CHAPTER III

TOPOLOGICAL PROPERTIES OF MAGNETIC AND VORTICITY FIELDS

The interior media of stars and planets are often virtually perfect conductors and possess magnetic fields. These fields are said to be "frozen" into the medium (for instance, plasma or magma) in spite of temperatures of a million degrees. Mathematically this means that any motion of the medium transports the fields by a diffeomorphism action preserving the mutual alocation of the fields' trajectories. Such a transform may diminish the field magnetic energy. The topological structure of the field provides obstacles to the full dissipation of the magnetic energy of the star or planet.

On the other hand, inhomogeneity of the medium's motion (e.g., the "differential rotation") stretches the particles and hence might amplify the magnetic energy (transforming part of the kinetic energy of the motion into magnetic energy). This competing mechanism is apparently responsible for the dynamo effect, generating a strong magnetic field from very small magnetic "seeds" (see Chapter V).

§1. Minimal energy and helicity of a frozen-in field

1.A. Variational problem for magnetic energy. In this chapter we will look for the energy minimum for the fields obtained from a given divergence-free vector field under the action of volume-preserving diffeomorphisms.

The *energy* of a vector field ξ defined in a domain M of the three-dimensional Euclidean space $\mathbb{R}^3$ is the integral $E = \int\limits_M (\xi, \xi)\, \mu$. (It differs by a factor of 2 from the energy used in preceding chapters, which simplifies noticeably the estimates below. Throughout Chapter III, the space $\mathbb{R}^3$ is always equipped with the standard metric, and μ is the volume form.)

A more general setting assumes that M is a Riemannian manifold, possibly with boundary. The fields are supposed to be divergence free with respect to the Riemannian volume form (and to obey some boundary conditions, such as tangency to the boundary of M, or equality of the field normal component at the boundary to a prescribed function). The energy $E = \langle\xi,\xi\rangle = \int_M (\xi,\xi)\,\mu$ is a *geometric* characteristic of the field relying on the choice of the Riemannian metric $(\ ,\ )$.

Our purpose is to estimate the energy by means of topological features of the field. Here a feature of the field is called *topological* if it persists under the action of diffeomorphisms preserving the volume element (but not necessarily the metric).

REMARK 1.1. In magnetohydrodynamics, where this variational problem naturally arises, the role of ξ is played by a magnetic field $\mathbf{B}$, frozen into a fluid of infinite conductivity (but of finite viscosity ν) filling a "star" M.

With an appropriate choice of units, the velocity field v and the magnetic field $\mathbf{B}$ satisfy the system of equations (cf. Section I.10)

$$\begin{cases} \frac{\partial v}{\partial t} + (v,\nabla)v = -\nabla p + \nu\Delta v + (\text{curl }\mathbf{B})\times\mathbf{B}, & \text{div } v = 0,\\ \frac{\partial \mathbf{B}}{\partial t} + \{v,\mathbf{B}\} = 0, \quad \text{div }\mathbf{B} = 0, \end{cases}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket of two vector fields. The covariant differentiation $(v,\nabla)v$, the Laplace operator $\Delta = -\text{curl curl}$, the vorticity curl $\mathbf{B}$, and the cross product $\times$, standard for $\mathbb{R}^3$, have natural generalizations to the case of an arbitrary Riemannian manifold M. The magnetic field $\mathbf{B}$ and the velocity field v are prescribed at the initial moment. The term $(\text{curl }\mathbf{B})\times\mathbf{B}$ represents the Lorentz force $\mathbf{j}\times\mathbf{B}$ acting on a current $\mathbf{j}$, which coincides (modulo the factor 4π) with curl $\mathbf{B}$ according to the Maxwell equation.

Physicists suggest that during evolution the kinetic energy dissipates due to the viscosity term $\nu\Delta v$, and the motion ceases "at the end," each particle approaching some terminal position. If this happens, the magnetic field, being frozen-in, will attain some terminal configuration. The energy of this terminal field must be a local minimum; otherwise the magnetic energy would have been converted into kinetic energy, and because of the Lorentz force, the fluid would move further until the hydrodynamical viscosity dissipated the excess of the magnetic energy above the minimum.

1.B. Extremal fields and their topology. The variational principle for magnetic fields is dual to that for the steady fluid flows (studied in Chapter II) in the following sense.

The energy functional that undergoes a minimization procedure is the same in both problems. The domain of this functional in the magnetic case consists of all fields diffeomorphic to a given one, while for the case of the ideal fluid the domain is replaced by the class of the isovorticed fields, i.e., by the fields with diffeomorphic vorticities. (The term "dual" above refers to the fact that the domain of diffeomorphic fields is an adjoint orbit in the Lie algebra of all divergence-free vector fields, whereas the isovorticed fields constitute a coadjoint orbit of that algebra; see Chapter I.)

The extremal fields in both of the variational problems coincide ([Arn9], for the proof see Section II.2). These fields have very peculiar topology (cf. Section II.1). Namely, the extremals are the divergence-free fields that commute with their vorticities. They are either Beltrami flows (i.e., the fields proportional to their own vorticities) or are "integrable" flows whose stream lines fill almost everywhere tori and annuli; see Fig.9 in Chapter II.

This analysis of topology of the extremal fields leaves little hope that the idealized physical model of the magnetic field relaxation, described above, is legitimate for any somewhat general initial conditions. Indeed, the initial magnetic field $\mathbf{B}$ can be chosen having no invariant magnetic surfaces. Then the terminal field, if there is one, cannot have invariant tori or annuli and must be a solenoidal field of a very special (Beltrami) type (see [Hen] for the first numerical evidence of chaos in the Beltrami flows). But such fields are too scarce, and one could hardly find a field with the prescribed topology of the magnetic lines amongst them.

