponding right-invariant metric.

To give the coordinate expression for the curvature, choose an orthonormal basis $e_1, \dots, e_n$ for the left-invariant vector fields. The Lie algebra structure can be described by an $n\times n\times n$ array of structure constants $\alpha_{ijk}$ where $[e_i, e_j] = \sum_k \alpha_{ijk} e_k$, or, equivalently, $\alpha_{ijk} = \langle [e_i, e_j], e_k\rangle$. This array is skew-symmetric in the first two indices. Then Theorem 2.1 claims the following:

THEOREM 2.1′(SEE [Mil4]). *The sectional curvature* $C_{e_1e_2}$ *is given by the formula*

$$C_{e_1e_2} = \sum_k \left(\frac{1}{2}\alpha_{12k}\left(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}\right)\right.$$
$$\left. - \frac{1}{4}\left(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12}\right)\left(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}\right) - \alpha_{k11}\alpha_{k22}\right). \tag{2.4}$$

REMARK 2.4. Before proving the theorem we give here an account of noteworthy facts about left-invariant metrics on Lie groups that can be formulated in a coordinate-free way (and some have infinite-dimensional counterparts). We refer to [Mil4] for all the details.

- If $\xi$ belongs to the center of a Lie algebra, then for every left-invariant metric, the inequality $C_{\xi\eta} \geq 0$ is satisfied for all $\eta$ (cf. Section VI.1.A on the Virasoro algebra).
- If a connected Lie group $G$ has a left-invariant metric with all sectional curvatures $C \leq 0$, then it is solvable (example: affine transformations of the line; see Section 1.A).
- If $G$ is unimodular (i.e., the operators $\mathrm{ad}_u$ are traceless for all $u \in \mathfrak{g}$), then any such metric with $C \leq 0$ must actually be flat ($C \equiv 0$) (cf. Remark II.4.14).
- Every compact Lie group admits a left invariant (and in fact, a bi-invariant) metric such that all sectional curvatures satisfy $C \geq 0$ (cf. Remark 2.2 above).
- If the Lie algebra of $G$ contains linearly independent vectors $\xi, \eta, \zeta$ satisfying $[\xi, \eta] = \zeta$, then there exists a left-invariant metric such that the Ricci curvature $r(\xi)$ is strictly negative, while the Ricci curvature $r(\zeta)$ is strictly positive. For instance, one can define such a metric on $SO(3)$, the configuration space of a rigid body, such that a certain Ricci curvature is negative!
- (Wallach) If the Lie group $G$ is noncommutative, then it possesses a left-invariant metric of strictly negative scalar curvature.
- If $G$ contains a compact noncommutative subgroup, then $G$ admits a left-invariant metric of strictly positive scalar curvature [Mil4].

PROOF OF THEOREM 2.1. To obtain explicit formulas for sectional curvatures of the group $G$ we start by expressing the covariant derivative in terms of the operation $B$ (or of the array $\alpha_{ijk}$).

LEMMA 2.5. *Let $\xi$ and $\eta$ be two left-invariant vector fields on the group $G$. Then the vector field $\nabla_\xi \eta$ is also left-invariant and at the point $e \in G$ is given by the following formula:*

$$\nabla_\xi \eta \mid_e = \frac{1}{2}([\xi, \eta] - B(\xi, \eta) - B(\eta, \xi)), \tag{2.5}$$

*where on the right-hand side, $\xi$ and $\eta$ are vectors in $\mathfrak{g} = T_e G$ defining the corresponding left-invariant fields on $G$.*

In coordinates,

$$\nabla_{e_i} e_j = \sum_k \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) e_k. \tag{2.5$'$}$$

PROOF OF LEMMA. Parallel transport on a Riemannian manifold preserves the inner product $\langle a, b\rangle$. On the other hand, the left-invariant product $\langle a, b\rangle$ of any left-invariant fields $a$ and $b$ is constant. Therefore, for any $c$, the operator $\nabla_c$ is antisymmetric on left-invariant fields:

$$\langle \nabla_c a, b\rangle + \langle \nabla_c b, a\rangle = 0.$$

Furthermore, for the covariant derivative on a Riemannian manifold the following "symmetry" condition holds (see Remark 1.10):

$$\nabla_c a - \nabla_a c = \{c, a\}.$$

Recall that on a Lie group, the Poisson bracket $\{\ ,\ \}$ of the left-invariant vector fields $a$ and $c$ coincides at $e \in G$ with the commutator $[\ ,\ ]_{\mathfrak{g}}$ in the Lie algebra:

$$\{a, c\} \mid_e = [a, c]_{\mathfrak{g}}. \tag{2.6}$$

The Poisson bracket of two right-invariant vector fields has the opposite sign (see Remark I.2.13 or [Arn4]).

Combining the above identities, we obtain the formula

$$\langle \nabla_\xi \eta, \zeta\rangle = \frac{1}{2}(\langle [\xi, \eta], \zeta\rangle - \langle [\eta, \zeta], \xi\rangle + \langle [\zeta, \xi], \eta\rangle),$$

easily seen to be equivalent to (2.5). For the orthonormal basis $e_1, \dots, e_n$, it immediately implies

$$\langle \nabla_{e_i} e_j, e_k\rangle = \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}).$$

This completes the proof of Lemma 2.5. □

Finally, the coordinate expression (2.4) for sectional curvature is a straightforward consequence of formulas (1.4), (2.5′), and (2.6). Theorem 2.1′ is proved. □

REMARK 2.6. Lemma 2.5 is deduced in [Arn4] from the Euler equation $\dot{\xi} = B(\xi, \xi)$ (see Chapter I).

Consider a neighborhood of the point 0 in the Lie algebra $\mathfrak{g}$ as a chart of a neighborhood of the unit element $e$ in the group using the exponential map $\exp : \mathfrak{g} \to G$. It sends $t\xi$ to the element $\exp(t\xi)$ of the one-parameter group starting at

$t = 0$ from $e$ with initial velocity $\xi$. (We leave aside the difficulties of this approach in the infinite-dimensional case, where the image of the exponential mapping does not contain the neighborhood of the group unit element.) The exponential mapping identifies the tangent spaces of the group $T_g G$ with the Lie algebra $\mathfrak{g}$.

The Euler equation implies that the geodesic line on the group has, in our coordinates, the following expansion in $t$:

$$\gamma(0, \xi, t) = t\xi + \frac{t^2}{2} B(\xi, \xi) + \mathcal{O}(t^3),\ t \to 0.$$

Then the approximate translation

$$P_{\gamma(\xi,t)}\eta = \frac{1}{t}\frac{d}{d\tau}\ |_{\tau=0}\ \gamma(0, \xi + \eta\tau, t) \in T_{\gamma(t)}\mathfrak{g} = \mathfrak{g}$$

of a vector $\eta \in T_0\mathfrak{g} = \mathfrak{g}$ is explicitly given by

$$P_{\gamma(\xi,t)}\eta = \eta + \frac{t}{2}(B(\xi, \eta) + B(\eta, \xi)) + O(t^2). \tag{2.7}$$

By definition, the covariant derivative (in the direction $\xi \in \mathfrak{g}$) of the left-invariant vector field on the group $G$ determined by the vector $\eta \in \mathfrak{g}$ is

$$\nabla_\xi \eta = \frac{d}{dt}\ |_{t=0}\ P_{\gamma^{-1}(\xi,t)} L_{\gamma(\xi,t)}\eta, \tag{2.8}$$

where on the left-hand side, $\eta$ stands for the corresponding left-invariant vector field on $G$.

Note that for any Lie group $G$ the operator of left translation by group elements $\exp a$ close to the identity (i.e., as $|a| \to 0$) acts on the Lie algebra $\mathfrak{g}$ (considered as a chart of the group) as follows:

$$L_a\xi = \xi + \frac{1}{2}[a, \xi] + O(a^2). \tag{2.9}$$

Indeed, the general case of any Lie group follows from the calculus on matrix groups:

$$\exp a \cdot \exp b = \exp\left(a + b + \frac{1}{2}[a, b] + O(a^2) + O(b^2)\right)$$

for any Lie algebra elements $a, b \to 0$. Setting $b = \xi t,\ t \to 0,\ |a| \to 0$, we get

$$\exp a \cdot \exp \xi t = \exp\left[a + \left(\xi + \frac{1}{2}[a, \xi] + O(a^2)\right) t + O(t^2)\right],$$

which is equivalent to (2.9).

Now, substituting into (2.8) the expressions (2.7) for $P_\gamma$ and (2.9) for the left translation $L_\gamma \eta$, we get the following:

$$\begin{aligned}\nabla_\xi \eta &= \frac{d}{dt}\,|_{t=0}\; P_{\gamma^{-1}(\xi,t)}(\eta + \frac{t}{2}[\xi,\eta] + \mathcal{O}(t^2)) \\ &= \frac{d}{dt}\,|_{t=0}\left(\eta + \frac{t}{2}([\xi,\eta] - B(\xi,\eta) - B(\eta,\xi)) + \mathcal{O}(t^2)\right).\end{aligned}$$

□

Theorem 2.1 can now be proven in the following coordinate-free way (see [Arn4]).

Proof of Theorem 2.1. Let $\xi, \eta$ be left-invariant vector fields on the group $G$. Then the fields $\{\xi,\eta\}$, $\nabla_\xi \eta$, and $\nabla_\eta \xi$ are left-invariant as well. The formula (2.5), combined with the notations (2.2), gives the following values of these vector fields at the identity $\mathrm{Id} \in G$:

$$\begin{aligned}\nabla_\xi \xi &= -2B_\xi, \qquad \nabla_\xi \eta = \alpha - \delta, \\ \nabla_\eta \eta &= -2B_\eta, \qquad \nabla_\eta \xi = -(\alpha + \delta).\end{aligned} \tag{2.10}$$

Now, in order to evaluate the terms in (1.4), we use these expressions along with the skew symmetry of $\nabla$ to obtain

$$\begin{aligned}\langle -\nabla_\xi \nabla_\eta \xi, \eta\rangle &= \langle \nabla_\xi \eta, \nabla_\eta \xi\rangle = -\langle \alpha - \delta, \alpha + \delta\rangle, \\ \langle \nabla_\eta \nabla_\xi \xi, \eta\rangle &= \langle \nabla_\xi \xi, \nabla_\eta \eta\rangle = -4\,\langle B_\xi, B_\eta\rangle.\end{aligned} \tag{2.11}$$

Moreover, by virtue of (2.6),

$$\begin{aligned}\langle \nabla_{\{\xi,\eta\}} \xi, \eta\rangle &= \langle \nabla_{[\xi,\eta]} \xi, \eta\rangle \\ &= \frac{1}{2}\langle [[\xi,\eta],\xi],\eta\rangle - \frac{1}{2}\langle B([\xi,\eta],\xi),\eta\rangle - \frac{1}{2}\langle B(\xi,[\xi,\eta]),\eta\rangle \\ &= -2\langle \alpha,\alpha\rangle + 2\langle \alpha,\beta\rangle.\end{aligned} \tag{2.12}$$

Finally, incorporating (2.10-2.12) into the definition (1.3-1.4) of sectional curvature, we get

$$C_{\xi\eta} = -\langle \alpha - \delta, \alpha + \delta\rangle - 4\langle B_\xi, B_\eta\rangle - 2\langle \alpha,\alpha\rangle + 2\langle \alpha,\beta\rangle,$$

which is equivalent to (2.1). This completes the proof. □

## §3. Riemannian geometry of the group of area-preserving diffeomorphisms of the two-torus

**3.A. The curvature tensor for the group of torus diffeomorphisms.** The coordinate-free formulas for the sectional curvature allow one to apply them to the infinite-dimensional case of groups of diffeomorphisms. The numbers that we obtain by applying the formula for the curvature of a Lie group to these infinite-dimensional groups are naturally called the curvatures of the diffeomorphism groups. We describe in detail the case of area-preserving diffeomorphisms of the two-dimensional torus and review the results for the two-sphere (the $S^2$ case is of special interest because of its relation to atmospheric flows and weather predictions), for the $n$-dimensional torus $T^n$ (the $T^3$ case is important for stability analysis of the $ABC$ flows), for a compact two-dimensional surface, and for any flat manifold.

We start with the two-dimensional torus $T^2$ equipped with the Euclidean metric: $T^2 = \mathbb{R}^2/\Gamma$, where $\Gamma$ is a lattice (a discrete subgroup) in the plane, e.g., the set of points with integral coordinates.

Consider the Lie algebra of divergence-free vector fields on the torus with a *single-valued stream function* (or a single-valued Hamiltonian function, with respect to the standard symplectic structure on $T^2$ given by the area form). The corresponding group $S_0\mathrm{Diff}(T^2)$ consists of area-preserving diffeomorphisms isotopic to the identity that leave the "center of mass" of the torus fixed.

REMARK 3.1. The subgroup $S_0\mathrm{Diff}(T^2)$ is a *totally geodesic submanifold* of the group $S\mathrm{Diff}(T^2)$ of all area-preserving diffeomorphisms; that is, any geodesic on $S_0\mathrm{Diff}(T^2)$ is a geodesic in the ambient group. This follows from the momentum conservation law: If at the initial moment the velocity field of an ideal fluid has a single-valued stream function, then at all other moments of time the stream function will also be single-valued. (Note that a similar statement holds in the more general case of a manifold $M$ with boundary: The two Lie subgroups in $S\mathrm{Diff}(M)$ corresponding to the Lie subalgebras of exact ($\mathfrak{g}_0$) and semiexact ($\mathfrak{g}_{se}$) vector fields (see Remark I.7.15) form totally geodesic submanifolds in the Lie group of all volume-preserving diffeomorphisms of $M$.)

The right-invariant metric on the group $S\mathrm{Diff}(T^2)$ is defined by the (doubled) kinetic energy: Its value at the identity of the group on a divergence-free vector field $v \in S\mathrm{Vect}(T^2)$ is $\langle v, v\rangle = \int_{T^2}(v,v)\ d^2x$. We will describe the sectional curvatures of the group $S_0\mathrm{Diff}(T^2)$ in various two-dimensional directions passing through the identity of the group. The curvatures of $S_0\mathrm{Diff}(T^2)$ and $S\mathrm{Diff}(T^2)$ in these directions are the same, since the submanifold $S_0\mathrm{Diff}(T^2)$ is totally geodesic.

The divergence-free vector fields that constitute the Lie algebra $S_0\mathrm{Vect}(T^2)$ of the group $S_0\mathrm{Diff}(T^2)$ can be described by their stream (i.e., Hamiltonian) functions with zero mean ($v = -H_y\partial/\partial x + H_x\partial/\partial y$). Thus in the sequel the set $S_0\mathrm{Vect}(T^2)$ will be identified with the space of real functions on the torus having average value zero. It is convenient to define such functions by their Fourier coefficients and to carry out all calculations over $\mathbb{C}$.

We now complexify our Lie algebra, inner product $\langle\ ,\ \rangle$, commutator $[\ ,\ ]$, and the operation $B$ in the algebra, as well as the Riemannian connection and curvature tensor $\Omega$, so that all these operations become (multi-)linear in the complex vector space of the complexified Lie algebra.

To construct a basis of this vector space we let $e_k$ (where $k$, called a *wave vector*, is a point of the Euclidean plane) denote the function whose value at a point $x$ of our plane is equal to $e^{i(k,x)}$.

This determines a function on the torus if the inner product $(k, x)$ is a multiple of $2\pi$ for all $x \in \Gamma$. All such vectors $k$ belong to a lattice $\Gamma^*$ in $\mathbb{R}^2$, and the functions $\{e_k \mid k \in \Gamma^*, k \neq 0\}$ form a basis of the complexified Lie algebra.

THEOREM 3.2 [Arn4,16]. *The explicit formulas for the inner product, commutator, operation $B$, connection, and curvature of the right-invariant metric on the group $S_0$Diff$(T^2)$ have the following form:*

$$\langle e_k, e_\ell\rangle = 0 \textit{ for } k+\ell \neq 0 \tag{3.1a}$$

$$\langle e_k, e_{-k}\rangle = k^2 S,$$

$$[e_k, e_\ell] = (k\times\ell)e_{k+\ell} \textit{ where } k\times\ell = k_1\ell_2 - k_2\ell_1, \tag{3.1b}$$

$$B(e_k, e_\ell) = b_{k,\ell}e_{k+\ell} \textit{ where } b_{k,\ell} = (k\times\ell)\frac{k^2}{(k+\ell)^2}, \tag{3.1c}$$

$$\nabla_{e_k}e_\ell = d_{\ell,k+\ell}e_{k+\ell} \textit{ where } d_{u,v} = \frac{(v\times u)(u,v)}{v^2}, \tag{3.1d}$$

$$\Omega_{k,\ell,m,n} := \langle\Omega(e_k, e_l)e_m, e_n\rangle = 0 \textit{ if } k+\ell+m+n \neq 0 \tag{3.1e}$$

$$\Omega_{k,\ell,m,n} = (a_{\ell n}a_{km} - a_{\ell m}a_{kn})S \textit{ if } k+\ell+m+n = 0,$$

$$\textit{where} \qquad a_{uv} = \frac{(u\times v)^2}{|u+v|}. \tag{3.1f}$$

In these formulas, $S$ is the area of the torus, and $u\times v$ the (oriented) area of the parallelogram spanned by $u$ and $v$. The parentheses $(u, v)$ denote the Euclidean scalar product in the plane, and angled brackets denote the scalar product in the Lie algebra.

We postpone the proof of this theorem, as well as of the corollaries below, until the next section. The formulas above allow one to calculate the sectional curvature in any two-dimensional direction.

EXAMPLE 3.3. Consider the parallel sinusoidal steady fluid flow given by the stream function $\xi = \cos(k, x) = (e_k + e_{-k})/2$. Let $\eta$ be any other real vector of the algebra $S_0\mathrm{Vect}(T^2)$ (i.e., $\eta = \Sigma x_\ell e_\ell$ with $x_{-\ell} = \bar{x}_\ell$).

THEOREM 3.4. *The curvature of the group $S_0 Diff(T^2)$ in any two-dimensional plane containing the direction $\xi$ is*

$$C_{\xi\eta} = -\frac{S}{4}\sum_\ell a_{k\ell}^2 |x_\ell + x_{\ell+2k}|^2, \tag{3.2}$$

*and therefore nonpositive.*

COROLLARY 3.5. *The curvature is equal to zero only for those two-dimensional planes that consist of parallel flows in the same direction as $\xi$, such that $[\xi, \eta] = 0$.*

COROLLARY 3.6. *The curvature in the plane defined by the stream functions $\xi = \cos(k, x)$ and $\eta = \cos(\ell, x)$ is*

$$C_{\xi\eta} = -(k^2 + \ell^2)\sin^2\alpha \cdot \sin^2\beta / 4S, \tag{3.3}$$

*where $S$ is the area of the torus, $\alpha$ is the angle between $k$ and $\ell$, and $\beta$ is the angle between $k + \ell$ and $k - \ell$.*

COROLLARY 3.7. *The curvature of the area-preserving diffeomorphism group of the torus $\{(x, y) \mod 2\pi\}$ in the two-dimensional directions spanned by the velocity fields $(\sin y, 0)$ and $(0, \sin x)$ is equal to $C = -1/(8\pi^2)$.*

REMARK 3.8. These calculations show that in many directions the sectional curvature is negative, but in a few it is positive. The stability of the geodesic is determined by the curvatures in the directions of *all* possible two-dimensional planes passing through the velocity vector of the geodesic at each of its points (the Jacobi equation).

Any fluid flow on the torus is a geodesic of our group. However, calculations simplify noticeably for a stationary flow. In this case the geodesic is a one-parameter subgroup of our group. Then the curvatures in the directions of all planes passing through velocity vectors of the geodesic at all of its points are equal to the curvatures in the corresponding planes going through the velocity vector of this geodesic at the initial moment of time. To prove it, (right-) translate the plane to the identity element of the group. Thus, stability of a stationary flow depends only on the curvatures in those two-dimensional planes in the Lie algebra that contain the velocity field of the steady flow.