It appears that for a correct description of the actual process it is necessary to take into account the magnetic viscosity, which violates the assumption that the field is frozen-in and implies "reconnection" of the magnetic lines. Such a process was not taken care of in our initial system of equations (one has to add the term $\mu\Delta B$ on the right-hand side of the second equation to capture this phenomenon).

QUESTION 1.2. To what extent can one use the extremal fields to study the behavior of the magnetic field $\mathbf{B}$ at large time scales? What phenomena should appear over the time interval during which the ordinary viscosity succeeds in extinguishing the motion of the fluid, but the magnetic viscosity would not yet extinguish the field $\mathbf{B}$?

1.C. Helicity bounds the energy. Let ξ be a divergence-free vector field defined in a simply connected domain $M \subset \mathbb{R}^3$ and tangent to the boundary of M.

DEFINITION 1.3. The *helicity* (or the *Hopf invariant*) of the field ξ in the domain $M \subset \mathbb{R}^3$ is

$$\mathcal{H}(\xi) = \langle \xi, \operatorname{curl}^{-1}\xi \rangle = \int_M (\xi, \operatorname{curl}^{-1}\xi)\, dV,$$

where $(\ ,\ )$ is the Euclidean inner product, and $A = \mathrm{curl}^{-1}\xi$ is a divergence-free vector potential of the field ξ, i.e., $\nabla \times A = \xi$, $\mathrm{div}\ A = 0$.

The integral is independent of the particular choice of A (which is defined up to addition of the gradient ∇f of a harmonic function, since M is simply connected). Indeed, integrating by parts, one obtains the following expression for the difference of the helicity values associated to two different choices of A:

$$\int_M (\xi, A_1)\,\mu - \int_M (\xi, A_2)\,\mu = \int_M (\xi, \nabla f)\,\mu = \int_M (f \mathrm{div}\ \xi)\,\mu + \int_{\partial M} (f \cdot \xi)\,dS = 0,$$

where the last term vanishes, since ξ is tangent to the boundary ∂M. Note that such a field $A = \mathrm{curl}^{-1}\xi$ exists and is defined uniquely in a simply connected M upon specification of the boundary conditions, e.g., A is tangent to the boundary of M (or, more generally, the normal to the boundary ∂M component (A, n) of the vector field A is fixed). If M is not bounded (say, $M = \mathbb{R}^3$), the field ξ is supposed to decay at infinity fast enough to make the integral above converge.

The helicity of a field measures the average linking of the field lines, or their relative winding (see details in Section 1.D below).

Though the idea of helicity goes back to Helmholtz and Kelvin (see [Kel]), its second birth in magnetohydrodynamics is due to Woltjer [Wol] and in ideal hydrodynamics is due to Moffatt [Mof1], who revealed its topological character (see also [Mor2]). The word “helicity” was coined in [Mof1] and has been widely used in fluid mechanics and magnetohydrodynamics since then. We refer to [Mof2, MoT] for nice historical surveys.

The principal feature of this concept is described in the following statement.

THEOREM 1.4 (HELICITY INVARIANCE). *The helicity $\mathcal{H}(\xi)$ is preserved under the action on ξ of a volume-preserving diffeomorphism of M.*

In this sense $\mathcal{H}(\xi)$ is a topological invariant: Though it is defined above with the help of a metric, every volume-preserving diffeomorphism carries a field ξ into a field with the same helicity. We will prove this theorem in a slightly more general setting at the end of this section just by giving a metric-free definition of the invariant. Now we get an immediate and important dividend:

THEOREM 1.5 [Arn9]. *For a divergence-free vector field ξ,*

$$E(\xi) \geq C \cdot |\mathcal{H}(\xi)|,$$

where C is a positive constant dependent on the shape and size of the compact domain M.

PROOF is a composition of the Schwarz inequality

$$\mathcal{H}^2(\xi) = \langle \xi, A\rangle^2 \ \leq \ \langle \xi, \xi\rangle \ \langle A, A\rangle$$

and the Poincaré inequality, applied to the vector field A (tangent to the boundary of M if $\partial M \neq \emptyset$):

$$\langle A, A\rangle = \int\limits_M (A, A) \ \mu \leq \frac{1}{C^2} \int\limits_M (\xi, \xi) \ \mu = \frac{1}{C^2}\langle \xi, \xi\rangle$$

for $A = \operatorname{curl}^{-1}\xi$, $E(\xi) = \langle \xi, \xi\rangle$. □

Various applications of this theorem can be found in [MoT, L-A].

REMARK 1.6. The inverse (nonlocal) operator curl^{-1} sends the space of divergence-free vector fields (tangent to the boundary) on a simply connected manifold onto itself. This operator is symmetric, and its spectrum accumulates at zero on both sides. The restriction of the operator $-\operatorname{curl}^2$ to the space of the divergence-free vector fields is called the Laplace–Beltrami operator on the divergence-free fields. Its components in the Euclidean $\mathbb{R}^3$ case are the Laplacians of the field components. Its spectrum is a sequence of real numbers divergent to $-\infty$.

This Laplacian $-\operatorname{curl}^2$ differs by the sign from the Laplace operator of topologists $d\delta + \delta d$ (see Sections 1.D and V.3.B below) restricted to the space of closed two-forms. Here a divergence-free vector field ξ on a Riemannian manifold is regarded as the corresponding closed 2-form $i_\xi \mu$.

COROLLARY 1.7. *The eigenfield of the operator* curl^{-1} *corresponding to the eigenvalue* λ *of the largest absolute value has minimal energy within the class of divergence-free fields obtained from this eigenfield by the action of volume-preserving diffeomorphisms.*

Indeed, for any field ξ the energy $E(\xi)$ can be minorized as follows:

$$E(\xi) = \langle \xi, \xi\rangle \ \geq \ \frac{1}{\lambda}\langle \operatorname{curl}^{-1} \xi, \xi\rangle,$$

and the inequality becomes the equality for the eigenfield with the eigenvalue λ. In general, the constant C of the preceding theorem can be taken equal to $|\lambda|$.

REMARK 1.8. The theorems above, as well as many results below, hold for the more general case of manifolds M whose first homology group vanishes: $H_1(M, \mathbb{R}) = 0$.

This statement also holds for an arbitrary closed three-dimensional Riemannian manifold if one confines oneself to divergence-free fields that are "null-homologous" (i.e., have a single-valued divergence-free potential).

EXAMPLE 1.9. The standard *Hopf vector field* on

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | \sum_{i=1}^{4} x_i^2 = 1\}$$

is defined by

$$\mathbf{v}(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3).$$

It corresponds to the maximal eigenvalue (=1/2) of the curl^{-1} operator on S^3 with the canonical induced metric and the orientation given by the inner normal. The trajectories of this field are the great circles along which $S^3 \subset \mathbb{C}^2$ intersects the complex lines $\mathbb{C}^1 \subset \mathbb{C}^2$ (see Fig.21 for $\mathbf{v}$-orbits under stereographic projection $S^3 \to \mathbb{R}^3$). These trajectories are pairwise linked. The Hopf field on S^3 has minimal energy among all the fields diffeomorphic to it, i.e., obtainable from it by the action of a volume-preserving diffeomorphism.

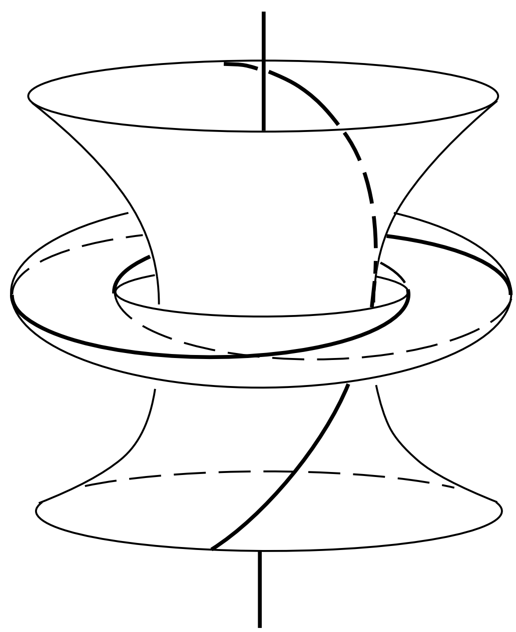


FIGURE 21. Hopf field in $\mathbb{R}^3$ (the stereographic projection from S^3). One circle becomes the vertical axis. Every two orbits are linked.

1.D. Helicity of fields on manifolds. We consider here an *ad hoc* definition of the helicity integral on manifolds [Arn9], establish its simplest properties (in particular, the topological invariance), and identify the result with Definition 1.3 above. An interesting topological meaning of the invariant will be discussed in the next two sections.

Let M be a three-dimensional manifold that is closed (compact, without boundary), oriented, and connected, and let μ be a volume element (i.e., a nonvanishing differential 3-form defining the correct orientation) on M. Notice that we fix a volume element on M, but we do not select any Riemannian metric.

DEFINITION 1.10. Every vector field ξ on M generates a differential 2-form ω_ξ according to the formula

$$\omega_\xi(\eta, \zeta) = \mu(\xi, \eta, \zeta)$$

for any vector fields η and ζ. The correspondence $\xi \mapsto \omega_\xi = i_\xi \mu$ is an isomorphism of the linear spaces of fields and 2-forms. The differential of ω_ξ, being a 3-form, can be expressed via the volume form as

$$d\omega_\xi = \varphi \cdot \mu,$$

where $\varphi : M \to \mathbb{R}$ is a smooth function. The function φ is called the *divergence* of the field $\xi : \varphi = \operatorname{div} \xi$. The velocity field of a flow that preserves the volume element on M is divergence free, and conversely every field with vanishing divergence on M is the velocity field of an incompressible flow.

REMARK 1.11. The origin of divergence is explained by the homotopy formula for the Lie derivative $L_\xi = i_\xi d + d i_\xi$. The Lie derivative L_ξ is the derivative of any differential form f along the vector field ξ, defined as the derivative of the form $g^{t*} f$ transported by the flow g^t of the vector field ξ, evaluated at the initial moment $t = 0$: $L_\xi f = \frac{d}{dt}|_{t=0}(g^{t*} f)$. The operation i_ξ is the substitution of the vector field ξ in the differential form as the first argument, and d is the (exterior) derivative. Applied to the form μ it gives $L_\xi \mu = i_\xi d\mu + d i_\xi \mu = d\omega_\xi = \varphi\mu$. Thus the function φ is the coefficient of stretching (or divergence) of the volume form by the field ξ.

DEFINITION 1.12. A divergence-free vector field ξ on M is said to be *null-homologous* if the 2-form ω_ξ corresponding to it is the differential of a globally defined 1-form α on M:

$$\omega_\xi = d\alpha.$$

The 1-form α will be called a *form-potential.* A field is null-homologous if and only if its flux across every closed surface is zero. In the case of a simply connected closed M every divergence-free vector field is null-homologous.