### 3.B. Curvature calculations.

PROOF OF THEOREM 3.2. Formula (3.1a) is an immediate consequence of the definition. Statement (3.1b) follows from the version of (2.6) for right-invariant fields: $\{a, c\}\mid_e = -[a, c]_{\mathfrak{g}}$. Moreover, combining (3.1b) with the definition of $B$, we come to the relation

$$\langle B(e_k, e_\ell), e_m\rangle = (\ell \times m)\langle e_{\ell+m}, e_k\rangle. \tag{3.4}$$

Further, application of (3.1a) shows that $B(e_x, e_\ell)$ is orthogonal to $e_m$ for $k+\ell+m \neq 0$. Thus, $B(e_k, e_\ell) = b_{k,\ell}e_{k+\ell}$. Expression (3.1c) follows from (3.1a) and (3.4) for $m = -k - \ell$.

Now, formulas (3.1b,c) along with expression (2.5) for the covariant derivative imply that

$$\nabla_{e_k} e_\ell = \frac{1}{2}(k \times \ell)\left(1 - \frac{k^2 - \ell^2}{(k+\ell)^2}\right) e_{k+\ell}.$$

This implies (3.1d) after the evident simplification

$$\frac{1}{2}\left(1 - \frac{k^2 - \ell^2}{(k+\ell)^2}\right) = \frac{(\ell, k+\ell)}{(k+\ell)^2}.$$

In order to find the curvature tensor (1.3), we first compute from (3.1d)

$$\nabla_{e_k}\nabla_{e_\ell} e_m = d_{\ell+m,k+\ell+m} d_{m,\ell+m} e_{k+\ell+m},$$
$$\nabla_{[e_k,e_\ell]} e_m = (k \times \ell) d_{m,k+\ell+m} e_{k+\ell+m}.$$

Along with (3.1a) this implies that $\Omega_{k,\ell,m,n} = 0$ for $k + \ell + m + n \neq 0$, and

$$\Omega_{k,\ell,m,n} = (d_{k+m,n} d_{n,k+n} - d_{\ell+m,n} d_{m,\ell+n} + (k \times \ell) d_{m,n}) n^2 S$$

for $k + \ell + m + n = 0$ (note that $d_{u,v}$ is symmetric in $u$ and $v$ due to (3.1d)). We leave to the reader the reduction of this identity to the form (3.1e). $\square$

PROOF OF THEOREM 3.4. To derive formula (3.2), we substitute $\xi = \frac{e_k + e_{-k}}{2}$ and $\eta = \sum_\ell x_\ell e_\ell$ into

$$\begin{aligned} C_{\xi\eta} &= \langle \Omega(\xi,\eta)\xi, \eta\rangle \\ &= \frac{1}{4}\sum_\ell (\Omega_{k,\ell,k,-2k-\ell} x_\ell x_{-2k-\ell} + \Omega_{-k,\ell,-k,2k-\ell} x_\ell x_{2k-\ell} \\ &\qquad + \Omega_{k,\ell,-k,-\ell} x_\ell x_{-\ell} + \Omega_{-k,\ell,k,-\ell} x_\ell x_{-\ell}). \end{aligned}$$

Taking into account the relations (3.1e-f) of Theorem 3.2, one obtains the coefficients of this quadratic form:

$$\Omega_{k,\ell,k,-2k-\ell} = \Omega_{-k,\ell,k,-\ell} = -a_{k,\ell}^2 S$$
$$\Omega_{k,\ell,-k,-\ell} = \Omega_{-k,\ell,-k,2k-\ell} = -a_{k,-\ell}^2 S.$$

Then the form $C_{\xi\eta}$ for the orthonormal vectors $\xi$ and $\eta$ becomes

$$\begin{aligned}\langle\Omega(\xi,\eta)\xi,\eta\rangle &= -\frac{S}{4}\sum_\ell [a_{k,\ell}^2(x_\ell x_{-2k-\ell} + x_\ell x_{-\ell}) + a_{k,-\ell}^2(x_\ell x_{2k-\ell} + x_\ell x_{-\ell})] \\ &= -\frac{S}{4}\sum_\ell a_{k,\ell}^2(x_\ell x_{-2k-\ell} + x_\ell x_{-\ell} + x_{\ell+2k}x_{-\ell} + x_{\ell+2k}x_{-\ell-2k}),\end{aligned}$$

where the last identity is due to $a_{k,-\ell}^2 = a_{k,\ell-2k}^2$ (see (3.1f)). Finally, from the reality condition $x_{-j} = \bar{x}_j$, we get

$$C_{\xi\eta} = -\frac{S}{4}\sum_\ell a_{k,\ell}^2(x_\ell\bar{x}_{\ell+2k} + x_\ell\bar{x}_\ell + x_{\ell+2k}\bar{x}_\ell + x_{\ell+2k}\bar{x}_{\ell+2k}),$$

which is equivalent to (3.2). □

Proof of Corollary 3.6. By definition the sectional curvature in the plane spanned by a pair of orthogonal vectors $\xi$ and $\eta$ is

$$C_{\xi,\eta} = \frac{\langle\Omega(\xi,\eta)\xi,\eta\rangle}{\langle\xi,\xi\rangle\langle\eta,\eta\rangle}.$$

For our choice of $\xi$ and $\eta$ we have $\langle\xi,\xi\rangle = k^2S/2$, $\langle\eta,\eta\rangle = l^2S/2$, and $x_\ell = x_{-\ell} = \frac{1}{2}$. Therefore, by virtue of Theorem 3.4 and (3.1f), one obtains

$$\langle\Omega(\xi,\eta)\xi,\eta\rangle = -\frac{S}{8}(a_{k,\ell}^2 + a_{k,-\ell}^2).$$

Moreover, the explicit expression (3.1f) gives the identity

$$a_{k,\ell}^2 + a_{k,-\ell}^2 = (k\times\ell)^4\left(\frac{1}{h_+^2} + \frac{1}{h_-^2}\right),$$

with $h_\pm := k\pm\ell$. This, in turn, can be written as

$$a_{k,\ell}^2 + a_{k,-\ell}^2 = \frac{(k\times\ell)^2(h_+\times h_-)^2}{2h_+^2h_-^2}(k^2+\ell^2),$$

where we made use of the evident relations $h_+^2 + h_-^2 = 2(k^2+\ell^2)$ and $h_+\times h_- = -2(k\times\ell)$. Putting all the above together and substituting $(k\times\ell)^2 = k^2\ell^2\sin^2\alpha$, $(h_+\times h_-)^2 = h_+^2h_-^2\sin^2\beta$, we obtain formula (3.3) of the corollary. □

## §4. Diffeomorphism groups and unreliable forecasts

**4.A. Curvatures of various diffeomorphism groups.** For an arbitrary compact $n$-dimensional closed Riemannian manifold $M$, the curvatures of the diffeomorphism group $S_0\mathrm{Diff}(M)$ were calculated by Lukatsky [Luk5] (see also [Smo2]). We refer to [Luk3] for computations of the curvature tensor for the diffeomorphism group of any compact two-dimensional surface (the case of the two-dimensional sphere $S^2$, important for meteorological applications, can be found in [Luk1, Yo]) and to [Luk4] for those of a locally flat manifold (the case of the flat $n$-dimensional torus was treated in [Luk2]; see also explicit formulas for $T^3$ in [KNH]).

Riemannian geometry and curvatures of the semidirect product groups, relevant for ideal magnetohydrodynamics, are considered in [Ono]. In [Mis3] the curvatures of the group of all diffeomorphisms of a circle are discussed. The latter group, as well as its extension called the *Virasoro group,* is the configuration space of the Korteweg–de Vries equation; see Section VI.1.A. The geometry of bi-invariant metrics and geodesics on the symplectomorphism groups was studied in [Don].

Another way of investigating the Riemannian geometry of the group of volume-preserving diffeomorphisms is to embed it as a submanifold in the group of all diffeomorphisms of the manifold and then to study the exterior geometry of the corresponding submanifold (see Section 5 below).

Keeping in mind applications to weather forecasting, we look first at the group $S\mathrm{Diff}(S^2)$ of diffeomorphisms preserving the area on a standard sphere $S^2 \subset \mathbb{R}^3 = \{x, y, z\}$. Consider the following two steady flows on $S^2$: the rotation field $u = (-y, x, 0)$ and the nonrealistic "tradewind current" $v = z \cdot (-y, x, 0)$, Fig.49 (the real tradewind current has the same direction in the northern and southern hemispheres).

It was proved in [Luk1] that the sectional curvatures $C_{uw}$ in all two-planes containing $u$ are nonnegative for every field $w \in S\mathrm{Vect}(S^2)$, while for two-planes containing the field $v$ the curvatures $C_{vw}$ are negative for "most" directions $w$. Notice that the nonrealistic tradewind current $v$ is a "spherical counterpart" of the parallel sinusoidal flow on the torus $\xi = (\sin y, 0)$ (see statements 3.4-3.7 above).

In the case of volume-preserving diffeomorphisms of a three-dimensional domain $M \subset \mathbb{R}^3$ of Euclidean space (most important for hydrodynamics), the Jacobi equation was used by Rouchon to obtain the following information on the sectional curvatures of the diffeomorphism group.

Definition 4.1 [Rou2]. A divergence-free vector field on $M$ is said to be a *perfect eddy* if it is equal to the velocity field of a solid rotating with a constant angular velocity around a fixed axis (in particular, the vorticity is constant). Such

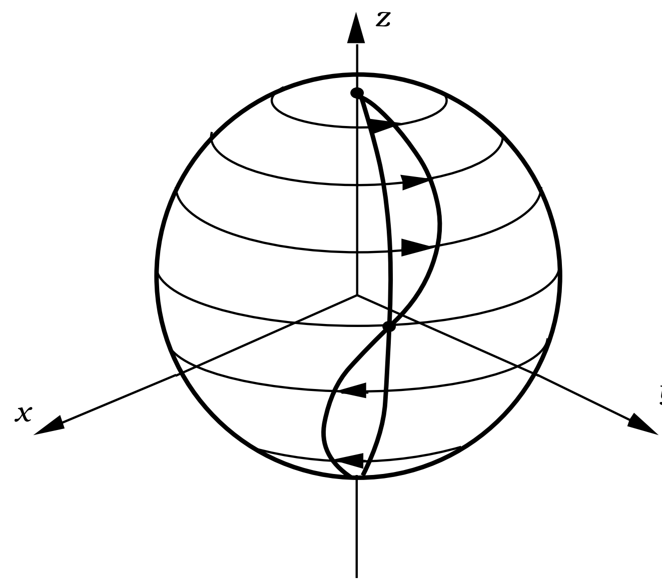



FIGURE 49. The velocity profile of the tradewind current on the sphere.

a fluid motion is possible if and only if the domain $M$ admits an axis of symmetry.

THEOREM 4.2 [Rou2]. *If the velocity field $v(t)$ of a flow of an ideal incompressible fluid filling a domain $M$ is a perfect eddy with constant vorticity, then the sectional curvature in every two-dimensional direction in $S\,\mathrm{Vect}(M)$ containing $v$ is nonnegative.*

*If the velocity $v(t)$ of an ideal fluid flow is not a perfect eddy, then for each time $t$ there always exist plane sections containing $v(t)$ (the velocity along the geodesic) where the sectional curvature is strictly negative.*

The result on nonnegativity of all the sectional curvatures holds also for rotations of spheres of arbitrary dimension [Mis1].

REMARK 4.3. One can expect that the negative curvature of the diffeomorphism group causes exponential instability of geodesics (i.e., flows of the ideal fluid) in the same way as for a finite-dimensional Lie group (see, e.g., computer simulations in [KHZ]). For instance, on an $n$-dimensional compact manifold with nonpositive sectional curvatures the Jacobi equation along the fluid motion with constant pressure function always has an unbounded solution [Mis1].

We emphasize that the instability discussed here is the exponential instability (also called the *Lagrangian instability*) of the *motion of the fluid*, not of its velocity field (compare with Section II.4). The above result shows that from a Lagrangian point of view, all solutions of the Euler equation in $M \subset \mathbb{R}^3$ (with the exception of the perfect eddy) are unstable.

On the other hand, a stationary flow can be a Lyapunov stable solution of the Euler equation, while the corresponding motion of the fluid is exponentially unstable. The reason is that a small perturbation of the fluid velocity field can induce an exponential divergence of fluid particles. Then, even for a well-predicted velocity field (the case of a stable solution of the Euler equation), we cannot predict the motion of the fluid mass without a great loss of accuracy.

REMARK 4.4. The curvature formulas for the diffeomorphism groups $S\mathrm{Diff}(M)$ simplify drastically for a *locally flat* (or *Euclidean*) manifold $M$, i.e., for a Riemannian manifold allowing local charts in which the Riemannian metric becomes Euclidean. Let $p : \mathrm{Vect}(M) \to S\mathrm{Vect}(M)$ be the orthogonal projection of the space of all smooth vector fields onto their divergence-free parts, where the orthogonality is considered with respect to the $L^2$-inner product on $\mathrm{Vect}(M)$. Let $q = \mathrm{Id} - p : \mathrm{Vect}(M) \to \mathrm{Grad}(M)$ be the orthogonal projection onto the space of the gradient vector fields.

Consider some local Euclidean coordinates $\{x_1, \ldots, x_n\}$ on $M$ and assign to a pair of vector fields $u$ and $v$ their covariant derivative in $M$:

$$\nabla_u v := \sum_i \left( \sum_j u_j(x) \frac{\partial v_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

THEOREM 4.5 [Luk4]. *Let $M$ be locally Euclidean. Then the sectional curvature $C_{uv}$ for an orthonormal pair of divergence-free vector fields $u, v \in S\,\mathrm{Vect}(M)$ is*

$$C_{uv} = -\langle q(\nabla_u v), q(\nabla_u v)\rangle + \langle q(\nabla_u u), \nabla_v v\rangle.$$

COROLLARY 4.6 [Luk4]. *If the vector field $\nabla_u u$ is divergence free (i.e., $q(\nabla_u u) \equiv 0$, for instance, $u$ is a simple harmonic on $T^n$), then the curvature is nonpositive:*

$$C_{uv} = -\langle q(\nabla_u v), q(\nabla_u v)\rangle.$$

REMARK 4.7. It is natural to describe the curvature tensor for the three-dimensional torus $T^3$ in the basis $e_k$ of $S_0\mathrm{Diff}(T^3)$, where $e_k = e^{i(k,x)}, k \in \mathbb{Z}^3 \backslash \{0\}$. An arbitrary velocity field $u(x)$ is represented as $u(x) = \sum_k u_k e_k$, where the Fourier components satisfy $(k, u_k) = 0$ (divergence free) and $u_{-k} = \bar{u}_k$ (reality condition). Then, according to [NHK], one has

$$\begin{aligned}&\langle \Omega(u_k e_k, v_\ell e_\ell) w_m e_m, z_n e_n \rangle \\ &= (2\pi)^3 \left( \frac{(u_k, m)(w_m, k)}{|k+m|} \cdot \frac{(v_\ell, n)(z_n, \ell)}{|\ell+n|} - \frac{(v_\ell, m)(w_m, \ell)}{|\ell+m|} \cdot \frac{(u_k, n)(z_n, k)}{|n+k|} \right).\end{aligned}$$

All the sectional curvatures in the three-dimensional subspace of the $ABC$ flows in $S_0\mathrm{Diff}(T^3)$ (see Section II.1) are equal to one and the same negative constant, i.e., the curvatures do not depend on $A, B$, and $C$ [KNH].

Fix a divergence-free vector field $v \in S\mathrm{Vect}(T^k)$ on the $k$-dimensional flat torus $T^k$. The average of the sectional curvatures of all tangential 2-planes in $S\mathrm{Vect}(T^k)$ containing $v$ is characterized by an infinite-dimensional analogue of the normalized Ricci curvature.

DEFINITION 4.8. Let $\Delta$ be the Laplace–Beltrami operator on vector fields from $S\mathrm{Vect}(T^k)$, and $\{e_i |\, i = 1, 2, \dots\}$ be its orthonormal ordered $(-\lambda_i \le -\lambda_{i+1})$ eigenbasis $(\Delta e_i = \lambda_i e_i)$. Define the *normalized Ricci curvature* in the direction $v$ by

$$\mathrm{Ric}(v) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} C_{ve_i}.$$

The normalized Ricci curvature in a given direction on a finite-dimensional manifold is the average of the sectional curvatures of all tangential 2-planes containing the direction (Definition 1.13). It differs from the classical Ricci curvature by the factor of *(dimension of manifold)*$-1$, and it makes sense as the dimension goes to infinity.

THEOREM 4.9 [Luk2]. *For a divergence-free vector field $v \in S\mathrm{Vect}(T^k)$ on the flat torus $T^k$,*

$$\mathrm{Ric}(v) = -\frac{k+1}{\mathrm{vol}(T^k)\cdot(k-1)k(k+2)} \|\sqrt{-\Delta}\; v\|^2_{L^2(T^k)},$$

*where $\sqrt{-\Delta}$ is the "positive" square root of the minus Laplace operator on vector fields.*

**4.B. Unreliability of long-term weather predictions.** To apply the curvature calculations above, we make the following simplifying assumption: The atmosphere is a two-dimensional homogeneous incompressible fluid over the *two-torus,* and the motion of the atmosphere is approximately a "tradewind current" parallel to the equator of the torus and having a sinusoidal velocity profile.

Though the *two-sphere* is a better approximation for the earth than the two-torus, the calculations carried out for a "tradewind current" over $S^2$ in [Luk1] show the same order of magnitude for curvatures in both groups $S_0\mathrm{Diff}(T^2)$ and $S\mathrm{Diff}(S^2)$. Hence, the same conclusions on the characteristic path length and instability of flows hold in both cases.

To estimate the curvature, we consider the "tradewind current" with velocity field $\xi(x, y) = (\sin y, 0)$ on the torus $T^2 = \{(x, y) \mod 2\pi\}$. Then, Theorem 3.4

shows that the curvature of the group $S_0\mathrm{Diff}(T^2)$ in the planes containing $\xi$ (with the wave vector $k = (0,1)$) varies within the limits $-2/S < C < 0$, where $S = 4\pi^2$ is the area of the torus. However, the lower limit here is obtained by a rather crude estimate. To make a rough estimate of the characteristic path length, we take a quarter of this limit as the value of the "mean curvature" $C_0 = -1/(2S)$. There exist many two-dimensional directions with curvatures of approximately this size.

Having agreed to start with this value $C_0$ for the curvatures, we obtain the characteristic path length $s = 1/\sqrt{-C_0} = \sqrt{2S}$; see Remark 1.16. (Recall that the characteristic path length is the average path length on which a small error in the initial condition grows by the factor of $e$.) Note that along the group $S\mathrm{Diff}(T^2)$, the velocity of motion corresponding to the "tradewind current" $\xi$ is equal to $\|\xi\|_{L^2(T^2)} = \sqrt{S/2}$ (since the average square value of the sine is $1/2$). Hence, the time it takes for our flow to travel the characteristic path length is equal to 2.

Now, the fastest fluid particles go a distance of 2 after this time, i.e., $1/\pi$ part of the entire orbit around the torus. Thus, if we take our value of the mean curvature, then the error grows by $e^{\pi} \approx 20$ after the time of one orbit of the fastest particle. Taking 100 km/hour as the maximal velocity of the tradewind current, we get 400 hours for the time of orbit, i.e., less than three weeks.

Thus, if at the initial moment, the state of the weather was known with small error $\varepsilon$, then the order of magnitude of the error of prediction after $n$ months would be

$$10^{kn}\varepsilon, \quad \text{where } k = \frac{30 \cdot 24}{400} \log_{10}(e^{\pi}) \approx 2.5.$$

For example, to predict the weather two months in advance we must have five more digits of accuracy than the prediction accuracy. In practice, this implies that calculating the weather for such a period is impossible.