REMARK 1.13. If M is endowed with a Riemannian metric $(\ ,\ )$ then the 1-form α can be identified with the vector field A for which

$$\alpha(\eta) = (A, \eta) \quad \text{for every field} \quad \eta.$$

Here $\xi = \text{curl } A$ (in the Euclidean case $\xi = \nabla \times A$), and the vector field A is called the *vector-potential* of ξ. We would like to make a point, however, that the forms ω and α (in contrast to the field A) do not depend on the Riemannian metric but rely only on the choice of the volume element μ.

DEFINITION 1.14. The *helicity (*or *Hopf invariant)* $\mathcal{H}(\xi)$ of a null-homologous field ξ on a three-dimensional manifold M (possibly with boundary) equipped with a volume element μ is the integral of the wedge product of the form ω_ξ and its form potential:

$$\mathcal{H}(\xi) = \int_M \alpha \wedge d\alpha = \int_M d\alpha \wedge \alpha, \text{ where } d\alpha = \omega_\xi.$$

THEOREM 1.15. *This definition is consistent, i.e., the value of $\mathcal{H}$ does not depend on the particular choice of the form-potential α, but only on the field ξ:*

i) for a manifold M without boundary, or

ii) for a simply connected manifold M with boundary, provided that the field ξ tangent to ∂M.

PROOF. *i)* First assume that M is without boundary. If $\beta = \alpha + \theta$ is another form potential for the same 2-form ω_ξ, then $d\theta = 0$, and therefore

$$\int_M \alpha \wedge d\alpha - \beta \wedge d\beta = \int_M \theta \wedge d\alpha = \int_M d(\theta \wedge \alpha) = \int_{\partial M = \emptyset} \theta \wedge \alpha = 0.$$

ii) Now $\partial M \neq \emptyset$. In the simply connected case, a variation θ of the form-potential is exact ($\theta = df$ for some function f on M), and the variation of $\mathcal{H}$ is given by

$$\int_M \theta \wedge d\alpha = \int_M df \wedge d\alpha = \int_M d(f \wedge d\alpha) = \int_{\partial M} f \wedge d\alpha = 0,$$

where $d\alpha$ vanishes on ∂M due to the condition $\xi || \partial M$. □

REMARK 1.16. In the presence of a Riemannian metric on M the helicity can be expressed as

$$\mathcal{H}(\xi) = \int_M \alpha \wedge \omega_\xi = \int_M \alpha \wedge i_\xi \mu = \int_M \alpha(\xi) \wedge \mu = \int_M (A, \xi)\ \mu = \langle\ \text{curl}^{-1}\xi, \xi\ \rangle,$$

where A is any vector-potential of ξ. (The shift of the substitution operator from μ to α is due to the fact that i_ξ is the (inner) differentiation: $i_\xi(\alpha \wedge \mu) = i_\xi \alpha \wedge \mu - \alpha \wedge i_\xi \mu$.) Therefore, consistent with Definition 1.3, $\mathcal{H}$ is the inner product of the field with its vector potential.

The above coordinate-free approach can be summarized in the following

COROLLARY 1.17. *The helicity of a null-homologous vector field ξ is preserved under the action of an arbitrary volume-preserving diffeomorphism of M. For a simply connected manifold M with boundary, the helicity of a divergence-free vector field tangent to the boundary does not change under the action of all volume-preserving diffeomorphisms of M that carry the boundary ∂M to itself.*

In particular, on a Riemannian manifold the inner product of a divergence-free field and its vector potential is preserved when the field is acted on by a volume-preserving diffeomorphism.

PROOF. The invariance of $\mathcal{H}$ under diffeomorphisms that preserve the volume element follows from the fact that $\mathcal{H}$ can be defined by using no structures other than the smooth structure of M and the volume element μ. $\square$

This observation constitutes the proof of the Helicity Invariance Theorem.

EXAMPLE 1.18 (= 1.9′). The helicity of the Hopf vector field on $S^3 \subset \mathbb{R}^4$ (defined in Example 1.9, Fig.21) is $\pi^2/2$. Indeed,

$$\mathcal{H}(\mathbf{v}) = \int_{S^3} (\mathbf{v}, \operatorname{curl}^{-1}\mathbf{v})\ \mu = \frac{1}{2}\int_{S^3} (\mathbf{v},\mathbf{v})\ \mu = \frac{1}{2}\int_{S^3} \mu = \frac{\operatorname{vol}\,(S^3)}{2} = \frac{2\pi^2}{2} = \pi^2,$$

since the eigenvalue of the curl^{-1} operator on S^3 is equal to $-1/2$, and the volume of S^3 is $2\pi^2$.