## §5. Exterior geometry of the group of volume-preserving diffeomorphisms

The group $S\mathrm{Diff}(M)$ of volume-preserving diffeomorphisms of a Riemannian manifold $M^n$ can be thought of as a subgroup of a larger object: the group $\mathrm{Diff}(M)$ of all diffeomorphisms of $M$ (cf. [E-M]). Just like its subgroup, the larger group is also equipped with a weak Riemannian metric (which is, however, no longer right-invariant):

$$\langle g_*\xi, g_*\eta\rangle = \int_M (\xi, \eta)_{g(x)} \mu(x), \tag{5.1}$$

where $\xi, \eta \in \operatorname{Vect}(M)$; $(\xi, \eta)_a$ is the inner product of $\xi$ and $\eta$ with respect to the metric $(\ ,\ )$ on $M$ at the point $a$; and $g \in \operatorname{Diff}(M)$.

Viewing the group of volume-preserving diffeomorphisms as a subgroup in the group of all diffeomorphisms of the manifold happens to be quite fruitful for various applications. To some extent the bigger group is "always flatter" than the subgroup. The source of many simplifications lies in the following fact.

THEOREM 5.1 [Mis1]. *The components of the curvature tensor $\Omega$ of the bigger group* $\operatorname{Diff}(M)$ *are the "mean values" of the curvature tensor components for the Riemannian manifold $M$ itself:*

$$\langle \Omega(u, v)w, z\rangle = \int_M \langle \Omega_x^M(u(x), v(x))w(x), z(x)\rangle_x \mu(x),$$

*where $\Omega_x^M$ is the curvature tensor of $M$ at $x \in M$; the volume form $\mu$ is defined by the metric, and $u, v, w, z, \in \operatorname{Vect}(M)$.*

Below we derive (following [Mis1, Shn4, Tod]) the second fundamental form of the embedding of the "curved" subgroup $S\operatorname{Diff}(M) \subset \operatorname{Diff}(M)$ into the "flatter" ambient group. Though not intrinsic in nature, it gives a nice shortcut to calculations of the curvatures.

For simplicity, let the manifold $M$ be the flat $n$-torus $T^n$. Represent a diffeomorphism $g \in \operatorname{Diff}(T^n)$ close to the identity in the form $g(x) = x + \xi(x)$.

PROPOSITION 5.2. *In the coordinates $\{\xi\}$, a $C^1$-small neighborhood of the identity* $\operatorname{Id} \in \operatorname{Diff}(T^n)$ *of the group* $\operatorname{Diff}(T^n)$ *equipped with the metric (5.1) is isometrically embedded in the Hilbert space $\mathbb{H} = \{\xi \in L^2(T^n, \mathbb{R}^n)\}$.*

PROOF OF PROPOSITION 5.2 is a straightforward calculation. □

Abusing notation, we will denote by $\mathbb{H}$ the (pre-)Hilbert space of smooth maps from the torus $T^n$ to $\mathbb{R}^n$ equipped with the $L^2$ inner product. Then a neighborhood of the identity of the group $\operatorname{Diff}(T^n)$ is isometric with a neighborhood of the origin in $\mathbb{H}$. The group $S\operatorname{Diff}(T^n)$ of volume-preserving diffeomorphisms of $T^n$ will be viewed as a submanifold $\mathcal{D}$ of $\mathbb{H}$ (Fig.50):

$$\mathcal{D} = S\operatorname{Diff}(T^n) = \{\xi \in L^2(T^n, \mathbb{R}^n) \mid \det\left[\operatorname{Id} + \frac{\partial \xi}{\partial x}\right] \equiv 1\} \subset \mathbb{H}.$$

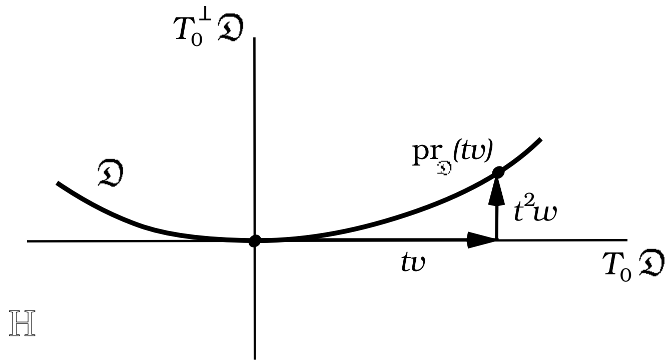



FIGURE 50. The embedding of the volume-preserving diffeomorphisms $\mathcal{D} = S\mathrm{Diff}(M)$ into the group of all diffeomorphisms $\mathbb{H} = \mathrm{Diff}(M)$.

DEFINITION 5.3. The *second fundamental form* $L$ (at $0 \in \mathcal{D}$) of the embedding $\mathcal{D} \subset \mathbb{H}$ is a quadratic map $L : T_0\mathcal{D} \to T_0^\perp\mathcal{D}$ from the tangent space $T_0\mathcal{D} \subset \mathbb{H}$ to its orthogonal complement $T_0^\perp\mathcal{D} \subset \mathbb{H}$. The value of the second fundamental form $L(v,v)$ at a vector $v \in T_0\mathcal{D}$ is equal to the acceleration of a point moving by inertia along $\mathcal{D}$ with initial velocity $v$ (see [K-N]).

In other words, $L$ measures (the second derivative of) the "distance" in $\mathbb{H}$ between the point $tv$ moving in the tangent space $T_0\mathcal{D}$ with constant velocity $v$ and the orthogonal projection $\mathrm{pr}_{\mathcal{D}}$ of this point to $\mathcal{D}$:

$$\mathrm{pr}_{\mathcal{D}}(tv) = tv + \frac{t^2}{2}L(v,v) + O(t^3) \text{ as } t \to 0. \tag{5.2}$$

The spaces $T_0\mathcal{D}$ and $T_0^\perp\mathcal{D}$ are more explicitly described as follows

$$T_0\mathcal{D} = S\mathrm{Vect}(T^n) = \{v \in \mathrm{Vect}(T^n) \mid \mathrm{div}\, v = 0\},$$
$$T_0^\perp\mathcal{D} = \mathrm{Grad}(T^n) = \{w \in \mathrm{Vect}(T^n) \mid w = \nabla p, \quad \text{for some} \;\; p \in C^\infty(T^n)\}$$

(we have included in $T_0\mathcal{D}$ the divergence-free fields shifting the center of mass of $T^n$, and hence $T_0^\perp\mathcal{D}$ consists of the gradients of all univalued functions).

Observe that for a vector field $v \in S\mathrm{Vect}(T^n)$ the transformation $x \mapsto x + tv(x)$ means that every point $x \in T^n$ moves uniformly with velocity $v(x)$ along the straight line passing through $x$. Such transformations are diffeomorphisms for smooth $v(x)$ and sufficiently small $t > 0$.

To demonstrate the machinery, we confine ourselves, for now, to the case $n = 2$ and give an alternative proof of Theorem 3.4 on curvatures $S\mathrm{Diff}_0(T^2)$ in two-dimensional directions containing the sinusoidal flow $\xi$ on the torus.

Proof of Theorem 3.4. A vector field $v \in S\mathrm{Vect}(T^2)$ can be described by the corresponding (univalued) stream (or Hamiltonian) function $\psi: \ v = \mathrm{sgrad}\,\psi$; that is, $v_1 = -\partial\psi/\partial x_2$ and $v_2 = \partial\psi/\partial x_1$.

Theorem 5.4. *The value of the second fundamental form $L$ at a vector field $v$ is*

$$L(v,v) = -2\nabla(\Delta^{-1}(\det[\mathrm{Hess}\ \psi])), \tag{5.3}$$

*where* Hess $\psi$ *is the Hessian matrix of the stream function $\psi$ of the field $v$, and $\Delta^{-1}$ is the Green operator for the Laplace operator $\Delta$ in the class of functions with zero mean on $T^2$.*

Proof of Theorem 5.4. The following evident relation shows how far the transformation $\mathrm{Id} + tv$ is from $\mathcal{D}$ (i.e., from being volume-preserving):

$$\det\left[\mathrm{Id} + t\frac{\partial v}{\partial x}\right] = 1 + t\ \mathrm{div}\ v + t^2(v_{1,1}v_{2,2} - v_{1,2}v_{2,1}) = 1 + t^2(v_{1,1}v_{2,2} - v_{1,2}v_{2,1}),$$

where $v_{i,j} := \partial v_i/\partial x_j$, and the last equality is due to $\mathrm{div}\ v = 0$. The transformation $\mathrm{Id} + tv$ does not belong to $\mathcal{D}$, and it changes the standard volume element on $T^2$ by a term quadratic in $t$.

Hence, we have to adjust $tv$ by adding to it a vector field $t^2 w \in T_0^{\perp}\mathcal{D}$ to suppress the divergence of $\mathrm{Id} + tv$. To compute the second fundamental form (see (5.2)) observe that its value at the vector $v$ is $L(v,v) = 2w$, where $w \in T_0^{\perp}\mathcal{D}$ is defined by the condition that the transform $x \mapsto x + tv + t^2 w$ is volume-preserving modulo $O(t^3)$.

The defining relation on the field $w$: $\mathrm{div}\ w = -(v_{1,1}v_{2,2} - v_{1,2}v_{2,1})$ follows immediately from the expansion

$$\det\left(\mathrm{Id} + t\frac{\partial v}{\partial x} + t^2\frac{\partial w}{\partial x}\right) = 1 + t^2(v_{1,1}v_{2,2} - v_{1,2}v_{2,1} + \ \mathrm{div}\ w) + O(t^3).$$

From the definition of $T_0^{\perp}\mathcal{D}$, the vector field $w$ is a gradient: $w = \nabla\varphi$. Therefore, $\mathrm{div}\ w = \nabla^2\varphi = \Delta\varphi$, and

$$L(v,v) = 2w = -2\nabla(\Delta^{-1}(v_{1,1}v_{2,2} - v_{1,2}v_{2,1})).$$

The introduction of the stream function $\psi$ for the field $v$ reduces the latter formula to the required form (5.3). □

The *symmetric fundamental form* $L(u,v)$ can now be obtained from the quadratic form $L(v,v)$ via polarization:

$$L(u,v) = (L(u+v, u+v) - L(u,u) - L(v,v))/2.$$

Finally, the sectional curvature $C_{uv}$ of the group $\mathcal{D}$ in the two-dimensional direction spanned by any two orthonormal vectors $u$ and $v$ is, according to the Gauss–Codazzi formula (Proposition VII.4.5 in [K-N]), given by

$$C_{uv} = \langle L(u,u), L(v,v)\rangle - \langle L(u,v), L(u,v)\rangle,$$

where $\langle\ ,\ \rangle$ is the inner product in the Hilbert space $\mathbb{H}$ (cf. also Theorem 4.5 above). For nonorthonormal vectors one has to normalize the curvature according to formula (1.5).

EXAMPLE 5.5. We will now calculate (using the second fundamental form) the sectional curvature in the direction spanned by the vector fields $u$ and $v$ with the stream functions $\phi = \cos ay$ and $\psi = \cos bx$ (where the wave vectors of Corollary 3.6 are $k = (0, a)$ and $\ell = (b, 0)$).

One easily obtains that $\det[\text{Hess }\phi] = \det[\text{Hess }\psi] \equiv 0$, while $\det[\text{Hess }(\phi+\psi)] = a^2b^2 \cos ay \cos bx$. Then the application of $\Delta^{-1}$ is equivalent to the multiplication of the above function by $-1/(a^2+b^2)$. Passing to the gradient, one sees that $L(u, v)$ is the vector field

$$L(u,v) = -\frac{a^2b^2}{a^2+b^2}\left((b \sin bx \cos ay)\frac{\partial}{\partial x} + (a \cos bx \sin ay)\frac{\partial}{\partial y}\right),$$

while both $L(u, u)$ and $L(v, v)$ vanish. Note also that $\langle u, u\rangle = a^2S/2$, $\langle v, v\rangle = b^2S/2$, where $S$ is the area of the torus. Finally, evaluating the $L^2$-norm of the vector field $L(u, v)$ over the torus, we come to the following formula for the sectional curvature:

$$C_{uv} = -\frac{\langle L(u,v), L(u,v)\rangle}{\langle u,u\rangle \cdot \langle v,v\rangle} = -\frac{a^2b^2}{(a^2+b^2)S},$$

which is in a perfect matching with (3.3).

We leave it to the reader to check that the substitution of the vector fields $u$ and $v$ with the stream functions $\xi = \frac{e_k+e_{-k}}{2}$ and $\eta = \Sigma x_\ell e_\ell$ (with $x_{-\ell} = \bar{x}_\ell$) into the curvature formula gives an alternative proof of Theorem 3.4 in full generality. $\square$

REMARK 5.6. Bao and Ratiu [B-R] have studied the *totally geodesic* (or *asymptotic*) directions on the Riemannian submanifold $S\text{Diff}(M) \subset \text{Diff}(M)$, i.e., those directions in the tangent spaces $T_gS\text{Diff}(M)$ (alternatively, divergence-free vector fields on $M$) for which the second fundamental form $L$ of $S\text{Diff}(M)$, relative to $\text{Diff}(M)$, vanishes. For an arbitrary manifold $M^n$, they obtained an explicit description of such directions $g_*v \in T_gS\text{Diff}(M)$ in the form of a certain first-order nonlinear partial differential equation on $v$. For the two-dimensional case this equation can be rewritten as an equation for the stream function. Assume for simplicity that $\partial M = \emptyset$ and $H^1(M) = 0$.

THEOREM 5.7 [B-R]. *For a two-dimensional Riemannian manifold $M$ the divergence-free vector field $g_* v \in T_g S\,\mathrm{Diff}(M)$ is a totally geodesic direction on $S\,\mathrm{Diff}(M)$ if and only if the stream function $\psi$ of the field $v$ satisfies the degenerate Monge–Ampère equation*

$$\det[\mathrm{Hess}\ \psi] = K \cdot m \cdot \|\nabla\psi\|^2/2,$$

*where $m$ is the determinant of the metric on $M$ in given coordinates $\{x_1, x_2\}$,* Hess *is the Hessian matrix of $\psi$ in these coordinates, and $K$ is the Gaussian curvature function on $M$.*

Their paper also contains examples of manifolds for which the Monge–Ampère equation has, or has no, solutions (see also [BLR] for a characterization of all manifolds $M$ for which the asymptotic directions are harmonic vector fields).

Asymptotic directions on $S\mathrm{Diff}(M^2)$ arise intrinsically in the context of a discrete version of the Euler equation of an incompressible fluid [MVe]. A solution of the discretized Euler equation is a recursive sequence of diffeomorphisms. The Monge–Ampère equation is the constraint on the initial condition ensuring that all the diffeomorphisms of the sequence preserve the area element on $M^2$.

REMARK 5.8. Consider an equivalence relation on the diffeomorphism group $\mathrm{Diff}(M)$, where two diffeomorphisms are in the same class if they differ by a volume-preserving transformation. We obtain a fibration of $\mathrm{Diff}(M)$ over the space of densities (i.e., the space of positive functions on $M$) with the fiber isomorphic to the set of volume-preserving diffeomorphisms $S\mathrm{Diff}(M)$. Let the manifold $M^n$ be one of the following: an $n$-dimensional sphere $S^n$, Lobachevsky space $\Lambda^n$, or Euclidean space $\mathbb{R}^n$. According to S.M. Gusein-Zade, there is no section of this $S\mathrm{Diff}(M)$-bundle over the space of densities that is invariant with respect to motions of the corresponding space $M^n$. However, there exists a unique connection in this bundle that is invariant with respect to motions on $M^n$. Parallel translation in this connection is essentially described in the proof of the Moser theorem (cf. Lemma III.3.5). The case of a two-dimensional sphere has interesting applications in cartography.

## §6. Conjugate points in diffeomorphism groups

Although in "most" of the two-dimensional directions the sectional curvatures of the diffeomorphism group $S\mathrm{Diff}(T^2)$ are negative, in some directions the curvature is positive.

EXAMPLE 6.1 [Arn4]. In the algebra $S\mathrm{Vect}(T^2)$ of all divergence-free vector fields on the torus $T^2 = \{(x,y) \mod 2\pi\}$ consider the plane spanned by the two stream functions $\xi = \cos(3px-y)+\cos(3px+2y)$ and $\eta = \cos(px+y)+\cos(px-2y)$. Then for the sectional curvature one has

$$C_{\xi\eta} = \frac{\langle \Omega(\xi,\eta)\xi,\eta\rangle}{\langle \xi,\xi\rangle\langle \eta,\eta\rangle} \to \frac{9}{8\pi^2} > 0 \quad \text{as} \quad p \to \infty.$$

It is tempting to conjecture, by analogy with the finite-dimensional case, that positivity of curvatures is related to the existence of the conjugate points on the group $S\mathrm{Diff}(T^2)$.

DEFINITION 6.2. A *conjugate point* of the initial point $\gamma(0)$ along a geodesic line $\gamma(t)$, $t \in [0,\infty)$ (on a Riemannian manifold $M$), is the point where the geodesic line hits an infinitely close geodesic, starting from the same point $\gamma(0)$. The conjugate points are ordered along a geodesic, and the first point is the place where the geodesic line ceases to be a local minimum of the length functional.

Strictly speaking, one considers zeroes of the first variation rather than the actual intersection of geodesics (see, e.g., [K-N]).

THEOREM 6.3 [Mis2]. *Conjugate points exist on the geodesic in the group $S\,\mathrm{Diff}(T^2)$ emanating from the identity with velocity $v = \mathrm{sgrad}\,\phi$ defined by the stream function $\phi = \cos 6x \cdot \cos 2y$.*

A segment of a geodesic line is no longer the shortest curve connecting its ends if the segment contains an interior point conjugate to the initial point (Fig.51). Indeed, the difference of lengths of a geodesic curve segment and of *any* $C^1$ $\epsilon$-close *curve* joining the same endpoints is of order $\epsilon^2$ (the geodesic being an extremal). The length of a *geodesic* $\epsilon$-close to the initial one and connecting the initial point $A$ with its conjugate point $C$ differs from the length of the initial geodesic between $A$ and $C$ by a quantity of higher order, $\epsilon^3$. The difference between $BD$ and $BC+CD$ is of order $\epsilon^2$. Hence $ADB$ is shorter than $ACB$.

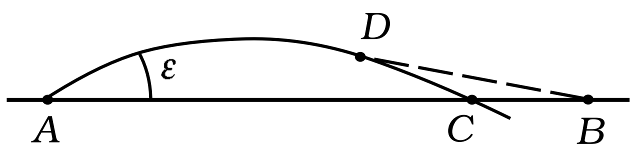



FIGURE 51. A geodesic ceases to be the length global minimum after the first conjugate point.

If a segment of a geodesic contains $k$ interior points conjugate to the initial point, then the quadratic form of the length second variation has $k$ negative squares.

REMARK 6.4 [Mis2]. Conjugate points can be found on the group $S\mathrm{Diff}(T^n)$, where $T^n$ is a flat torus of arbitrary dimension $n$ (this is a simple corollary of the two-dimensional example above).

More examples are provided by geodesics on the group of area-preserving diffeomorphisms of a surface of positive curvature (example: uniform rotation of the standard two-dimensional sphere; see Section 4.A).

On the other hand, those geodesics in $S\mathrm{Diff}(M)$ that are also geodesics in $\mathrm{Diff}(M)$ have no conjugate points whenever $M$ is a Riemannian manifold of nonpositive sectional curvature [Mis1]. Such geodesics have asymptotic directions on $S\mathrm{Diff}(M)$ and correspond to the solutions of the Euler equation in $M$ with constant pressure functions.

REMARK 6.5. It has been neither proved nor disproved that the Morse index of a geodesic line corresponding to a smooth stationary flow is finite (for any finite portion of the geodesic). It is interesting to consider whether the conjugate points might accumulate in this situation.

Note that the existence of a small geodesic segment near the initial point that is free of conjugate points follows from the nondegeneracy of the geodesic exponential map at the initial point and in its neighborhood. One might also ask what the shortest path is to a point on the geodesic that is separated from the initial point by a point conjugate to it. (One hopes that the overall picture in this classical situation is not spoiled by the pathology of the absence of the shortest path between special diffeomorphisms, discovered by Shnirelman in [Shn1]; see Section 7.D.)

## §7. Getting around the finiteness of the diameter of the group of volume-preserving diffeomorphisms[1]

Consider a volume-preserving diffeomorphism of a bounded domain. In order to reach the position prescribed by the diffeomorphism, every fluid particle has to move along some path in the domain. The distance of this diffeomorphism from the identity is the averaged characteristic of the path lengths of the particles.

It turns out that the diameter of the group of volume-preserving diffeomorphisms of a three-dimensional ball is finite, while for a two-dimensional domain it is infinite. This difference is due to the fact that in three (and more) dimensions there is enough

[1] This section was written by A. Shnirelman.

space for particles to move to their final places without hitting each other. On the other hand, the motion of the particles in the plane might necessitate their rotation about one another. The latter phenomenon of "braiding" makes the system of paths of particles necessarily long, in spite of the boundedness of the domain (and hence, of the distances between the initial and final positions of each particle).

In this section we describe some principal properties of the group of volume-preserving diffeomorphisms $\mathcal{D}(M) := S\mathrm{Diff}(M)$ as a metric space along with their dynamical implications.

**7.A. Interplay between the internal and external geometry of the diffeomorphism group.**

DEFINITION 7.1. Let $M^n$ be a Riemannian manifold with volume element $dx$. Introduce a *metric* on the group $\mathcal{D}(M) = S\mathrm{Diff}(M^n)$ as follows. To any path $g_t$, $t_1 \le t \le t_2$, on the group $\mathcal{D}(M)$ we associate its *length:*

$$\ell\{g_t\}_{t_1}^{t_2} = \int_{t_1}^{t_2} \|\dot g_t\|_{L^2(M)} dt = \int_{t_1}^{t_2} \left(\int_{M^n} \left\| \frac{\partial g_t(x)}{\partial t} \right\|^2 dx\right)^{1/2} dt. \tag{7.1}$$

For two fluid configurations $f, h \in \mathcal{D}(M)$, we define the *distance* between them on $\mathcal{D}(M)$ as the infimum of the lengths of all paths connecting $f$ and $h$:

$$\mathrm{dist}_{\mathcal{D}(M)}(f,h) = \inf_{\substack{\{g_t\}\subset \mathcal{D}(M)\\ g_0=f,\, g_1=h}} \ell\{g_t\}_0^1.$$

This definition makes $\mathcal{D}(M)$ into a *metric space*. Now the *diameter* of $\mathcal{D}(M)$ is the supremum of distances between its elements:

$$\mathrm{diam}\ (\ \mathcal{D}(M)\ ) = \sup_{f,h\in\mathcal{D}(M)} \mathrm{dist}_{\mathcal{D}(M)}(f,h).$$

The metric on the group of volume-preserving diffeomorphisms $\mathcal{D}(M)$ defined here is induced by the right-invariant metric on the group defined at the identity by the kinetic energy of vector fields (compare formula (7.1) with Example I.1.3).

We start with the study of the following three intimately related problems:

A. (DIAMETER PROBLEM) Is the diameter of the group $\mathcal{D}(M)$ of volume-preserving diffeomorphisms infinite or finite? In the latter case, how can it be estimated for a given manifold $M^n$?

Let $M$ be a bounded domain in the Euclidean space $\mathbb{R}^n$ (with the Euclidean volume element $dx$). In this case the diffeomorphism group $\mathcal{D}(M)$ is naturally embedded into the Hilbert space $L^2(M^n, \mathbb{R}^n)$ of vector functions on $M^n$. This embedding defines an isometry of $\mathcal{D}(M)$ with its (weak) Riemannian structure onto its image equipped with the Riemannian metric induced from the Hilbert space.

DEFINITION 7.2. The *standard* distance $\mathrm{dist}_{L^2}$ between two diffeomorphisms $f, h \in \mathcal{D}(M) \subset L^2(M, \mathbb{R}^n)$ is the distance between them in the ambient Hilbert space $L^2(M, \mathbb{R}^n)$:

$$\mathrm{dist}_{L^2}(f, h) = \|f - h\|_{L^2(M,\mathbb{R}^n)}.$$

B. (RELATION OF METRICS) What is the relation between the distance $\mathrm{dist}_{\mathcal{D}(M)}$ in the group $\mathcal{D}(M)$ defined above and the standard distance $\mathrm{dist}_{L^2}$ pulled back to $\mathcal{D}(M)$ directly from the space $L^2(M, \mathbb{R}^n)$?

Evidently, $\mathrm{dist}_{\mathcal{D}(M)} \geq \mathrm{dist}_{L^2}$. But does there exist an estimate of $\mathrm{dist}_{\mathcal{D}(M)}(g, h)$ through $\mathrm{dist}_{L^2}(g, h)$? In particular, is it true that if two volume-preserving diffeomorphisms are close in the Hilbert space, then they can be joined by a short path within the group $\mathcal{D}(M)$?

C. (SHORTEST PATH) Given two volume-preserving diffeomorphisms, does there exist a path connecting them in the group $\mathcal{D}(M)$ that has minimal length? If so, it is a geodesic; i.e., after an appropriate reparametrization it becomes a solution of the Euler equation for an ideal fluid in $M^n$. Finding the shortest path between two arbitrary fluid configurations promises to be a good method for constructing fluid flows.

REMARK 7.3. Similar problems can be posed for diffeomorphisms of an arbitrary Riemannian manifold $M^n$, if we first isometrically embed $M^n$ in $\mathbb{R}^q$ for some $q$. Such an embedding for which the Euclidean metric in $\mathbb{R}^q$ descends to the given Riemannian metric on $M^n$ exists by virtue of the Nash theorem [Nash] (where one can take $q = 3n(n + 9)/2$ for a compact $M^n$, and $q = 3n(n + 1)(n + 9)/2$ for a noncompact $M^n$).

**7.B. Diameter of the diffeomorphism groups.** In what follows we confine ourselves to the simplest domain $M^n$, namely, to a unit cube: $M^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 < x_i < 1\}$. We thus avoid the topological complications due to the topology of $M$.

THEOREM 7.4 [Shn1]. *For a unit $n$-dimensional cube $M^n$ where $n \geq 3$, the diameter of the group of smooth volume-preserving diffeomorphisms $\mathcal{D}(M)$ is finite in the right-invariant metric $\mathrm{dist}_{\mathcal{D}(M)}$.*

THEOREM 7.4′ [Shn5]. $\quad \mathrm{diam}\ (\ \mathcal{D}(M^n)\ ) \leq 2\sqrt{\frac{n}{3}}.$

These theorems generalize to the case of an arbitrary simply connected manifold $M$. However, the diameter can become infinite if the fundamental group of $M$ is not trivial [ER2]. The two-dimensional case is completely different:

THEOREM 7.5 [ER1,2]. *For an arbitrary manifold $M$ of dimension $n = 2$, the diameter of the group $\mathcal{D}(M)$ is infinite.*

One can strengthen the latter result in the following direction.

DEFINITION 7.6. A diffeomorphism $g : M \to M$ of an arbitrary domain (or a Riemannian manifold) $M$ is called *attainable* if it can be connected with the identity diffeomorphism Id by a piecewise-smooth path $g_t \subset \mathcal{D}(M)$ of finite length.

THEOREM 7.7 [Shn2]. *Let $M^n$ be an $n$-dimensional cube and $n \geq 3$. Then every element of the group $\mathcal{D}(M)$ is attainable. In the case $n = 2$, there are unattainable diffeomorphisms of the square $M^2$. Moreover, the unattainable diffeomorphisms can be chosen to be continuous up to the boundary $\partial M^2$ and identical on $\partial M^2$.*

A diffeomorphism $g$ may be unattainable if its behavior near the boundary $\partial M^n$ of $M$ is complicated enough. We will give an example of an unattainable diffeomorphism in Section 8.A. Note that only attainable diffeomorphisms are physically reasonable, since the fluid cannot reach an unattainable configuration in a finite time.

The statement above allows one to specify Theorem 7.5:

THEOREM 7.8 (=7.5′). *For two-dimensional $M$, the subset of attainable diffeomorphisms in $\mathcal{D}(M)$ is of infinite diameter.*

It is not known whether all the attainable diffeomorphisms form a subgroup in $\mathcal{D}(M^2)$ (i.e., whether the inverse of an attainable diffeomorphism is attainable). It is not always true that if a path $\{g_t\} \subset \mathcal{D}(M^2)$ has finite length, then the length of the path $\{g_t^{-1}\} \subset \mathcal{D}(M^2)$ is finite. The group $\mathcal{D}(M^2)$ splits into a continuum of equivalence classes according to the following relation: Two diffeomorphisms are in the same class if they can be connected to each other by a path of finite length. Every equivalence class has infinite diameter.

The proofs of the two-dimensional results are rather transparent and are discussed in Sections 8.A-B. Various approaches to the three- (and higher-) dimensional case are, on the contrary, all quite intricate, and only the ideas are discussed below.

**7.C. Comparison of the metrics and completion of the group of diffeomorphisms.** The main difference between the geometries of the groups of diffeomorphisms in two and three dimensions is based on the observation that for a long path on $\mathcal{D}(M^3)$, which twists the particles in space, there always exists a "shortcut" untwisting them by "making use of the third coordinate." More precisely, the following estimate holds.

THEOREM 7.9 [Shn1]. *Given dimension $n \geq 3$, there exist constants $C > 0$ and $\alpha > 0$ such that for every pair of volume-preserving diffeomorphisms $f, h \in \mathcal{D}(M^n)$ of the unit cube,*

$$\operatorname{dist}_{\mathcal{D}(M)}(f, h) \leq C(\operatorname{dist}_{L^2}(f, h))^{\alpha}.$$

THEOREM 7.9′ [Shn5]. *The exponent $\alpha$ in this inequality is not less than $2/(n+4)$.*

This property means that the embedding of $\mathcal{D}(M^n)$ into $L^2(M, \mathbb{R}^n)$, $n \geq 3$, is "Hölder-regular." Apparently, it is far from being smooth, i.e., $\alpha < 1$. Certainly, the Hölder property (Theorem 7.9) implies the finiteness of the diameter (Theorem 7.4).

No such estimate is true for $n = 2$. Namely, for every pair of positive constants $c, C$ there exists a diffeomorphism $g \in \mathcal{D}(M^2)$ such that $\operatorname{dist}_{\mathcal{D}(M)}(g, \mathrm{Id}) > C$, but $\operatorname{dist}_{L^2}(g, \mathrm{Id}) < c$. This complements Theorem 7.5 but requires, of course, a separate proof.

Theorems 7.4 and 7.9 imply the following simple description of the completion of the metric space $\mathcal{D}(M^n)$ in the case $n \geq 3$.

COROLLARY 7.10. *For $n \geq 3$ the completion of the group $\mathcal{D}(M^n)$ in the metric $\operatorname{dist}_{\mathcal{D}(M)}$ coincides with the closure $\bar{\mathcal{D}}(M^n)$ of the group in $L^2(M^n, \mathbb{R}^n)$.*

PROOF OF COROLLARY. Each Cauchy sequence $\{g_i\}$ in $\mathcal{D}(M^n)$ (with respect to the metrics $\operatorname{dist}_{\mathcal{D}(M^n)}$) is a Cauchy sequence in $L^2(M^n, \mathbb{R}^n)$, and therefore it converges to some element $g \in L^2(M^n, \mathbb{R}^n)$.

Conversely, if $g$ belongs to the closure of $\mathcal{D}(M^n)$ in $L^2(M^n, \mathbb{R}^n)$, then there exists a sequence of diffeomorphisms $\{g_i\} \subset \mathcal{D}(M^n)$ that converges to it in $L^2(M^n, \mathbb{R}^n)$. Therefore, by virtue of Theorem 7.9, $\{g_i\}$ is a Cauchy sequence in $\mathcal{D}(M^n)$, and thus $g$ lies in the completion of $\mathcal{D}(M^n)$. □

THEOREM 7.11 [Shn1]. *The completion $\bar{\mathcal{D}}(M^n)$ $(n \geq 3)$ of the group $\mathcal{D}(M^n)$ consists of all measure-preserving endomorphisms on $M$, i.e., of such Lebesgue measurable maps $f : M^n \to M^n$ that for every measurable subset $\Omega \subset M^n$,*

$$\operatorname{mes}\ f^{-1}(\Omega) = \operatorname{mes}\ \Omega.$$

The idea for a proof of this theorem is as follows. Divide the unit cube $M^n$ into $N^n$ equal small cubes having linear size $N^{-1}$. Consider the class $\mathcal{D}_N$ of piecewise-continuous mappings, translating in a parallel way each small cube $\varkappa$ into another small cube $\sigma(\varkappa)$, where $\sigma$ is some permutation of the set of small cubes. First one proves that every measure-preserving map $f : M \to M$ may be approximated with arbitrary accuracy in $L^2(M^n, \mathbb{R}^n)$ by a permutation of small cubes for sufficiently large $N$. In turn, every such permutation may be approximated in $L^2$ by a smooth volume-preserving diffeomorphism. Conversely, if a map $g$ belongs to the closure of the diffeomorphism group $g \in \bar{D}(M^n)$, then $g = \lim g_i$, $g_i \in \mathcal{D}(M^n)$, almost everywhere in $M^n$, and hence $g$ is a measure-preserving endomorphism; see [Shn1] for more detail.

In the two-dimensional case, the completion of $\mathcal{D}(M^2)$ in the metric $\operatorname{dist}_{\mathcal{D}(M)}$ is a proper subset of the $L^2$-closure $\bar{D}(M^2)$; no good description of this completion is known.

**7.D. The absence of the shortest path.** We will see below how the facts presented so far in this section imply a negative answer to the question of existence of the shortest path in the diffeomorphism group:

THEOREM 7.12 [Shn1]. *For a unit cube $M^n$ of dimension $n \geq 3$, there exist a pair of volume-preserving diffeomorphisms that cannot be connected within the group $\mathcal{D}(M)$ by a shortest path, i.e., for every path connecting the diffeomorphisms there always exists a shorter path.*

Thus, the attractive variational approach to constructing solutions of the Euler equations is not directly available in the hydrodynamical situation. This is not to say that the variational approach is wrong, but merely that our understanding is still incomplete and further work is required.

REMARK 7.13. The proof of Theorem 7.12 is close in spirit to the Weierstrass example of a variational problem having no solution. Weierstrass proposed his example in his criticism of the use of the Dirichlet principle for proving the existence of a solution of the Dirichlet problem for the Laplace equation.

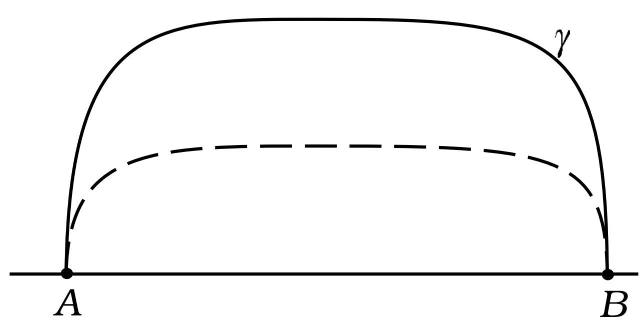



FIGURE 52. There is no smooth shortest curve between $A$ and $B$ that would be orthogonal to the segment $AB$ at the endpoints.

This example illustrates that in some cases a functional cannot attain its infimum. Consider two points $A$ and $B$ in the plane. We are looking for a smooth curve $\gamma$ of minimal length connecting $A$ and $B$ such that its tangents at the points $A$ and $B$ are orthogonal to the line $(AB)$, Fig.52. It is clear that if $\gamma$ is different from the segment $[AB]$, then it may be squeezed toward the line $(AB)$ (say, by the factor $1/2$), and this transformation reduces its length. However, if $\gamma$ coincides with the segment $[AB]$, it does not satisfy the boundary conditions. Thus, the infimum cannot be attained within the class of admissible curves.

Weierstrass's criticism encouraged Hilbert to establish a solid foundation for the Dirichlet variational principle.

We proceed to describe an example of a diffeomorphism $g \in \mathcal{D}(M^3)$ of the three-dimensional cube $M^3$ that cannot be connected to the identity diffeomorphism Id by a shortest path.

Let $(x_1, x_2, z)$ be Cartesian coordinates in $\mathbb{R}^3$, and let $M^3 = \{0 < x_1, x_2, z < 1\}$. Consider an arbitrary diffeomorphism $g \in \mathcal{D}(M^3)$ of the form

$$g(x, z) = (h(x), z),$$

where $h$ is an area-preserving diffeomorphism of the square $M^2$ and $x := (x_1, x_2)$.

THEOREM 7.12$'$. *If*

$$\operatorname{dist}_{\mathcal{D}(M^3)}(\mathrm{Id}, g) < \operatorname{dist}_{\mathcal{D}(M^2)}(\mathrm{Id}, h),$$

*then* Id $\in \mathcal{D}(M^3)$ *cannot be connected with the diffeomorphism $g$ by a shortest path in* $\mathcal{D}(M^3)$.

PROOF. Rather than the length, we shall estimate an equivalent quantity, the action along the paths.

DEFINITION 7.14. The *action* along a path $g_t$, $t_1 \le t \le t_2$, on the group of diffeomorphisms $\mathcal{D}(M^n)$ of a Riemannian manifold $M^n$ is the quantity

$$\jmath\{g_t\}_{t_1}^{t_2} = \frac{1}{2}\int_{t_1}^{t_2} \|\dot{g}_t\|^2_{L^2(M)}\, dt = \frac{1}{2}\int_{t_1}^{t_2}\int_{M^n} \Big\|\frac{\partial g_t(x)}{\partial t}\Big\|^2 dy dt.$$

The action and the length are related via the inequality

$$\ell^2 \le 2\jmath\,(t_2 - t_1). \tag{7.2}$$

Unlike the length $\ell\{g_t\}_{t_1}^{t_2}$, the action $\jmath\{g_t\}_{t_1}^{t_2}$ depends on the parametrization, and the equality in (7.2) holds if and only if the parametrization is such that

$$\|\dot{g}_t\|^2_{L^2(M)} = \int_M \Big\|\frac{\partial g_t(x)}{\partial t}\Big\|^2 dx \equiv \text{ const}.$$

This allows us, in the sequel, to pass freely from one notion to the other.

Suppose there exists a shortest path $g_t$ connecting Id and $g$; we shall construct another path that has smaller length.

First squeeze the flow $g_t$ by a factor of 2 along the $z$-direction: Instead of a family of volume-preserving diffeomorphisms of the three-dimensional unit cube $M^3$, we now have volume-preserving diffeomorphisms of the parallelepiped

$$P_1 = \{(x, z) \in M^3 \mid 0 < z < 1/2\}.$$

Now consider the new (discontinuous) flow $\bar{g}_t$ in the cube $M^3 = P_1 \cup P_2$ that is the above squeezed flow on each of the halves $P_1$ and $P_2$; see Fig.53. (Here $P_2$ is specified by the condition $1/2 < z < 1$.) It is easy to see that the flow $\bar{g}_t$ is incompressible in $M^3$ and is, in general, discontinuous on the (invariant) plane $z = 1/2$. Notice also that the flow $\bar{g}_t$ satisfies the same boundary conditions as $g_t$ : $\bar{g}_0 = \mathrm{Id}$, $\bar{g}_1 = g$.