EXAMPLE 1.19. With any smooth map $\pi : S^3 \to S^2$ one can associate the following integer number, called the *Hopf invariant* of π. Fix on the sphere S^2 an arbitrary area form ν normalized by the condition $area(S^2) := \int_{S^2} \nu = 1$. Such a form is closed on the sphere S^2, and hence its pullback $\pi^*\nu$ is exact on S^3 (since $H^2(S^3) = 0$). That is, there exists a 1-form α such that $d\alpha = \pi^*\nu$. Then the Hopf invariant of π is

$$\mathcal{H}(\pi) = \int_{S^3} \alpha \wedge \pi^*\nu.$$

PROPOSITION 1.20. *$\mathcal{H}(\pi)$ is an integer.*

PROOF HINT: Choose the form ν to be a δ-type form on S^2 supported at one point only. Compare the result with the topological definition of the Hopf invariant below. $\square$

Given a volume form on S^3, the number $\mathcal{H}(\pi)$ is the helicity of the divergence-free vector field ξ defined by the condition $i_\xi\mu = \pi^*\nu$. The orbits of this field are closed, being the preimages of points of S^2 under the mapping π. The above

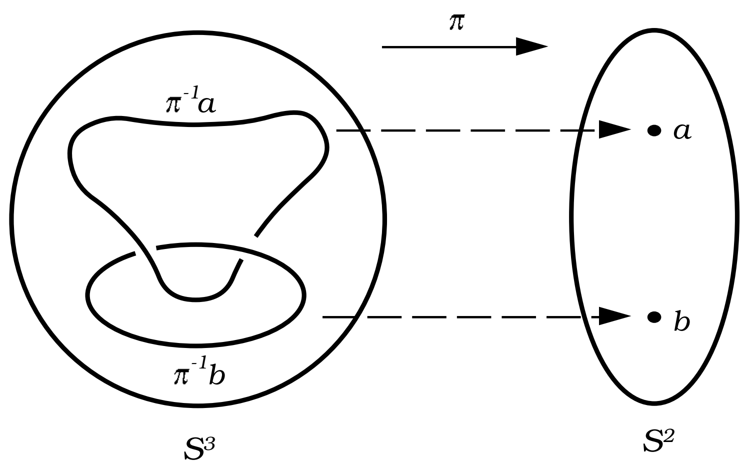


FIGURE 22. Hopf invariant for a map $S^3 \to S^2$.

definition of the helicity is a generalization of the Hopf invariant to the case where an exact 2-form on S^3 (or on M^3) is not necessarily a pullback for any map π.

An equivalent (topological) definition of the Hopf invariant for a map $S^3 \to S^2$ is the linking number in S^3 of the preimages of two generic points in S^2 (Fig.22). The equivalence of the topological and integral definitions plays a key role in what follows in this chapter.

Theorem 1.5 claims that for a map $\pi : S^3 \to S^2$ with nonzero Hopf invariant $\mathcal{H}(\pi)$, (a multiple of) the absolute value of this invariant bounds below the energy of the corresponding vector field. The latter field is directed along the fibers of the map π. The length of the vectors is defined by the volume form on S^3 and the pullback of the S^2 area element.

REMARK 1.21. L.D. Faddeev proposed another but relevant variational problem for the mappings π from $\mathbb{R}^3$ to S^2. Consider the functional on such mappings that is a (weighted) sum of two terms. The first term is the Dirichlet integral (of the squared derivative) of the map π. The second term is the energy of the corresponding vector field directed along the fibers of the map. Then this functional is bounded below by (a multiple of) $|\mathcal{H}(\pi)|^{3/4}$, where $\mathcal{H}(\pi)$ is the Hopf invariant of the map $\pi : \mathbb{R}^3 \to S^2$ [V-K]. The proof uses a version of the Sobolev inequality [Sob1]; cf. Theorem 5.3 below and its proof, which employs the same inequality.

Furthermore, some recent computer experiments for the relaxation process of an initial mapping with nonzero Hopf invariant exhibit the following phenomenon. In the equivariant case (of S^1 acting by rotations on R^3 and S^2), one observes an "energy gap" over the poles, where the rotation axis intersects the sphere. It would be very interesting to explain this singularity structure.

The addition of the Dirichlet integral to the energy is similar to the addition of the Lagrange multiplier in the problem of energy minimization. We could start with the action of all diffeomorphisms, and then consider the conditional minimum for the action of only volume-preserving ones.

REMARK 1.22. The Hopf invariant equips the Lie algebra of divergence-free vector fields on a closed simply connected three-dimensional manifold with a *bilinear form* :

$$\mathcal{H}(\xi, \eta) = \langle \xi, \ \mathrm{curl}^{-1}\eta \rangle,$$

where $\mathrm{curl}^{-1}\eta$ is a vector-potential of the field η.

This form is invariant with respect to the natural action of volume-preserving diffeomorphisms on vector fields (i.e., with respect to the adjoint representation of the group $S\mathrm{Diff}(M)$ in its Lie algebra; see Chapter I). Moreover, the form $\mathcal{H}$ is *symmetric*, since

$$\mathcal{H}(\xi, \eta) = \int_M i_\xi \mu \wedge d^{-1}(i_\eta \mu) = \int_M d^{-1}(i_\xi \mu) \wedge i_\eta \mu = \int_M i_\eta \mu \wedge d^{-1}(i_\xi \mu) = \mathcal{H}(\eta, \xi).$$

The positive and negative subspaces of the form $\mathcal{H}$ are both infinite-dimensional; see [Arn9, Smo1]. Thus $\mathcal{H}$ generates a *bi-invariant pseudo-Euclidean* (indefinite) metric on the corresponding group $S\mathrm{Diff}(M)$. For the case of a non-simply connected M one has to confine oneself to the subalgebra of all null-homologous vector fields within the Lie algebra of all divergence-free vector fields on M (see Section IV.8.D for more detail).

In this case one may also hope to define the generalized Hopf invariants with values in some modules over the fundamental group, but this way has not yet been duly explored.

§2. Topological obstructions to energy relaxation

2.A Model example: Two linked flux tubes. The helicity obstruction to the energy relaxation is clearly seen in the example of a magnetic field confined to two linked solitori, Fig.23a,b. Assume that the field vanishes outside those tubes and the field trajectories are all closed and oriented along the tube axes inside.