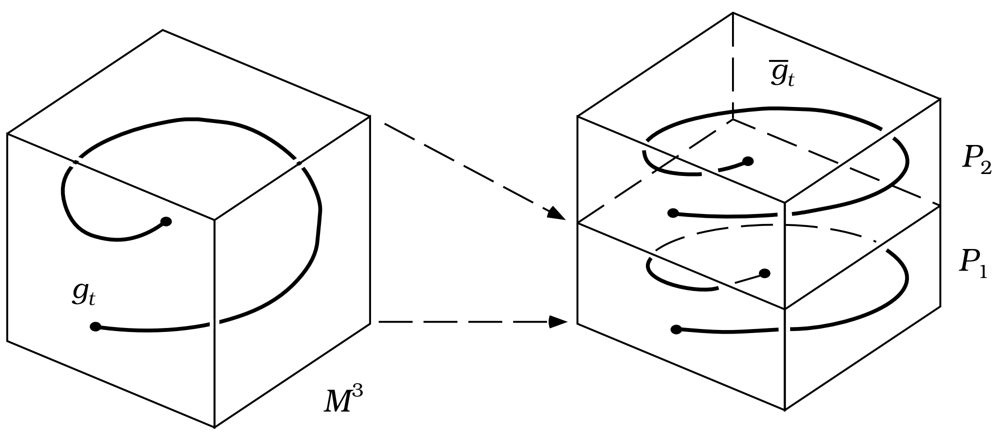



FIGURE 53. Each of the two parallelepipeds contains the former flow in the cube squeezed by a factor of 2.

Compare the actions along the paths $g_t$ and $\bar{g}_t$. Define the horizontal $\mathfrak{j}_H$ and vertical $\mathfrak{j}_V$ components of the action $\mathfrak{j} = \mathfrak{j}_H + \mathfrak{j}_V$ for $g_t$ as follows:

$$\mathfrak{j}_H\{g_t\}_0^1 = \frac{1}{2}\int_0^1 dt \int_{M^3} \|\frac{\partial x'(x,z,t)}{\partial t}\|^2\ d^2x\,dz,$$
$$\mathfrak{j}_V\{g_t\}_0^1 = \frac{1}{2}\int_0^1 dt \int_{M^3} \|\frac{\partial z'(x,z,t)}{\partial t}\|^2\ d^2x\,dz,$$

and similarly for the path $\bar{g}_t$. (Here $g_t(x, z) = (x', z')$.) From the definition of $\bar{g}_t$, we easily obtain

$$\mathfrak{j}_H\{\bar{g}_t\}_0^1 = \mathfrak{j}_H\{g_t\}_0^1, \qquad \text{while} \qquad \mathfrak{j}_V\{\bar{g}_t\}_0^1 = \frac{1}{2}\mathfrak{j}_V\{g_t\}_0^1.$$

Therefore, the action along the path $\bar{g}_t$ is smaller than that along the path $g_t$ : $\mathrm{j}\{\bar{g}_t\}_0^1 < \mathrm{j}\{g_t\}_0^1$ if the vertical component of the action is positive, $\mathrm{j}_V\{g_t\}_0^1 > 0$.

The last condition, $\mathrm{j}_V\{g_t\}_0^1 > 0$, follows from the assumption of the theorem. Indeed, if the vertical component of the action vanishes ($\mathrm{j}_V\{g_t\}_0^1 = 0$), then $\partial z'(x,z,t)/\partial t \equiv 0$, and the map $x \mapsto x'(x,z,t)$ for any fixed $z$ and $t$ is an area-preserving diffeomorphism of the square $M^2$. In this case the action along the flow $g_t$ is

$$\mathrm{j}\{g_t\}_0^1 = \int_0^1 dz \left( \int_0^1 dt \int_{M^2} \frac{1}{2} \left\| \frac{\partial x'(x,z,t)}{\partial t} \right\|^2 d^2x \right),$$

which implies that there is a value $z_0 \in [0,1]$ such that

$$\mathrm{j}\{g_t\}_0^1 \geq \int_0^1 dt \int_{M^2} \frac{1}{2} \left\| \frac{\partial x'(x,z_0,t)}{\partial t} \right\|^2 d^2x.$$

However, this is impossible, since by the assumption, the distance in $\mathcal{D}(M^2)$ from Id to the diffeomorphism $x \mapsto x'(x,z_0,t) \mid_{t=1} = h(x)$ is greater than the distance in $\mathcal{D}(M^3)$ from Id to $g$, the length of the shortest path $g_t$ in $\mathcal{D}(M^3)$ connecting Id and $g$.

Hence $\mathrm{j}_V\{g_t\}_0^1 > 0$, and we have constructed a discontinuous flow $\bar{g}_t$ (connecting Id and the diffeomorphism $g$) whose action is less than that of $g_t$: $\mathrm{j}\{\bar{g}_t\}_0^1 < \mathrm{j}\{g_t\}_0^1$. Now Theorem 7.12$'$ follows from the following lemma on a smooth approximation.

LEMMA 7.15. *For every $\varepsilon > 0$ there exists a smooth flow $\varphi_t \subset \mathcal{D}(M^3)$, $0 \leq t \leq 1$, that starts at the identity $\varphi_0 = \mathrm{Id}$, reaches an $\varepsilon$-vicinity of $g$ in the standard $L^2$-metric, $dist_{L^2(M^3)}(\varphi_1, g) < \varepsilon$, and whose action approximates the action along the discontinuous path $\bar{g}_t$ : $|\mathrm{j}\{\bar{g}_t\}_0^1 - \mathrm{j}\{\varphi_t\}_0^1| < \varepsilon$.*

To complete the proof, we replace the short but discontinuous path $\bar{g}_t$ by its smooth approximation $\varphi_t$. It starts at the identity and ends up at $\varphi_1$, $L^2$-close to $g$. By Theorem 7.9, there exists a path $f_t$ in $\mathcal{D}(M^3), 1 \leq t \leq 2$, connecting $\varphi_1$ and $g$, and such that

$$\ell\{f_t\}_0^1 \leq C\varepsilon^\alpha, \qquad \alpha > 0,$$

and hence the length $\ell\{f_t\}_0^1$ tends to 0 together with $\varepsilon$. It follows that the composite path $\varphi_t \cup f_t$ has length not exceeding $\ell\{\bar{g}_t\} + \varepsilon + C\varepsilon^\alpha$. Finally, observe that for sufficiently small $\varepsilon$, the composite path is shorter than $g_t$, because $\ell\{\bar{g}_t\}_0^1 < \ell\{g_t\}_0^1$. This completes the proof of Theorem 7.12$'$ modulo Lemma 7.15. □

PROOF OF THEOREM 7.12. By virtue of Theorem 7.5, for every $C > 0$ there is a diffeomorphism $h$ of the square $M^2$ such that $\mathrm{dist}_{\mathcal{D}(M^2)}(\mathrm{Id}, h) > C$. On the other hand, if $g$ is a diffeomorphism of the 3-dimensional cube $M^3$ having the form

$$g(x,z) = (h(x), z)\ ,\ x \in M^2, z \in (0,1),$$

then by Theorem 7.4′, $\mathrm{dist}_{\mathcal{D}(M^3)}(\mathrm{Id}, g) \le 2$. Hence, if $C > 2$, this diffeomorphism $g$ cannot be connected with Id by a shortest path. This completes the proof of Theorem 7.12. $\square$

PROOF OF LEMMA 7.15. For a (discontinuous) flow $\bar{g}_t$, we define (almost everywhere) its Eulerian velocity field by $v(x,t) = \partial \bar{g}_t(\bar{g}_t^{-1}(x))/\partial t$. For a small $\delta > 0$ let $M_\delta$ be the set $M$ with the $\delta$-neighborhood of the boundary $\partial M$ removed, and $\rho_\delta$ the dilation mappings $M \to M_\delta$. Denote by $v_\delta(x,t) = (\rho_\delta)_* v(x,t)$ the image of the field $v$ under the dilations. By setting $v_\delta \equiv 0$ outside $M_\delta$, we obtain an $L^2$-vector field in the whole of $\mathbb{R}^3$ that is incompressible in the generalized sense, i.e., the vector field $v_\delta$ is $L^2$-orthogonal to every gradient vector field.

Now define the smooth field

$$w_\delta(x,t) := \int_{\mathbb{R}^3} v_\delta(y,t)\phi_\delta(x-y)dy,$$

as a convolution of the field $v_\delta$ with a mollifier $\phi_\delta(x) = \delta^{-3}\phi(x/\delta)$, where $\phi(x) \in C_0^\infty(\mathbb{R}^3)$ and $\int \phi(x)dx = 1$. The field $w_\delta(x,t)$ has compact support $w_\delta(x,t) \in C_0^\infty(M)$ for all $t$, and as $\delta \to 0$, it converges to the field $v(x,t)$ uniformly on every compact set in $M$ outside $\partial M$ and outside the plane $z = \frac{1}{2}$. Moreover, $w_\delta \to v$ in $L^2(M)$. This implies that for sufficiently small $\delta$, the smooth flow $\varphi_t$ obtained by integrating the vector field $w_\delta$ satisfies the conditions of Lemma 7.15. $\square$

**7.E. Discrete flows.** The proofs of Theorems 7.4, 7.7, and 7.9 are similar and are based on the following discrete approximation of the group $\mathcal{D}(M^n)$ (cf. also [Lax, Mos2]). Split the cube $M^n \subset \mathbb{R}^n$ into $N^n$ identical subcubes. Let $M_N$ be the set of all these cubes. Denote by $\mathcal{D}_N$ the group of all permutations of the set $M_N$; this is a discrete analogue of the group $\mathcal{D}(M)$ of volume-preserving diffeomorphisms.

Two subcubes $\varkappa, \varkappa' \in M_N$ are called *neighboring* if they have a common $(n-1)$-dimensional face. A permutation $\sigma \in \mathcal{D}_N$ is called *elementary* if each subcube $\varkappa \in M_N$ is either not affected by $\sigma$, or $\sigma(\varkappa)$ is a neighbor of $\varkappa$. A sequence of elementary permutations $\sigma_1, \dots, \sigma_k$ is called a *discrete flow*; the number $k$ is called its *duration*. We say that the discrete flow $\sigma_1, \dots, \sigma_k$ *connects* the configurations $\sigma, \sigma' \in \mathcal{D}_N$ if $\sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_1 \circ \sigma = \sigma'$.

The following, purely combinatorial, theorem is the cornerstone of the study. It can be regarded as a discrete version of Theorem 7.4.

THEOREM 7.16. *For every dimension $n$ there exists a constant $C_n > 0$ such that for every $N$ every two configurations $\sigma, \sigma' \in \mathcal{D}_N$ can be connected by a discrete flow $\sigma_1, \dots, \sigma_k$ whose duration $k$ is less than $C_n \cdot N$.*

The proof is tedious yet elementary; see [Shn1].

To formulate the discrete analogue of Theorem 7.9, we define the length of a discrete flow (not to be confused with its duration).

DEFINITION 7.17. Let $\sigma_1, \dots, \sigma_k$ be a discrete flow in $M_N$. Let each permutation $\sigma_j$ take $m_j$ subcubes into neighboring ones and leave $N^n - m_j$ subcubes in place. The *length* $\ell\{\sigma_j\}_1^k$ of the discrete flow is (cf. (7.1))

$$\ell\{\sigma_j\}_1^k = \sum_{j=1}^{k} \frac{(m_j/N^n)^{1/2}}{N}.$$

(The reasoning is transparent: A permutation $\sigma_j$ is approximated by a vector field of magnitude $1/N$ supported on a set of volume $m_j/N^n$. Then the summands are "$L^2$-norms" of the permutations.)

The *distance* in $\mathcal{D}_N$ between configurations $\sigma, \sigma'$ is defined as

$$\operatorname{dist}_{\mathcal{D}_N}(\sigma, \sigma') = \min \ell\{\sigma_j\}_1^k,$$

where min is taken over all discrete flows (of arbitrary duration) connecting $\sigma$ and $\sigma'$. The $L^2$*-distance* between $\sigma, \sigma' \in \mathcal{D}_N$ is defined as

$$\operatorname{dist}_{L^2}(\sigma, \sigma') = \left( \sum_{\varkappa \in M_N} \frac{\|\sigma(\varkappa) - \sigma'(\varkappa)\|^2}{N^n} \right)^{\frac{1}{2}},$$

where $\|\sigma(\varkappa) - \sigma'(\varkappa)\|$ is the distance in $\mathbb{R}^3$ between the centers of the corresponding small cubes.

The following is the analogue of Theorem 7.9 on the relation of metrics.

THEOREM 7.18. *For every dimension $n$ there are constants $C_n > 0$ and $\alpha_n > 0$ such that for every $N$ and every pair of permutations $\sigma, \sigma' \in \mathcal{D}_N$,*

$$\operatorname{dist}_{\mathcal{D}_N}(\sigma, \sigma') \le C_n (\operatorname{dist}_{L^2}(\sigma, \sigma'))^{\alpha_n}.$$

The proof is an inductive multistep construction of a "short" discrete flow connecting two given $L^2$-close discrete configurations; see [Shn1]. The explicit construction in the three-dimensional case ($n = 3$) yields $\alpha_3 \ge 1/64$.

**7.F. Outline of the proofs.** The proof of Theorems 7.4 and 7.9 proceeds as follows.

Let $g \in \mathcal{D}(M^n)$ with $n \ge 3$. We construct a path $g_t \subset \mathcal{D}(M^n)$, $0 \le t \le 1$, that connects the identity and $g$ ( $g_0 = \mathrm{Id}$, $g_1 = g$), and such that

$$\ell\{g_t\}_0^1 \le C(\operatorname{dist}_{L^2}(\mathrm{Id}, g))^{\alpha}.$$

First of all, we prove that $g$ can be approximated by some permutation $\sigma \in \mathcal{D}_N$ of small cubes for sufficiently large $N$. Furthermore, for every $\varepsilon > 0$ there exists a discontinuous, piecewise-smooth flow $\xi_\tau$, $0 \le \tau \le 1$, connecting $\sigma$ and $g$, and such that $L\{\xi_\tau\}_0^1 < \varepsilon$. (More precisely, for every $\tau$ the mapping $\xi_\tau : M^n \to M^n$ is smooth in every small cube $\varkappa \in M_N$, discontinuous on interfaces between neighboring cubes $\varkappa$, and measure-preserving.)

Construct the "short" discrete flow $\sigma_1, \dots, \sigma_k$ connecting Id and $\sigma$ in $\mathcal{D}_N$ and satisfying the conclusion of Theorem 7.18. One can show that there exists a discontinuous flow $\eta_t$, $0 \le t \le 1$, that interpolates the flow $\sigma_1, \dots, \sigma_k$ at the moments $t = j/k$ for all $j = 1, \dots, k$ ( $\eta_{t=j/k} = \sigma_j \circ \sigma_{j-1} \circ \dots \circ \sigma_1$ ) and has the same order of length:

$$\ell\{\eta_t\}_0^1 \le \text{ const} \cdot \ell\{\sigma_j\}_1^k.$$

Therefore, the composition of the paths $\eta_t$ and $\xi_t$ is a discontinuous flow connecting Id and $g$ and having a controllable length.

The final step is the smoothening of the latter flow provided by the following

LEMMA 7.19. *Let $g \in \mathcal{D}(M^n)$, and let $\zeta_t : M^n \to M^n$ be a discontinuous measure-preserving flow such that $\zeta_0 = \mathrm{Id}$, $\zeta_1 = g$, and $\zeta_t$ is smooth in every small cube $\varkappa \in M_N$. If $n \ge 3$, then for every $\varepsilon > 0$ there exists a smooth flow $g_t : M^n \to M^n$, $0 \le t \le 1$, with the same boundary conditions $g_0 = \mathrm{Id}, g_1 = g$ as $\zeta_t$, and such that $\ell\{g_t\}_0^1 < \ell\{\zeta_t\}_0^1 + \varepsilon$.*

The flow $g_t$ coincides with $\zeta_t$ in every cube $\varkappa_\delta = \{x \in \varkappa \mid \mathrm{dist}_{\mathbb{R}^n}(x, \partial\varkappa) > \delta\}$ for some small $\delta$. The most subtle point in extending the flow is to define it on the $\delta$-neighborhood of all subcube faces, the "froth-like" domain $K_\delta := \bigcup_{\varkappa \in M_N} (\varkappa \setminus \varkappa_\delta)$. Here we use the fact that the fundamental group of the domain $K_\delta$ is trivial ($\pi_1(K_\delta) = 0$), which is true if $n \ge 3$. Theorem 7.9 (and Theorem 7.4 as a particular case) follows; see [Shn1] and analogous arguments in the proof of Theorem III.3.3 in Section III.3 for the details.

The proof of Theorem 7.7 is similar, but more complicated; we refer to [Shn2].

**7.G. Generalized flows.** We now return to the problem of finding the shortest paths in the group of volume-preserving diffeomorphisms $\mathcal{D}(M^n)$. We already know that there exist pairs of diffeomorphisms that cannot be connected by a smooth flow of minimal length. Is there, however, some wider class of flows (say, discontinuous or measurable) where the minimum is always attainable? This problem has been resolved by Y. Brenier [Bre1]. He found a natural class of "generalized incompressible flows" for which the variational problem is always solvable.

The generalized flows ($GF$) are a far-reaching generalization of the classical flows, where fluid particles are not only allowed to move independently of each other, but also their trajectories may meet each other: The particles may split and collide. The only restrictions are that the density of particles remains constant all the time and that the mean kinetic energy is finite. The formal definition of the GF is presented below.

Let $X = C([0,1]; M^n)$ be the space of all parametrized continuous paths $x(t)$ in $M^n$. Fix a diffeomorphism $g \in \mathcal{D}(M^n)$.

DEFINITION 7.20. *A generalized flow (GF) in $M^n$ connecting the diffeomorphisms* Id *and $g$ is a probabilistic measure $\mu\{dx\}$ in the space $X$ satisfying the following conditions:*

(i) *For every Lebesgue-measurable set $A \subset M^n$, and every $t \in [0,1]$,*

$$\mu\{x(t) \mid x(t_0) \in A\} = \text{ mes } A$$

*(this may be called incompressibility).*

(ii) *For $\mu$-almost all paths $x(t)$, the action along each of them is finite*

$$\mathrm{j}\{x(\cdot)\} = \frac{1}{2}\int_0^1 \|\frac{\partial x(t)}{\partial t}\|^2 dt < \infty,$$

*and so is the "total action"*

$$\mathrm{j}\{\mu\} = \int_X \mathrm{j}\{x(\cdot)\}\mu\{dx\} < \infty$$

*(finiteness of action).*

(iii) *For $\mu$-almost all paths $x(t)$, the endpoints $x(0)$ and $x(1)$ are related by means of the diffeomorphism $g$: $x(1) = g(x(0))$ (boundary condition).*

Thus, a generalized flow $\mu\{dx\}$ can be thought of as a random process. In general this process is neither Markov nor stationary. This notion is very similar to the notion of a polymorphism, appearing in the work of Neretin [Ner2]. Polymorphisms arise as a natural domain for the extensions of representations of diffeomorphism groups.

Every smooth flow $g_t \subset \mathcal{D}(M)$ may be regarded as a generalized flow if we associate to $g_t$ the measure $\mu_{(g_t)}\{dx\}$ such that for every measurable set $Y \subset X$ its $\mu_{(g_t)}$-measure is equal to the measure of points whose trajectories belong to $Y$:

$$\mu_{(g_t)}(Y) = \text{ mes}\{a \in M^n \big| \{g_t(a)\} \in Y\}.$$

This measure is concentrated on the $n$-dimensional set of trajectories $g_t(a)$ of the flow $g_t$.