To minimize the energy of a vector field with closed orbits by acting on the field by a volume-preserving diffeomorphism, one has to shorten the length of most trajectories. (Indeed, the orbit periods are preserved under the diffeomorphism action; therefore, a reduction of the orbits' lengths shrinks the velocity vectors

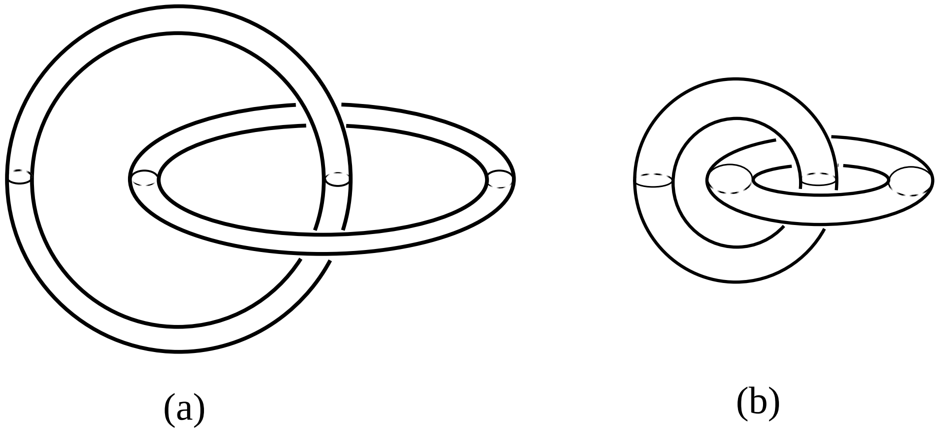


FIGURE 23. a) A magnetic field is confined to two linked solitori. b) Relaxation fattens the tori and shrinks the field orbits.

along the orbits.) In turn, the shortening of the trajectories implies a fattening of the solitori (since the acting diffeomorphisms are volume-preserving).

For a linked configuration, as in Fig.23b, the solitori prevent each other from endless fattening and therefore from further shrinking of the orbits. Therefore, heuristically, in the volume-preserving relaxation process the magnetic energy of the field supported on a pair of linked tubes is bounded from below and cannot attain too small values [Sakh].

Below we show that the helicity of a field measures the rate of the mutual winding (or "helix-likeness") of the field trajectories around each other. To visualize this notion (and the paradigm "helicity bounds energy" of the preceding section), we first concentrate on the degenerate situation above (see [Mof1]).

Let a magnetic (that is, divergence-free) field ξ be identically zero except in two narrow linked flux tubes whose axes are closed curves C_1 and C_2. The magnetic fluxes of the field in the tubes are Q_1 and Q_2 (Fig.24).

Suppose further that there is no net twist within each tube or, more precisely, that the field trajectories foliate each of the tubes into pairwise unlinked circles.

LEMMA 2.1. *The helicity invariant of such a field is given by*

$$\mathcal{H}(\xi) = 2\; lk(C_1, C_2) \cdot Q_1 \cdot Q_2, \tag{2.1}$$

where $lk(C_1, C_2)$ is the linking number of C_1 and C_2.

DEFINITION 2.2. The (Gauss) *linking number* $lk(\Gamma_1, \Gamma_2)$ of two oriented closed curves Γ_1, Γ_2 in $\mathbb{R}^3$ is the signed number of the intersection points of one curve with an arbitrary (oriented) surface bounded by the other curve (Fig.25). The sign of

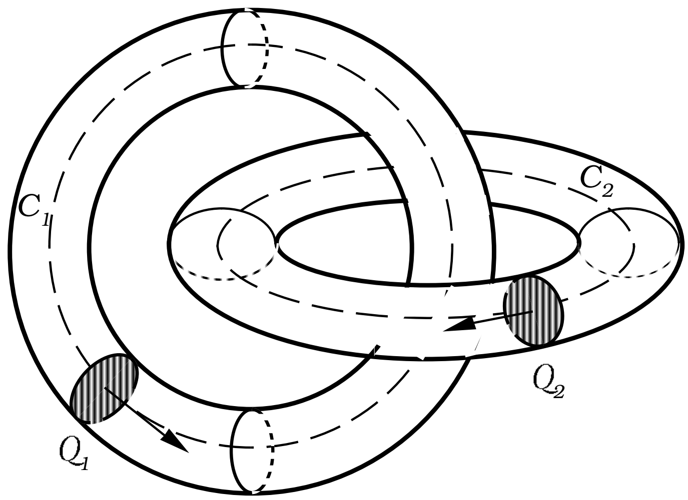


FIGURE 24. C_1, C_2 are axes of the tubes; Q_1, Q_2 are the corresponding fluxes.

each intersection point is defined by the orientation of the 3-frame that is formed at this point by the velocity vector of the curve and by the 2-frame orienting the surface.

The linking number of curves is symmetric: $lk(\Gamma_1, \Gamma_2) = lk(\Gamma_2, \Gamma_1)$.

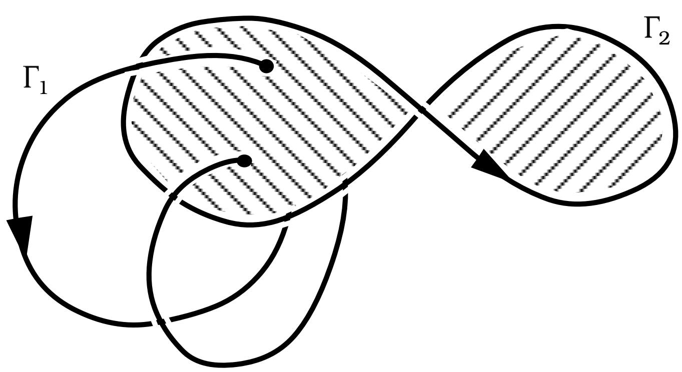


FIGURE 25. The linking number of Γ_1 and Γ_2 is the signed number of intersections of Γ_1 with a surface bounded by Γ_2.