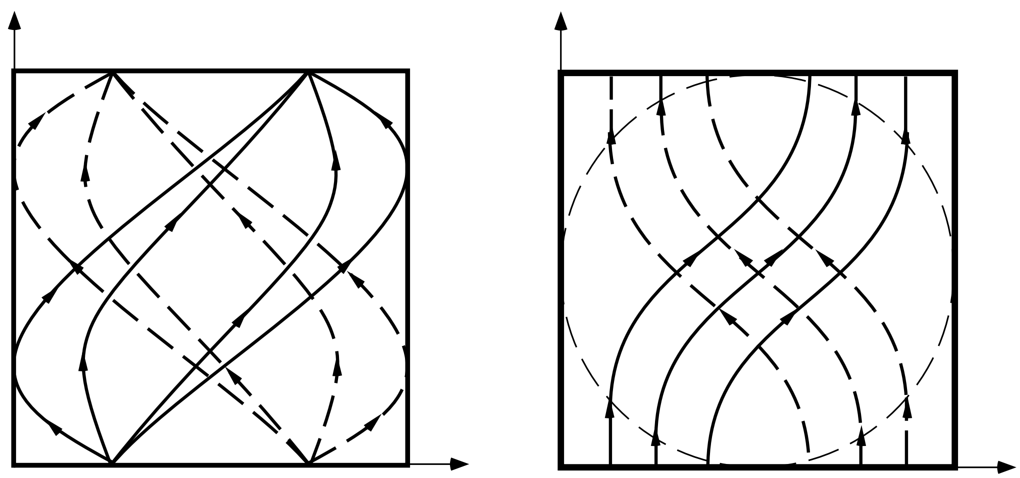
Another example of a $GF$ is a *multiflow*, that is, a convex combination of $GF$s, corresponding to smooth flows. In other words, in the multiflows different portions of fluid move (penetrating each other, see Fig.54) in different directions within the same volume! Generic $GF$s are much more complicated than multiflows.

THEOREM 7.21 [Bre1]. *For every diffeomorphism $g \in \mathcal{D}(M^n)$ there always exists a generalized flow $\mu$ (with the boundary conditions* Id *and $g$) that realizes the minimum of action,*

$$\mathrm{j}\{\mu\} = \min_{\mu'} \mathrm{j}\{\mu'\},$$

*where the minimum is taken over generalized flows $\mu'$ connecting the identity* Id *and the diffeomorphism $g$.*

Thus, although in our example the infimum cannot be assumed among smooth flows, there exists a generalized flow minimizing the action (as well as the length); see also [Roe].



(a) (b)

FIGURE 54. Trajectories of particles in the $GF$, corresponding to (a) the flip of the interval $[0, 1]$ and (b) the interval-exchange map.

In fact, Theorem 7.21 is even more general, since it applies equally to discontinuous and to orientation-changing maps $g$. In the latter case the minimizing $GF$ is especially interesting because the fluid is being turned "inside out"! No measurable flow, or even multiflow, can produce such a transformation. These problems are nontrivial even in the one-dimensional case.

Here are two beautiful examples, found by Brenier. Let $g_1(x)$ and $g_2(x)$ be the transformations of the segment $[0,1]$, defined, respectively, as a flip-flop map $g_1(x) := 1 - x,\ x \in [0,1]$, and as an interval-exchange map $g_2 :\ [0, 1/2] \leftrightarrow [1/2, 1]$.

Figures 54a and 54b display the trajectories of fluid particles for the minimal flows connecting Id with $g_1$ and $g_2$, respectively. In the first case, each fluid particle splits at $t = 0$ into a continuum of trajectories of "smaller" particles; they move independently, pass through all the points of the segment, and then coalesce at $t = 1$. In the second case, the $GF$ is a multiflow (more precisely, a 2-flow). For more examples of exotic minimal $GF$s, see [Bre1,2].

An important question is to what extent these minimal $GF$s may be regarded as generalized solutions of the Euler equation. The similarity between these generalized flows and the "true" solutions extends very far: For example, for every $GF$ $\mu$ there exists a scalar function $p(x,t)$, playing the role of pressure [Bre2], such that for almost every fluid particle its acceleration at almost every $(x,t)$ is equal to $-\nabla_x p$ ! The minimal $GF$s are generalized solutions of the mass transport problem (of the so-called Kantorovich problem). However, their hydrodynamical meaning is not yet completely understood.

**7.H. Approximation of fluid flows by generalized ones.** Generalized flows have proved to be a powerful and flexible tool for studying the structure of the space $\mathcal{D}(M^n)$ of volume-preserving diffeomorphisms. The key role here is played by the following approximation theorem.

THEOREM 7.22 [Shn5]. *Let $g \in \mathcal{D}(M^n),\ n \geq 3$, be a smooth volume-preserving diffeomorphism. Then each generalized flow $\mu\{dx\}$ connecting* Id *and $g$ can be approximated (together with the action) by smooth incompressible flows: There exists a sequence of smooth incompressible flows $g_t^{(k)}$ connecting* Id *and $g$ such that as $k \to \infty$,*

(i) *the measures $\mu_{g_t^{(k)}}$ weak*-converge in $X$ to the measure $\mu$;*

(ii) *the actions along $g_t^{(k)}$ converge to the total action along $\mu\{dx\}$:*

$$\mathfrak{j}\{g_t^{(k)}\}_0^1 \to \mathfrak{j}\{\mu\}_0^1.$$

Here weak*-convergence means that for every bounded continuous functional $\varphi\{x(\cdot)\}$ on $X$,

$$\langle \varphi\{x\},\ \mu_{g_t^{(k)}}\{dx\}\rangle \to \langle \varphi(x),\ \mu\{dx\}\rangle, \quad \text{as} \quad k \to \infty.$$

We shall not present here the (lengthy) proof of this theorem, referring to [Shn5] instead. An immediate consequence of this theorem and formula (7.2) is the following estimate on the distances in $\mathcal{D}(M^n),\ n \geq 3$.

COROLLARY 7.23. *If $n \geq 3$, then for every diffeomorphism $g \in \mathcal{D}(M^n)$,*

$$\operatorname{dist}(\mathrm{Id}, g) = \inf(2 \cdot \mathrm{j}\{\mu\}_0^1)^{1/2},$$

*where the infimum is taken over all generalized flows $\mu$ connecting* Id *and $g$.*

Thus, to estimate the distance between Id and $g \in \mathcal{D}(M^n)$, we may try to construct a $GF$ connecting Id and $g$ and having the smallest possible action. Then Lemma 7.23 guarantees a majorant for the distance.

EXAMPLE 7.24 (=THEOREM 7.4). Let us estimate the diameter diam $\mathcal{D}(M^n)$ of the group of volume-preserving diffeomorphisms of the $n$-dimensional unit cube. An accurate computation of all the intermediate constants in the proof of Theorem 7.4 for $n = 3$ yields diam $\mathcal{D}(M^3) < 100$, which is very far from reality. Here we prove

THEOREM 7.4′ [Shn5]. *If the dimension $n \geq 3$, then* diam $\mathcal{D}(M^n) \leq 2\sqrt{n/3}$.

PROOF. We use a construction close to that of Y. Brenier, who proved that all fluid configurations on the torus are attainable by $GF$s [Bre1].

The required $GF$ is constructed as follows. At $t = 0$ every fluid particle in the cube splits into a continuum of particles moving in all directions. Having originated at a point $y$, this "cloud" (of cubical form) expands, and at $t = 1/2$ it fills out the whole cube $M^n$ with a constant density. During the second half of the motion ($1/2 \leq t \leq 1$) the "cloud" shrinks and collapses at $t = 1$ to the point $g(y)$. All "clouds" expand and shrink simultaneously, and the overall density remains constant for all $t$.

More accurately, suppose that the cube $M^n$ is given by the inequalities $|x_i| < \frac{1}{2}$, $i = 1, \ldots, n$. Let $\Gamma$ be a discrete group of motions generated by the reflections in the faces of $M^n$. For each initial point $y \in M^n$ and a velocity vector $v \in M^n$, we define the corresponding billiard trajectory in $M^n$ for the time $0 \leq t \leq 1/2$, i.e., the path $x_{y,v}(t) \subset M$, where

$$x_{y,v}(t) := \Gamma(y + 4vt) \cap M^n.$$

Given a point $y$, the end-point mapping $\phi_y : v \to x_{y,v}(1/2)$ is a $2^n$-fold covering of $M^n$, and moreover, $\phi_y$ is volume-preserving. These billiard trajectories are trajectories of the "microparticles" into which every initial point $y$ splits. At $t = \frac{1}{2}$ the microparticles fill $M^n$ uniformly, and after this moment they move along other billiard trajectories, gathering at the point $g(y)$ at the end. All particles split and move independently in the same manner; incompressibility is fulfilled automatically.

Let $M_y, M_v, M_z, M_u$ be 4 copies of the cube $M^n$ with coordinates $y, v, z, u$, respectively. Define a set $\Omega \subset M_y \times M_v \times M_z \times M_u$ that consists of all four-tuples $\omega = (y, v, z, u) \in \Omega$ such that $z = g(y)$ and such that the end-points of the corresponding trajectories coincide: $x_{y,v}(1/2) = x_{z,u}(1/2)$.

Denote by $d\omega = 2^{-n} dy dv$ the normed volume element on $\Omega$. Then the required $GF$ $\mu$ is the following random process in $M^n$ with probability space $(\Omega, d\omega)$:

$$x(t, \omega) = \begin{cases} x_{y,v}(4t), & 0 \le t \le 1/2, \\ x_{g(y),u}(4 - 4t), & 1/2 \le t \le 1, \end{cases}$$

where $\omega = (y, v, g(y), u) \in \Omega$. The action of this $GF$ is

$$\mathrm{j}\{\mu\}_0^1 = \frac{1}{2} \int_{M_v} 16v^2 \; dv = \frac{n}{2} \int\limits_{-1/2}^{1/2} 16x^2 \; dx = \frac{2n}{3}.$$

By virtue of Corollary 7.23 this implies that the distance between the identity Id and the diffeomorphism $g$ (which has been chosen arbitrarily) is majorated as follows

$$\mathrm{dist}(\mathrm{Id}, g) \le \sqrt{2 \cdot \mathrm{j}\{\mu\}_0^1} = 2\sqrt{n/3}.$$

Hence, the diameter of the group $\mathcal{D}(M^n)$ has the same upper bound. $\square$

Analogous (though much longer) reasoning proves Theorem 7.9′, which minorates the Hölder exponent $\alpha_n$, $n \ge 3$, for the embedding of $\mathcal{D}(M^n)$ into $L^2(M, \mathbb{R}^n)$ : $\alpha_n \ge 2/(n+4)$; see [Shn5]. Given $g$, we construct explicitly the $GF$ satisfying that estimate. Our constructions are possibly not optimal, and a natural question is to find the best possible estimate for the diameter and for the exponent $\alpha_n$. Is the latter equal to or less than 1? Both possibilities are interesting.

**7.I. Existence of cut and conjugate points on diffeomorphism groups.** One more application of the techniques of generalized flows is the proof of the existence of cut points on the space $\mathcal{D}(M^n)$; cf. Section 6.

Definition 7.25. Let $g_t \subset \mathcal{D}(M^n)$ be a geodesic trajectory on the group of diffeomorphisms. We call a point $g_{t_c}$ on the trajectory the *first cut* of the initial point $g_0$ along $g_t$ if the geodesic $g_t$ has minimal length among all curves connecting $g_0$ and $g_\tau$ for all $\tau < t_c$, and it ceases to minimize the length as soon as $\tau > t_c$ (i.e., for every $\tau > t_c$ there exists a curve $g'_t$ connecting $g_0$ and $g_\tau$ whose length is less than the length of the segment $\{g_t |\ 0 < t < \tau\}$). We call a point $g_{t_c}$ the *first local cut* if it is the first cut, and the curve $g'_t$ may be chosen arbitrarily close to the geodesic segment $\{g_t |\ 0 < t < \tau\}$ for every $\tau > t_c$.

On a complete finite-dimensional Riemannian manifold, the cut point of a point $g_0$ comes no later than the first conjugate point of $g_0$. The example of a flat torus shows that one can have cut points but no conjugate points. But in the finite-dimensional case, the first local cut point is always a conjugate point; so, all cut points on the torus are nonlocal, which is evident. In the case of diffeomorphism groups the precise relationship between cut points and conjugate points has yet to be clarified.

In the two-dimensional case ($n = 2$) the conjugate points on certain geodesics in $\mathcal{D}(T^2)$ were found by G. Misiołek [Mis2]; see Section 6. It is curious that for $n \geq 3$ there are local cut (and, probably, conjugate?) points on *every* sufficiently long geodesic curve. This is a consequence of the following result.

THEOREM 7.26 [Shn5]. *Let $\{g_t|\ 0 \leq t \leq T\} \subset \mathcal{D}(M^n)$ be an arbitrary path on the group and $n \geq 3$. If the length of the path exceeds the diameter of the group, $\ell\{g_t\}_0^T >$ diam $\mathcal{D}(M^n)$, then there exists a path $\{g'_t|\ 0 \leq t \leq T\} \subset \mathcal{D}(M^n)$ with the same endpoints $g_0$ and $g_T$ that*

(i) *is uniformly close to $g_t$ (i.e., for every $\varepsilon > 0$ there exists a path $g'_t$ such that $\mathrm{dist}_{\mathcal{D}(M)}(g_t, g'_t) < \varepsilon$ for every $t \in [0,T]$) and*
(ii) *has a smaller length: $\ell\{g'_t\}_0^T < \ell\{g_t\}_0^T$.*

In other words, if the geodesic segment $g_t$ is long enough ($\ell\{g_t\}_0^T >$ diam $\mathcal{D}(M^n)$), then there exists a local cut point $g_{t_c}$ with $t_c < T$. A shorter path can be chosen arbitrarily close to the initial geodesic, which on a complete finite-dimensional manifold would imply the existence of a conjugate point.

PROOF. Let $h_t,\ 0 \leq t \leq T$, be a path in $\mathcal{D}(M^n)$ connecting $g_0$ and $g_T$ and such that

$$\ell\{h_t\}_0^T < \text{ diam } \mathcal{D}(M^n) + \frac{\delta}{2} < \ell\{g_t\}_0^T$$

for some small $\delta > 0$. Such a path exists by the definition of diameter.

Assume that the parametrization of the paths $g_t$ and $h_t$ is chosen in such a manner that $\|\dot{g}_t\| =$ const, $\|\dot{h}_t\| =$ const, and hence we have the inequality $\mathrm{j}\{h_t\}_0^T < \mathrm{j}\{g_t\}_0^T$ for the actions as well.

Let $\mu_{g_t}, \mu_{h_t}$ be the $GF$s corresponding to the classical flows $g_t$, $h_t$. Consider the convex combination $\bar{\mu}$ of the measures $\mu_{g_t}, \mu_{h_t}$ in the space $X$ : $\bar{\mu} := (1-\lambda)\mu_{g_t} + \lambda\mu_{h_t}$ for some $0 < \lambda < 1$. Then the total action for the generalized flow $\bar{\mu}$ is

$$\mathrm{j}\{\bar{\mu}\}_0^T = (1-\lambda)\ \mathrm{j}\{g_t\}_0^T + \lambda\ \mathrm{j}\{h_t\}_0^T < \mathrm{j}\{g_t\}_0^T.$$

To return to the classical flows we use approximation Theorem 7.22. It guarantees that there exists a smooth flow $f_t, 0 \leq t \leq T$, connecting $g_0$ and $g_T$ and weakly*-approximating the $GF$ $\mu$ together with its action, so that $\jmath\{f_t\}_0^T < \jmath\{g_t\}_0^T$. The flow $f_t$, certainly, depends on $\lambda$, and it is easy to see that for small $\lambda$ the flow $f_t$ is $L^2$-close to $g_t$:

$$\operatorname{dist}_{L^2}(f_t, g_t) < C \cdot \lambda^{1/2}.$$

Hence, for sufficiently small $\lambda$, these two flows are close on the group by virtue of Theorem 7.9: $\operatorname{dist}_{\mathcal{D}(M^n)}(f_t, g_t) < \varepsilon$ for all $0 \le t \le T$. This completes the proof of Theorem 7.26. □

For other applications of the generalized flows and for more detail we refer to [Shn5, Shn8].

## §8. Infinite diameter of the group of Hamiltonian diffeomorphisms and symplecto-hydrodynamics

The picture changes drastically when we turn from the group of volume-preserving diffeomorphisms of three- (and higher-) dimensional manifolds to area-preserving diffeomorphisms of surfaces. Practically none of the aspects under consideration in the preceding section (such as metric properties and diameter of the group, existence of solutions for the variational problem of Cauchy and Dirichlet types, or completion of the group and description of attainable diffeomorphisms) can be literally transferred to this case. It is natural to describe the properties of the groups of area-preserving diffeomorphisms of surfaces in the more general setting of diffeomorphisms of arbitrary symplectic manifolds.

DEFINITION 8.1. A *symplectic manifold* $(M, \omega)$ is an even-dimensional manifold $M^{2n}$ endowed with a nondegenerate closed differential two-form $\omega$.

A *group of symplectomorphisms* consists of all diffeomorphisms $g : M \to M$ that preserve the two-form $\omega$ (i.e., $g^*\omega = \omega$). We will be considering symplectomorphisms belonging to the identity connected component in the symplectomorphism group, and particularly those symplectomorphisms that can be obtained as the time-one map of a Hamiltonian flow. By the Hamiltonian flow we mean the flow of a time-dependent Hamiltonian vector field (having a single-valued Hamiltonian function). Such symplectic diffeomorphisms of $M$ are called *Hamiltonian*. Denote the group of Hamiltonian diffeomorphisms by $\operatorname{Ham}(M)$ and the corresponding Lie algebra of Hamiltonian vector fields by $\operatorname{ham}(M)$.

For a two-dimensional manifold the symplectic two-form $\omega$ is an area form, and the group of area- and mass center-preserving diffeomorphisms $S\mathrm{Diff}_0(M^2)$ coincides with the group $\mathrm{Ham}(M)$. The role of the group $\mathrm{Ham}(M)$ in plasma dynamics is similar to that of the group $S\mathrm{Diff}(M)$ in ideal fluid dynamics.

The study of geodesics of right-invariant metrics on symplectomorphism groups is an interesting and almost unexplored domain. It might be called symplecto-hydrodynamics, and it is a rather natural generalization of two-dimensional hydrodynamics. The relation becomes even more transparent for complex or almost complex manifolds, where the metric $\langle\ ,\ \rangle$ is related to the symplectic structure $\omega$ by means of the relation $\langle \xi, \eta\rangle = \omega(\xi, i\,\eta)$.

The symplecto-hydrodynamics in higher dimensions differs drastically from that in dimension two. For instance, every bounded domain on the plane can be embedded in any other domain of larger area by a symplectomorphism (i.e., by a diffeomorphism preserving the areas). Already in dimension four this is not always the case: Even some ellipsoids in a symplectic space cannot be embedded in a ball of larger volume by a symplectomorphism [Gro]. For example, the ellipsoid $\frac{1}{a^2}(p_1^2+q_1^2)+\frac{1}{b^2}(p_2^2+q_2^2)\le 1$ cannot be sent into a ball $p_1^2+q_1^2+p_2^2+q_2^2\le R^2$ of bigger volume if $R<\max(a,b)$. Moreover, a "symplectic camel" (a bounded domain in the symplectic four-dimensional space) cannot go through the eye of a needle (a small hole in the three-dimensional wall), while in volume-preserving hydrodynamics such a percolation through an arbitrarily small hole is always possible in any dimension.

Thus the preservation of the symplectic structure $\omega$ of the phase space $M$ by the Hamiltonian phase flow implies some peculiar restrictions on the resulting diffeomorphisms, making symplectomorphisms scarce among the volume-preserving maps in dimensions $\ge 4$. (Moreover, the group of symplectomorphisms of a symplectic manifold is $C^0$-closed in the group of all diffeomorphisms of the manifold, i.e., in general, a volume-preserving diffeomorphism cannot be approximated by symplectic ones [El1, Gro]). These restrictions might even imply some unexpected phenomena in statistical mechanics, where, in spite of the symplectic nature of the problem, one usually takes into account the first integrals and volume preservation only and freely permutes the particles of the phase space. One may also hope that symplecto- (contacto-, conformo-) hydrodynamics will find other physically interesting applications. In this section we will describe a few results known in symplecto-hydrodynamics.

We will concentrate mostly on two main metrics with which the group $\mathrm{Ham}(M)$ can be equipped. The first one is the right-invariant metric, which arises from the kinetic energy and is responsible for hydrodynamic applications (we follow [ER1,2]). The second one is the bi-invariant metric introduced in [Hof] (and studied in [E-P, LaM]), which has turned out to be a powerful tool in symplectic geometry and topology.

**8.A. Right-invariant metrics on symplectomorphisms.** Let $(M^{2n},\omega)$ be a compact *exact* symplectic manifold. This means that the symplectic form $\omega$ is a differential of a 1-form $\theta : \omega = d\theta$.

Such a manifold neccessarily has a nonempty boundary. Otherwise the integral

$$\int_M \omega^n = \int d(\theta \wedge \omega^{n-1})$$

would vanish, which is impossible since the $2n$-form $\mu = \omega^n$ is a volume form on $M$. We fix a Riemannian metric on $M$ with the same volume element.

DEFINITION 8.2. The right-invariant $L^p$-*metric* on $\mathrm{Ham}(M)$ is determined by the $L^p$-norm ($p \geq 1$) on Hamiltonian vector fields $\mathrm{ham}(M)$ at the identity of the group (for hydrodynamics, one needs the $L^2$-case corresponding to the kinetic energy of a fluid). Given a path $\{g_t |\ t \in [0,1]\} \subset \mathrm{Ham}(M)$, we define its $L^p$-*length* $\ell_p(\{g_t\})$ by the formula

$$\ell_p(\{g_t\}) = \int_0^1 \left\|\frac{dg_t}{dt}\right\|_{L^p} dt = \int_0^1 \left(\int_M \left\|\frac{dg_t}{dt}\right\|^p \mu\right)^{1/p} dt.$$

The length functional $\ell_p$ gives rise to the *distance function* $\mathrm{dist}_p$ on $\mathrm{Ham}(M)$ by

$$\mathrm{dist}_p(f,g) = \inf \ell_p(\{g_t\}),$$

where the infimum is taken over all paths $g_t$ joining $g_0 = f$ and $g_1 = g$. Finally, define the *diameter* of the group by

$$\mathrm{diam}_p(\mathrm{Ham}(M)) := \sup_{f,g \in \mathrm{Ham}(M)} \mathrm{dist}_p(f,g).$$

THEOREM 8.3(=7.5$''$) [ER2]. *The diameter* $\mathrm{diam}_p(\mathrm{Ham}(M))$ *of the group of Hamiltonian diffeomorphisms* $\mathrm{Ham}(M)$ *is infinite in any right-invariant* $L^p$*-metric.*

Note that the strongest result is that for the $L^1$-norm, since

$$\ell_p(*) \geq C(M,p) \cdot \ell_1(*).$$

REMARK 8.4. Contrary to the volume-preserving case (for dim $M \geq 3$), the infiniteness of the diameter of the symplectomorphism group has a local nature, and it is not related to the topology of the underlying manifold. The source of the

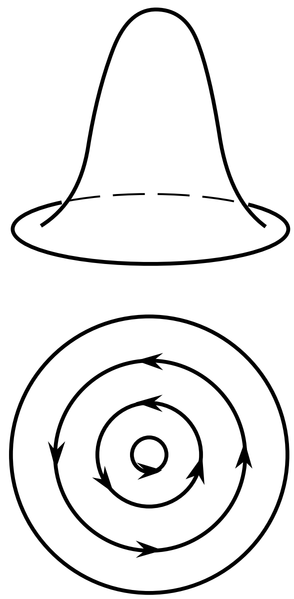


FIGURE 55. Profile of the Hamiltonian function and the trajectories of the corresponding flow, which is "a long path" on the group $\mathrm{Ham}(B^2)$.

distinction between these two cases is in the different topologies of the corresponding groups of linear transformations. The fundamental group of the group of linear symplectic transformations $\mathrm{Ham}(2n)$ is infinite, while it is finite in the volume-preserving case of $SL(2n)$ for $n > 1$.

To give an example of a "long path" in a group of Hamiltonian diffeomorphisms, we consider the unit disk $B^2 \subset \mathbb{R}^2$ with the standard volume form. Then such a path on $\mathrm{Ham}(B^2)$ is given, for instance, by the Hamiltonian flow with Hamiltonian function $H(x, y) = (x^2+y^2-1)^2$ for a long enough period of time (Fig.55). The final symplectomorphism is sufficiently far away from the identity diffeomorphism $\mathrm{Id} \in \mathrm{Ham}(B^2)$, since two-dimensionality prevents "highly twisted clusters of particles" to untwist via a short path.

This allows one to present the following example of an unattainable diffeomorphism of the square [Shn2]. It corresponds to the time-one map of the flow whose Hamiltonian function is depicted in Fig.56. It has "hills" of infinitely increasing height and with supports on a sequence of disks convergent to the boundary of the square.

We will prove Theorem 8.3 for the case of the $L^2$-norm and the group of symplectomorphisms of the ball $B^{2n}$ that are fixed on the boundary $\partial B$ [ER1]. The main ingredient of the proof is the notion of the Calabi invariant.

**8.B. Calabi invariant.** Consider the group $\mathrm{Ham}_\partial(B)$ of the Hamiltonian dif-

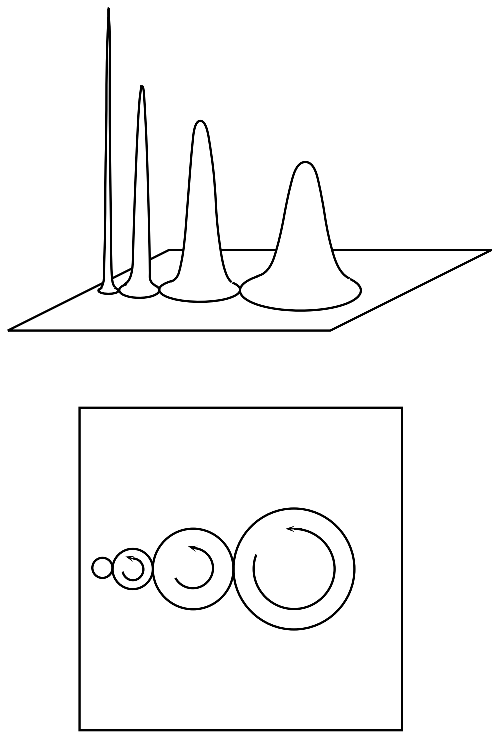


FIGURE 56. An unattainable diffeomorphism of the square.

feomorphisms of the ball $(B^{2n}, \omega)$ stationary on the sphere $\partial B$ ($\omega$ being the differential of a 1-form $\theta$, say, the standard symplectic structure $\omega = \sum dp_i \wedge dq_i$ in $\mathbb{R}^{2n}$).

PROPOSITION 8.5. *Given a 1-form $\theta$ on the ball $B$ and a Hamiltonian diffeomorphism $g \in \mathrm{Ham}_\partial(B)$ fixed on the sphere $\partial B$, there exists a unique function $h : B^{2n} \to \mathbb{R}$ vanishing together with its gradient on $\partial B$ and such that*

$$\theta - g^*\theta = dh. \tag{8.1}$$

PROOF. The 1-form $\theta - g^*\theta$ is closed ($d(\theta - g^*\theta) = d\theta - g^*d\theta = \omega - g^*\omega = 0$) and hence exact in the ball $B^{2n}$. Therefore, it is the differential of some function $h$. The vanishing property for $h$ is provided by the condition that $g$ is steady on the boundary. □

LEMMA–DEFINITION 8.6. *The integral of the function $h$ over the ball $B$ does not depend on the choice of the 1-form $\theta$ satisfying $d\theta = \omega$. The* Calabi invariant *of the Hamiltonian diffeomorphism $g$ is this integral divided by $(n+1)$:*

$$\mathrm{Cal}\,(g) := \frac{1}{n+1} \int_B h\,\omega^n.$$

PROOF. The form $\theta$ is defined modulo the differential of a function. Under the change $\theta \mapsto \tilde{\theta} = \theta + df$, the function $h$ becomes $\tilde{h} = h + (f - g^* f)$, since the differential commutes with pullbacks. The forms $(g^* f)\omega^n$ and $f\,\omega^n$ have the same integrals, since the map $g$ preserves the symplectic structure $\omega$. Then the integral of $h$ is preserved:

$$\int_B \tilde{h}\,\omega^n = \int_B h\,\omega^n + \int_B (f - g^* f)\,\omega^n = \int_B h\,\omega^n = \mathrm{Cal}\,(g).$$

$\square$

Lemma–definition 8.6 also holds for an arbitrary symplectomorphism $g$ of the ball, fixed on the boundary. However, for Hamiltonian diffeomorphisms, there is the following alternative description of the Calabi invariant.

Let $\mathrm{ham}_\partial(B)$ be the Lie algebra of the group $\mathrm{Ham}_\partial(B)$ of Hamiltonian diffeomorphisms of the ball. It consists of the Hamiltonian vector fields vanishing on the boundary sphere $\partial B$. We shall identify it with the space of Hamiltonian functions $H$, normalized by the condition that $H$ and its differential both vanish on $\partial B$. (Notice that the definition of the Hamiltonian function corresponding to a vector field from $\mathrm{ham}_\partial(B)$ is the infinitesimal version of relation (8.1).)

Let $g \in \mathrm{Ham}_\partial(B)$ be a Hamiltonian diffeomorphism of the $n$-dimensional ball $B$. Consider any path $\{g_t |\, 0 \le t \le T,\ g(0) = \mathrm{Id},\ g(T) = g\}$ on the group $\mathrm{Ham}_\partial(B)$ connecting the identity element with $g$. The path may be regarded as the flow of a time-dependent Hamiltonian vector field on $B$ whose normalized Hamiltonian function $H_t$ (defined on $B^{2n} \times [0,T]$) vanishes on $\partial B$ along with its differential.

THEOREM 8.7 [Ca]. *The integral of the Hamiltonian function $H_t$ over $B^{2n} \times [0,T]$ is equal to the Calabi invariant of the symplectomorphism $g$:*

$$\mathrm{Cal}\,(g) = \int_0^T \left( \int_B H_t\, \omega^n \right) dt. \tag{8.2}$$

*In particular, this integral does not depend on the connecting path, that is, on the choice of time-dependent Hamiltonian $H_t$, provided that the time-one map $g(T) = g$ is fixed.*

Geometrically, the Calabi invariant is the volume in the $(2n+2)$-dimensional space $\{(x,t,z)\} = B^{2n} \times [0,T] \times \mathbb{R}$ under the graph of the function $(x,t) \mapsto z = H_t(x)$.

LEMMA 8.8. *The Calabi invariant* $\mathrm{Cal}: \mathrm{Ham}_\partial(B) \to \mathbb{R}$ *is the group homomorphism of the group of Hamiltonian diffeomorphisms of* $B$ *(fixed on the boundary) onto the real line.*

PROOF OF LEMMA. Let $g_1, g_2 \in \mathrm{Ham}_\partial(B) \to \mathbb{R}$ be two Hamiltonian diffeomorphisms, and $g = g_2 \circ g_1$. We have to show that the corresponding functions $h$ and $h_i$, $i = 1, 2$, vanishing on the boundary $\partial B$ and determined by the condition (8.11), satisfy the relation

$$\int h\,\omega^n = \int h_1\,\omega^n + \int h_2\,\omega^n.$$

The latter holds, since

$$dh = \theta - g^*\theta = \theta - g_2^*\theta + g_2^*\theta - (g_2 \circ g_1)^*\theta = dh_2 + g_2^*(dh_1),$$

and because $g_2$ preserves the symplectic form $\omega$. □

REMARK 8.9. The kernel of the Calabi homomorphism (formed by the Hamiltonian diffeomorphisms whose Calabi invariant vanishes) is a simple group, the commutant of $\mathrm{Ham}_\partial(B)$; see [Ban]. (The *commutant of a group* consists of the products of commutators of the group elements.)

The fact that the Hamiltonian diffeomorphisms whose Calabi invariant vanishes form a connected normal subgroup is evident (one can multiply the time-dependent Hamiltonian by a constant). The Lie algebra of this subgroup consists of the Hamiltonian vector fields whose normalized Hamiltonian functions have zero integral. The fact that the subgroup consists of the products of commutators in $\mathrm{Ham}_\partial(B)$ is similar to the following. The Hamiltonian functions with vanishing integral are representable as finite sums of Poisson brackets of functions from $\mathrm{ham}_\partial(B)$.

The subgroup of Hamiltonian diffeomorphisms with vanishing Calabi invariant has an infinite diameter, just as the ambient group of all Hamiltonian diffeomorphisms of the ball [ER2].

More generally, the Calabi invariant is the homomorphism of the group $\mathrm{Symp}_\partial(B)$ of all symplectomorphisms of the ball (fixed on the boundary) to $\mathbb{R}$. We do not know whether the group of symplectomorphisms $\mathrm{Symp}_\partial(B)$ of a $2n$-dimensional ball (fixed on the boundary) and this normal subgroup $\{\mathrm{Cal}(g) = 0\}$ are simply connected. The group $\mathrm{Symp}_\partial(B)$ is known to be contractible for $n = 1$ [Mos1] and for $n = 2$ [Gro].

PROOF OF THEOREM 8.7. Let $\{g_t \in \mathrm{Ham}_\partial(B) \mid 0 \le t \le T,\ g(0) = \mathrm{Id},\ g(T) = g\}$ be a path of Hamiltonian diffeomorphisms with the time-dependent Hamiltonian function $H_t$.

Owing to the homomorphism property of Cal , it is enough to prove the relation

$$\int_B h\,\omega^n = (n+1)\int_0^T \left(\int_B H_t\ \omega^n\right) dt$$

for an "infinitesimally short" period of time $[0,T]$. In other words, we differentiate this relation in $t$ at $t=0$, and will prove the identity

$$\int_B \left(\frac{d}{dt}h\right)\ \omega^n = (n+1)\int_B H_0\ \omega^n.$$

Note that the time derivative at $t=0$ of the left-hand side of formula (8.1) for the diffeomorphism $g_t$ is, by definition, minus the Lie derivative of the 1-form $\theta$ along the Hamiltonian vector field $v = \frac{d}{dt}|_{t=0}g_t$ generated by the function $H = H_0$:

$$-L_v\theta = d\left(\frac{d}{dt}h\right). \tag{8.3}$$

We apply the homotopy formula $L_v = i_v d + d i_v$ (see Section I.7.B) to the Lie derivative $L_v\theta$ and use the definition of the Hamiltonian function $-dH = i_v\omega$, where $\omega = d\theta$:

$$-L_v\theta = -i_v d\theta - d i_v\theta = d(H - i_v\theta). \tag{8.4}$$

From formulas (8.3-4) one finds the derivative $d/dt\,h$:

$$\frac{d}{dt}h = H - i_v\theta.$$

(Actually, formulas (8.3-4) allow one to reconstruct the derivative up to an additive constant only, which turns out to be zero by virtue of the vanishing boundary conditions for all $H$, $v$, and $h$.)

Now, Theorem 8.7 follows from the following lemma.

Lemma 8.10.

$$-\int_B (i_v\theta)\ \omega^n = n\int_B H\ \omega^n.$$

Proof of Lemma. Owing to the properties of the inner derivative operator $i_v$, we have

$$\begin{aligned} -\int_B i_v\theta\wedge\omega^n &= -\int_B \theta\wedge i_v(\omega^n) \\ &= -n\int_B \theta\wedge i_v\omega\wedge\omega^{n-1} = n\int_B \theta\wedge dH\wedge\omega^{n-1}. \end{aligned}$$

Moving the exterior derivative $d$ to the 1-form $\theta$ gives

$$n\int_B d\theta \wedge H \wedge \omega^{n-1} - n\int_{\partial B} \theta \wedge H \wedge \omega^{n-1} = n\int_B H\,\omega^n,$$

since the function $H$ vanishes on $\partial B$. This completes the proof of Lemma 8.10 and Theorem 8.7. $\square$

REMARK 8.11. Theorem 8.7 can be reformulated as follows. Define the *Calabi integral* of a Hamiltonian function $H$ as $\int_B H\,\omega^n$. This formula defines a linear function (or an exterior 1-form) on the Lie algebra $\mathrm{ham}_\partial(B)$.

The *Calabi form* on the corresponding group $\mathrm{Ham}_\partial(B)$ is the right-invariant differential form coinciding with the Calabi integral on the Lie algebra $\mathrm{ham}_\partial(B)$. The Calabi form is actually a *bi-invariant* (i.e., both left- and right-invariant) 1-form on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}_\partial(B)$. It immediately follows from the fact that the Calabi integral, defined on $\mathrm{ham}_\partial(B)$, is invariant under the adjoint representation of this group $\mathrm{Ham}_\partial(B)$ in the corresponding Lie algebra $\mathrm{ham}_\partial(B)$. In turn, the latter holds because a symplectomorphism sends the Hamiltonian vector field of $H$ to the Hamiltonian vector field of the transported function, while preserving the form $\omega^n$. Hence, the symplectomorphism action preserves the integral.

The *Calabi invariant* $\mathrm{Cal}\,(g)$ of a Hamiltonian diffeomorphism $g$ in $\mathrm{Ham}_\partial(B)$ is the integral of the Calabi form along a path $g_t$ in $\mathrm{Ham}_\partial(B)$ joining the identity diffeomorphism with $g$ .

THEOREM 8.7$'$. *The Calabi form is exact: The integral depends only on the final point $g$ and not on the connecting path.*

Although we have already proved the exactness of the Calabi form in slightly different terms, we present here a shortcut to prove its closedness. It would imply exactness if we knew that the group of Hamiltonian diffeomorphisms is simply connected, i.e., that every path connecting the identity with $g$ in $\mathrm{Ham}_\partial(B)$ is homotopical (or at least homological) to any other. Unfortunately, we do not know whether this is the case in all dimensions (see Remark 8.9).

PROOF OF CLOSEDNESS. We start with a well-known general fact:

LEMMA 8.12. *For any right-invariant differential form $\alpha$ on a Lie group,*

$$d\alpha(\xi,\eta) = \alpha([\xi,\eta])$$

*for every pair of vectors $\xi, \eta$ in the Lie algebra.*

(This formula follows from the definition of the exterior differential $d$; see Section I.7.B). Therefore, the differential of the Calabi form is minus the integral of the Poisson bracket of two Hamiltonian functions.

LEMMA 8.13. *Let $H$ be a Hamiltonian function defined on the ball $B$ and constant on the boundary $\partial B$. Then the Poisson bracket of $H$ with another Hamiltonian function $F$ has zero integral over $B$.*

EXAMPLE 8.14. For two smooth functions $F$ and $H$ in a bounded domain $D$ of the plane $(p, q)$,

$$\int_D \{F, H\}\ dp \wedge dq = 0,$$

provided that $H$ is constant on $\partial D$ (i.e., that the Hamiltonian field of $H$ is tangent to $\partial D$). Indeed,

$$\int_D \{F, H\}\ dp \wedge dq = \int_D dF \wedge dH = -\int_{\partial D} H\,dF = -H \int_{\partial D} dF = 0,$$

since $H$ is constant on the boundary $\partial D$.

PROOF OF LEMMA 8.13. The Poisson bracket $\{F, H\}$ is (minus) the derivative of $F$ along the Hamiltonian vector field of $H$. Consider the $2n$-form in $B$ that is the result of transporting the $2n$-form $F\omega^n$ by the flow of $H$. This flow leaves the ball $B$ invariant, since $H$ is constant on the boundary $\partial B$, and the corresponding Hamiltonian flow is tangent to $\partial B$.

Then the integral of the Poisson bracket $\{F, H\}$ is equal to (minus) the time derivative of the integral over $B$ of this resulting form. But this Hamiltonian flow preserves $\omega^n$ and hence preserves the integral. Thus, the time derivative of the integral of the transported form vanishes, and so does the integral of the Poisson bracket. □

REMARK 8.15. Similarly, for any two smooth functions on a *closed* compact symplectic manifold, the integral of their Poisson bracket vanishes. Here one might replace one of the functions by a closed (nonexact) differential 1-form; it does not change the proof. Moreover, every function on a connected closed symplectic manifold whose integral vanishes can be represented as a sum of Poisson brackets of functions on this manifold [Arn7].