PROOF OF LEMMA. The helicity volume integral $\mathcal{H}(\xi) = \langle \mathrm{curl}^{-1}\xi,\ \xi \rangle = \int (A, \xi)\, \mu$ over the tubes (here $A = \mathrm{curl}^{-1}\xi$) descends to the sum of the corresponding line integrals:

$$\mathcal{H}(\xi) = Q_1 \int_{C_1} (A,\ dC_1) + Q_2 \int_{C_2} (A,\ dC_2).$$

Indeed, the volume element μ in each tube is the product of the line element dC_i and the area element dS_i of the tube cross section. In turn, the integral of the

ξdS_i over the corresponding cross section is the flux Q_i. Hence,

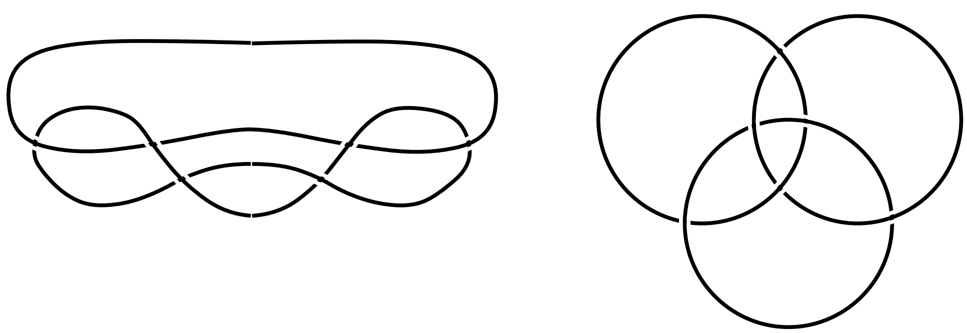
$$\int_{i^{th}\ tube} (A, \xi)\ dS_i\ dC_i = \int_{C_i} \int_{S_i} (A, (\xi dS_i)\ dC_i) = Q_i \int_{C_i} (A,\ dC_i).$$

Furthermore, the circulation $\int_{C_1}(A, dC_1)$ of the field A over the curve C_1 is the full flux of curl $A = \xi$ through a surface bounded by the axis curve C_1. The latter flux is equal to $Q_2 \cdot lk(C_1, C_2)$: Every crossing of the surface by the second tube contributes to the signed amount of Q_2 into the full flux. Note that the first tube itself does not contribute into that flux through its axis C_1, due to the assumption on the net twist within the tubes.

The same argument applied to the second circulation integral doubles the result: $\mathcal{H}(\xi) = 2\ lk(C_1, C_2) \cdot Q_1 \cdot Q_2$. □

A generalization of this example to the case of an arbitrary divergence-free vector field ξ is described in Section 4.

2.B. Energy lower bound for nontrivial linking. The linking number is a rather rough invariant of a linkage. The signed number entering the definition of lk can turn out to be zero for configurations of curves linked in an essential way (see, e.g., the so-called Whitehead link in Fig.26a). However, the heuristic observation of the beginning of Section 2.A for the energy bound still holds.



(a) (b)

FIGURE 26. Nontrivial links with vanishing pairwise linking numbers. a) Whitehead link, b) Borromean rings.

The heuristics above are supported by the following result of M. Freedman [Fr1]: Any essential linking between circular packets of ξ-integral curves implies a lower bound to E.

DEFINITIONS 2.3. A link L, i.e., a smooth embedding of n circles into a 3-dimensional manifold, is *trivial* if it bounds n smoothly and disjointly embedded disks. Otherwise, the link is called *essential.*

A vector field ξ on M is said to be *modeled on* L if there is a ξ-invariant tubular neighborhood of $L \subset M$ foliated by integral curves of ξ that is diffeomorphic to $\bigcup_{i=1}^{n} D_i^2 \times S^1$ foliated by circles $\{point\} \times S^1$ (here D^2 is a 2-dimensional disk).

THEOREM 2.4 [Fr1]. *If ξ is a divergence-free vector field on a closed* 3*-manifold M that is modeled on an essential link (or knot) L, then there is a positive lower bound to the energy of fields obtained from ξ by the action of volume preserving diffeomorphisms of M.*

Under the additional assumption on a field to be strongly modeled on a link, the lower energy bound for a field in $\mathbb{R}^3$ was obtained in [FH1] explicitly. A divergence-free field ξ is *strongly modeled on* L if there is a volume-preserving embedding that carries the field $\frac{\partial}{\partial\theta}$ directed along the circles in $\bigcup_{i=1}^{n}(D^2 \times S^1)_i$ into ξ within a tubular neighborhood of L. The neighborhood consists of several solid tori of equal volume, which we denote by V.

THEOREM 2.5 [FH1]. *The energy of a vector field ξ strongly modeled on an essential link L in $\mathbb{R}^3$ satisfies the inequality*

$$E(\xi) > \left(\frac{\sqrt{6/125}}{\pi^2}\right)^{4/3} \cdot V^{5/3} \approx 0.00624\ V^{5/3}.$$

Note that given any link, one may construct a field modeled (and even strongly modeled) on it. The exponent $5/3$ has the following origin. The Euclidean dilation with a factor l multiplies the image field by l and the volume element by l^3. Thus the total energy gains the factor l^5, while the volume is multiplied by the factor l^3. Hence, the ratio $E/V^{5/3}$ is purely geometrical and independent of scaling in the Euclidean case.