The closedness of the Calabi form (the invariance of integral (8.2) under the deformations of the path) follows immediately from Lemmas 8.12 and 8.13: The differential of the Calabi form is the integral of (minus) the Poisson bracket of any two functions from $\mathrm{Symp}_\partial(B)$, which always vanishes. □

Now we are ready to prove the infiniteness of the diameter of the symplectomorphism group.

PROOF OF THEOREM 8.3 FOR $\mathrm{Ham}_\partial(B^{2n})$. Let the ball $B^{2n}$ be equipped with the standard symplectic structure, and let $\mu = \omega^n$ denote the corresponding volume form. The $\ell_2$-length of a path $\{g_t\}$ in the right-invariant metric on $\mathrm{Ham}_\partial(B)$ ($g_t$ being the flow of a time-dependent Hamiltonian function $H_t(x)$ joining the endpoints $g_0 = \mathrm{Id}$ and $g_1 = g$) is given by

$$\ell(\{g_t\}) = \int_0^1 \left( \int_B \|\frac{dg_t(x)}{dt}\|^2 \ \mu \right)^{1/2} dt$$
$$= \int_0^1 \left( \int_B \|\nabla H_t(g_t)\|^2 \ \mu \right)^{1/2} dt = \int_0^1 \|\nabla H_t\|_{L_2(B)}\, dt.$$

Then the desired estimate follows from the Poincaré and Schwarz inequalities:

$$\ell(\{g_t\}) = \int_0^1 \|\nabla H_t\|_{L^2(B)}\, dt \geq c_1 \int_0^1 \|H_t\|_{L^2(B)}\, dt$$
$$\geq c_2 \int_0^1 \|H_t\|_{L^1(B)}\, dt \geq c_2 \int_0^1 \int_B H_t \ \mu\, dt = c_2 \cdot \mathrm{Cal}\ (g).$$

Owing to the surjectivity of the map $\mathrm{Cal} : \ \mathrm{Ham}_\partial(B) \to \mathbb{R}$, one can find a Hamiltonian diffeomorphism (see Remark 8.4) with an arbitrarily large Calabi invariant and therefore arbitrarily remote from the identity. □

Analogous statements for the right-invariant metric generated by the $L^1$-norm on vector fields $\mathrm{ham}(M)$ and for nonexact symplectomorphisms (Theorem 8.3 in full generality) require noticeably more work [ER2].

**8.C. Bi-invariant metrics and pseudometrics on the group of Hamiltonian diffeomorphisms.** The group Ham $(\mathbb{R}^{2n})$ of (compactly supported) Hamiltonian diffeomorphisms of the standard space $\mathbb{R}^{2n}$ (or the group of symplectomorphisms of a ball that are stationary in a neighborhood of its boundary) admits interesting *bi-invariant* metrics (see [Hof, E-P, H-Z, LaM, Plt, Don]). The right-invariant metrics discussed above are defined in terms of the norm of *vector fields*, which requires an additional ingredient, a metric on $\mathbb{R}^{2n}$. On the contrary, the bi-invariant metrics are defined solely in terms of the *Hamiltonian functions*.

DEFINITION 8.16 [E-P]. Any $L^p$-norm ($1 \le p \le \infty$) on the space $C_0^\infty(\mathbb{R}^{2n})$ of compactly supported Hamiltonian functions assigns the *length* $l_p$ to any smooth curve on the group $\mathrm{Ham}(M)$. Given the Hamiltonian function $H_t \in C_0^\infty(\mathbb{R}^{2n})$ of a flow from $f$ to $g$, we define

$$l_p(f,g) := \int_0^1 \|H_t\|_{L^p(\mathbb{R}^{2n})}\ dt.$$

The length functional generates a *pseudometric* $\rho_p$ on the group $\mathrm{Ham}(M)$ (i.e., a symmetric nonnegative function on $\mathrm{Ham}(M) \times \mathrm{Ham}(M)$ obeying the triangle inequality).

The pseudometrics $\rho_p$ are *bi-invariant.* This immediately follows from invariance of the $L^p$-norm under the adjoint group action: The integral

$$\|H\|^p_{L^p(\mathbb{R}^{2n})} = \int_{\mathbb{R}^{2n}} |\ H(x)\ |^p\ \omega^n$$

persists under symplectic changes of the variable $x$. More generally, one can start with an arbitrary symplectically invariant norm on the algebra ham $(M)$.

In particular, the distance $\rho_\infty(\mathrm{Id}, f)$ in the $L^\infty$-(pseudo)metric between any Hamiltonian diffeomorphism $f \in \mathrm{Ham}(\mathbb{R}^{2n})$ and the identity element Id reads

$$\rho_\infty(\mathrm{Id}, f) = \inf_H \int_0^1 \sup_x\ |\ H(x,t)\ |\ dt,$$

where the infimum is taken over all Hamiltonian functions $H(x,t)$ corresponding to flows starting at Id and ending at $f$. By definition, $\rho_\infty(f,g) := \rho_\infty(\mathrm{Id}, fg^{-1})$.

This bi-invariant (pseudo-)metric $\rho$ is equivalent to the one introduced by Hofer [Hof]:

$$\rho'_\infty(\mathrm{Id}, f) = \inf_H \int_0^1 (\sup_x H(x,t) - \inf_x H(x,t))\ dt. \tag{8.5}$$

We use the notation $\rho_\infty$ in the sequel for both $\rho_\infty$ and $\rho'_\infty$.

THEOREM 8.17 [Hof]. *The (pseudo-)metric $\rho_\infty$ is a genuine bi-invariant metric on $\mathrm{Ham}(M)$; i.e., in addition to positivity and the triangle inequality, the relation $\rho_\infty(f,g) = 0$ implies that $f = g$.*

Lalonde and McDuff [LaM] showed that $\rho_\infty$, defined by the same formula (8.5) for any symplectic manifold $(M,\omega)$, is a true metric on the group $\mathrm{Ham}(M)$. They used it to prove Gromov's nonsqueezing theorem in full generality for maps of arbitrary symplectic manifolds into a symplectic cylinder.

It turns out, however, that the limit case $p = \infty$ is the only $L^p$-norm on Hamiltonians that generates a metric. For $1 \leq p < \infty$ there are distinct symplectomorphisms with vanishing $\rho_p$-distance between them [E-P]. The features of the (pseudo)metrics above are deduced from special properties of the following symplectic invariant, first introduced by Hofer for subsets of $\mathbb{R}^{2n}$ and called the displacement energy.

Let $\rho$ be a bi-invariant (pseudo)metric on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M)$ of an open symplectic manifold $M$.

DEFINITION 8.18. The *displacement energy* $e(A)$ of a subset $A \subset M$ is the (pseudo)-distance from the identity map to the set of all symplectomorphisms that push $A$ away from itself: $e(A) = \inf\ \{\ \rho(\mathrm{Id}, f)\ \}$, where the infimum is taken over all $f \in \mathrm{Ham}(M)$ such that $f(A) \cap A = \emptyset$. (If there is no such $f$, we set $e(A) := \infty$.)

THEOREM 8.19 [E-P]. *Let $\rho$ be a bi-invariant metric on* $\mathrm{Ham}(M)$. *Then the displacement energy of every open bounded subset $A \subset M$ is nonzero: $e_\rho(A) \neq 0$.*

For instance, for a disk $B \subset \mathbb{R}^2$ of radius $R$, the displacement energy in Hofer's metric is $\pi R^2$ [Hof]. Furthermore, the displacement energy is nonzero for every compact *Lagrangian submanifold* of $M$ [Che]. (A submanifold $L$ of a symplectic manifold $(M^{2n}, \omega)$ is called *Lagrangian* if $\dim L = n$ and the restriction of the 2-form $\omega$ to $L$ vanishes.)

PROOF OF THEOREM 8.19. First notice that for the group commutator $[\phi, \psi]$ of any two elements $\phi, \psi \in \mathrm{Ham}(M)$ one has

$$\rho(\mathrm{Id}, [\phi, \psi]) \leq 2 \min(\rho(\mathrm{Id}, \phi), \rho(\mathrm{Id}, \psi)). \tag{8.6}$$

This follows from the bi-invariance of the metric $\rho$ and the triangle inequality.

Choose arbitrary diffeomorphisms $\phi, \psi \in \mathrm{Ham}(M)$ such that their supports are in $A$ and $[\phi, \psi] \neq \mathrm{Id}$. Then Theorem 8.19 will be proved with the following lemma:

LEMMA 8.20. $\rho(\mathrm{Id}, [\phi, \psi]) \leq 4 e_\rho(A)$.

PROOF OF LEMMA. Assume that a Hamiltonian diffeomorphism $h \in \mathrm{Ham}(M)$ displaces $A:\ h(A) \cap A = \emptyset$. Then the diffeomorphism $\theta := \phi h^{-1} \phi^{-1} h$ has the same restriction to $A$ as $\phi$. Hence, $\phi^{-1}\psi\phi = \theta^{-1}\psi\theta$. Utilizing the bi-invariance and the inequality (8.6), we have

$$\rho(\mathrm{Id}, [\phi, \psi]) = \rho(\psi, \phi^{-1}\psi\phi) = \rho(\psi, \theta^{-1}\psi\theta) \leq 2\rho(\mathrm{Id}, \theta) \leq 4\rho(\mathrm{Id}, h).$$

Minimization over $h$ completes the proof. □

COROLLARY 8.21 [E-P]. *The (pseudo)metric* $\rho_p$ *on* $\mathrm{Ham}(M)$ *generated by the* $L^p$*-norm on* $C_0^\infty(M)$ *is not a metric for* $p<\infty$.

PROOF OF COROLLARY. Let $B\subset M$ be an embedded ball and $\{g_t^H\}$ a (compactly supported) Hamiltonian flow that pushes $B$ away from itself: $g_1^H(B)\cap B=\emptyset$. This flow is generated by a function $H\in C_0^\infty(M\times[0,1])$. Introduce a new Hamiltonian function $K(.,t)$ by smoothly cutting off $H(.,t)$ outside a neighborhood $U_t\subset M$ of the moving boundary $g_t^H(\partial B)$; see Fig.57.

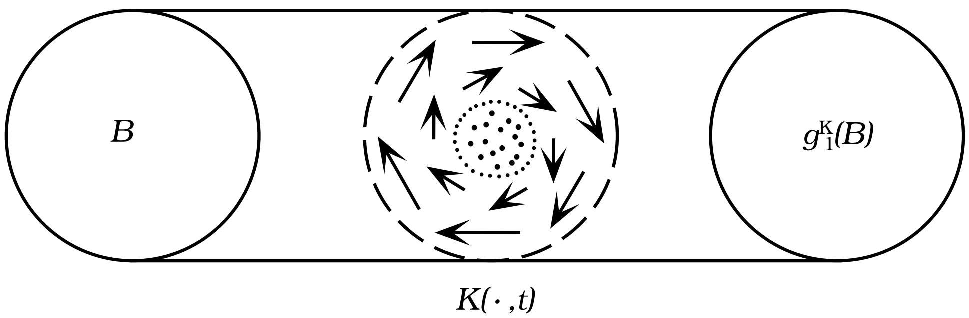



FIGURE 57. Displacement of a ball with rotation.

The flows of $K$ and $H$ coincide when restricted to the boundary $\partial B$: $g_t^K(\partial B)=g_t^H(\partial B)$ for every $t$, and therefore $g_1^K(B)\cap B=\emptyset$.

Shrinking the neighborhoods $U_t$, one can make the $L^p$-norm of $K(.,t)$ (and hence the distance $\rho_p(\mathrm{Id},g_1^K)$) arbitrarily small for every $p\neq\infty$. Thus the displacement energy of $B$ associated to $\rho_p$, $p\neq\infty$ vanishes, and Theorem 8.19 is applicable.

□

Informally, one can push a ball away from itself with an arbitrarily low energy, but the tradeoff is an extremely fast rotation of the shifted ball near the boundary: The function $K(.,t)$ has steep slopes (and hence a large gradient) in a neighborhood of $\partial B$.

Let us confine ourselves to the case of compactly supported Hamiltonian diffeomorphisms in $\mathbb{R}^{2n}$.

REMARK 8.22 [E-P]. For every diffeomorphism $\phi\in\mathrm{Ham}(\mathbb{R}^{2n})$ and $1<p<\infty$, one has $\rho_p(\mathrm{Id},\phi)=0$. If $p=1$, then $\rho_1(\mathrm{Id},\phi)=|\mathrm{Cal}(\phi)|$.

The situation is completely different for the bi-invariant metrics of $L^\infty$-type. We refer the reader to [H-Z] for an account of other peculiar properties of symplectomorphism groups.

In particular, consider the embedding of the group of Hamiltonian diffeomorphisms, say, of the ball $B \subset \mathbb{R}^{2n}$, into the group of all compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$:

THEOREM 8.23 [Sik]. *The subgroup of all Hamiltonian diffeomorphisms* $\mathrm{Ham}_\partial(B)$ *of a unit ball (steady near the boundary) has a finite diameter (in Hofer's metric) in the group of all compactly supported Hamiltonian diffeomorphisms of* $\mathbb{R}^{2n}$.

For the diffeomorphisms with support in the ball of radius $R$, the diameter is majorated by $16\pi R^2$ [H-Z]. Furthermore, for Hofer's metric the following analogue of the $S\mathrm{Diff}(M^3)$- and $L^2$- metric estimates holds (cf. Theorem 7.9):

THEOREM 8.24 [Hof]. *The metric* $\rho_\infty$ *is continuous in the* $C^0$*-topology: for every* $\psi \in \mathrm{Ham}(\mathbb{R}^{2n})$,

$$\rho_\infty(\mathrm{Id}, \psi) \leq 128\,(\textit{diameter of supp}(\psi))\,|\,\mathrm{Id} - \psi|_{C^0},$$

*where* $\rho_\infty$ *is given by (8.5).*

The diameter result changes drastically if we consider the group $\mathrm{Ham}_\partial(B)$ by itself. For every symplectic manifold $M$ with boundary its group of Hamiltonian diffeomorphisms $\mathrm{Ham}_\partial(M)$ stationary at the boundary has an infinite diameter in Hofer's metric (compare with Theorem 8.3 asserting the infiniteness of the diameter in the right-invariant $L^2$-metric). This follows from the existence of symplectomorphisms with arbitrarily large Calabi invariant, which bounds below Hofer's metric; see, e.g., [ER2]. (A more subtle statement is that the diameter of the commutant subgroup of $\mathrm{Ham}_\partial(M)$, the group of Hamiltonian diffeomorphisms with zero Calabi invariant, is also infinite; see [LaM].)

PROBLEM 8.25. Is the diameter of the group of Hamiltonian diffeomorphisms of the two-dimensional sphere finite in Hofer's metric?

**8.D. Bi-invariant indefinite metric and action functional on the group of volume-preserving diffeomorphisms of a three-fold.** Though divergence-free vector fields do not have analogues of Hamiltonian functions if the dimension of the manifold is at least 3, the group of volume-preserving diffeomorphisms of a simply connected three-dimensional manifold can be equipped with a bi-invariant (yet indefinite) metric.

Let $M$ be a simply connected compact three-dimensional manifold equipped with a volume form $\mu$. To define a bi-invariant metric, one needs to fix on the Lie algebra $S\mathrm{Vect}(M)$ of divergence-free vector fields a quadratic form that is invariant under the adjoint action of the group $S\mathrm{Diff}(M)$ (i.e., under a change of variables preserving the volume form). Such a form has already been introduced in Chapter III (Section III.1.D) as the *Hopf invariant* (or the helicity functional, or the asymptotic linking number) of a divergence-free vector field.

Recall that we start with a divergence-free vector field $v$ on $M^3$ and define the differential two-form $\alpha = i_v\mu$, which is exact on $M$. The Hopf invariant $\mathcal{H}(v)$ is the indefinite quadratic form

$$\mathcal{H}(v) = \int_M d^{-1}\alpha \wedge \alpha.$$

The group $S\mathrm{Diff}(M)$ can be equipped with a right-invariant indefinite "finite signature" metric $\rho$ by right translations of $\mathcal{H}$ into every tangent space on the group.

The quadratic form $\mathcal{H}(v)$ is invariant under volume-preserving changes of variables by virtue of the coordinate-free definition of $\mathcal{H}$. It follows that the corresponding indefinite metric $\rho$ on the group $S\mathrm{Diff}(M)$ is bi-invariant. This metric has infinite inertia indices $(\infty, \infty)$, due to the spectrum of the $d^{-1}$ (or $\mathrm{curl}^{-1}$) operator (see [Arn9, Smo1]). The properties of this metric, apart from those discussed in Chapter III, are still obscure.

A similar phenomenon is encountered in symplectic topology (or symplectic Morse theory; see, e.g., [A-G, Arn22, Cha, Vit, Gro]). The action functional on the space of contractible loops in a symplectic manifold also has inertia indices $(\infty, \infty)$.

Remark 8.26. For a non-simply connected three-manifold $M$ equipped with a volume form $\mu$, the definition of the helicity invariant can be extended to null-homologous vector fields (i.e., the fields belonging to the image of the curl operator):

$$\mathcal{H}(v, w) = \int_M (i_v\mu) \wedge d^{-1}(i_w\mu).$$

Proposition 8.27. *The null-homologous vector fields form a subalgebra of the Lie algebra of divergence-free vector fields on $M$.*

Proof. For any two divergence-free vector fields $v$ and $w$ on $M$, their commutator $\{v, w\}$ is null-homologous : $i_{\{v,w\}}\mu = d\,(i_v i_w \mu)$. □

Corollary 8.28. *The subgroup of volume-preserving diffeomorphisms of $M$ corresponding to the subalgebra of null-homologous vector fields is endowed with a bi-invariant "finite signature" metric.*

The subalgebra of null-homologous vector fields is also a Lie ideal in the ambient Lie algebra of divergence-free fields. Moreover, the null-homologous vector fields form the *commutant* (i.e., the space spanned by all finite sums of commutators of elements) *of the Lie algebra* of all divergence-free vector fields on an arbitrary compact connected manifold $M^n$ with a volume form ([Arn7]; see also [Ban] for the symplectic case).

REMARK 8.29. Consider a divergence-free vector field on a three-dimensional manifold that is exact and has a vector-potential. We can associate to this field some kind of Morse complex by the following construction. Associate to a closed curve in the manifold the integral of the vector-potential along this curve (if the curve is homologous to zero, it is the flux of the initial field through a Seifert surface bounded by our curve).

We have defined a function on the space of curves. The critical points of this function are the closed trajectories of the initial field. Indeed, if the field is not tangent to the curve somewhere, its flux through the small transverse area would be proportional to the area, and the first variation cannot vanish.

The positive and negative inertia indices of the second variation of this functional are both infinite. Indeed, in the particular case of a vertical field in a manifold fibered into circles over a surface, our functional is the oriented area of the projection curve. The latter is exactly the nonperturbed functional of the Rabinowitz–Conley–Zehnder theory; see [H-Z].

From this theory we know that the infiniteness of both indices is not an obstacle to the application of variational principles. We may, therefore, hope that the study of the Morse theory of our functional might provide some interesting invariants of the divergence-free vector field. In hydrodynamical terms these would be invariants of the class of isovorticed fields, that is, of coadjoint orbits of the volume-preserving diffeomorphism group.

The Morse index of a closed trajectory changes when the trajectory collides with another one, that is, when a Floquet multiplier is equal to 1. For the $n$-fold covering of the trajectory, the index changes when the Floquet multiplier traveling along the unit circle crosses an $n^{\text{th}}$-root of unity. Thus, one may hope to have a rather full picture of the Morse complex at least for the curves in the total space of a circle bundle that are sufficiently close to the fibers.