REMARK 2.6. Theorem 2.5 suggests the following construction of a set of invariants of topological or smooth 3-manifolds. The invariants are parametrized by the isotopy classes of knots and links in the manifold. They might also be regarded as the invariants of embeddings of 1-dimensional manifolds into 3-dimensional ones.

Consider the ratio $E/V^{5/3}$ for a vector field strongly modeled on the knot or on the link of a given isotopy class in a Riemannian manifold. Take the infimum over all such fields and over all the Riemannian metrics. The resulting number is an invariant of the smooth (perhaps, even topological) isotopy class of the pair (link, 3-manifold).

Further, one might take the infimum over all the compact 3-manifolds for a homotopically trivial link to get an invariant of the classical link or knot. (Is this infimum equal to the infimum of the above ratio for Euclidean 3-space or for the 3-sphere? Is the supremum over all the 3-manifolds finite?)

One might also start with a compact Riemannian manifold of volume 1 and with a link of k solid tori of volume V each. If kV is smaller than 1, the infimum of $E/V^{5/3}$ over the metrics of total volume 1 is a function of V, which is still an invariant of the embedding. We do not know whether these invariants are nontrivial, i.e., whether they distinguish any 3-manifolds or embeddings (cf. Remark 6.7).

Freedman and He have informed us that Theorem 2.5 can be generalized to arbitrary Riemannian manifolds. The limit of the coefficient $C(V)$ for small volumes V is the same constant as in the Euclidean case $C = \left(\sqrt{6/125}/\pi^2\right)^{4/3}$ given by Theorem 2.5.

The strongly modeled fields have very simple behavior near the link and are far from being generic within divergence-free vector fields. It would be of interest to completely get rid of the condition on a special tubular neighborhood.

PROBLEM 2.7. Is there an energy lower bound for a field having a set of closed trajectories forming an essential link on a Riemannian manifold (without an assumption on a neighborhood of closed orbits)?

REMARK 2.8. The strongest result in this direction was obtained in [FH2] (see Section 5), where the condition on a field to be modeled on a link was weakened to the requirement for a field in $\mathbb{R}^3$ to have invariant tori confining the link components. Such fields form an ample set near the integrable divergence-free flows. This follows from the KAM theory of Hamiltonian perturbations of integrable Hamiltonian systems.

In particular, if a closed field orbit is *elliptic* (and generic), i.e., its Poincaré map has two eigenvalues of modulus 1, then this orbit is confined to a set of nested tori invariant under the field (see, e.g., [AKN]). Thus, *every such orbit forming an essential knot provides the lower bound for the energy of the corresponding field.* Indeed, the energy of any of the invariant solid tori confining this knotted orbit cannot diminish to zero, according to [FH2]. One can argue that a vector field with a knotted *hyperbolic* closed orbit (whose Poincaré map has real eigenvalues of the modulus different from 1) may not have a positive lower bound for the energy (cf. the next section).

REMARK 2.9. The different estimates for the magnetic energy, should magnetic solid tori form a trivial or nontrivial link, have a striking counterpart in the theory of Brownian motion.

Let K be a knot in S^3, and $\{z(t)|\ t > 0\}$ the standard Brownian motion on S^3 starting at some point $O \notin K$ at a distance $d(O, K) = \tau > 0$ from K. If K is unknotted, then there exists almost surely a sequence $t_1 < t_2 < \dots$ such that $t_n \to \infty$ and for which $d(z(t_n), O) \leq \tau/2$. Furthermore, the loop that we obtain by following the Brownian path up to $z(t_n)$ and then joining $z(t_n)$ to O by a short path $\Delta(z(t_n), O)$ is homotopic to O in $S^3 \setminus K$ [Var]. In other words, (almost surely) the Brownian path returns close to its starting point untangled with respect to K, and it does this infinitely many times.

The exact opposite happens when K is knotted: There almost surely exists a $T > 0$ such that whenever the distance $d(z(t), O)$ is small enough, $d(z(t), O) \leq \tau/2$ and $t > T$, the homotopy class of the above loop is not trivial [Var]. In this sense the Brownian motion can tell whether K is an essential knot or not. Heuristically, this means that the Brownian motion distinguishes the existence of a hyperbolic metric on the universal covering to $S^3 \setminus K$ (see Thurston's theorem on the hyperbolic structure on the complement to a nontrivial knot or link [Th2]).

§3. Sakharov–Zeldovich minimization problem

Assume now that a divergence-free field has a trivial topology in that all field trajectories are closed and pairwise unlinked. An example of such a field is the rotation field in a 3-dimensional ball (Fig.27). The energy lower bounds considered in Section 2 are valid for essential links and are not applicable here. On the contrary, in this case the field energy can be reduced almost to zero by a keen choice of volume-preserving diffeomorphisms [Zel2, Sakh, Arn9, Fr2].

THEOREM 3.1. *The energy of the rotation field in a 3-dimensional ball can be made arbitrarily close to zero by the action of a suitable diffeomorphism that preserves volumes and fixes the points in a neighborhood of the ball boundary.*

REMARK 3.2. This result, formulated by A. Sakharov and Ya. Zeldovich [Sakh, Zel2], is based on the following reasoning. Divide the whole ball into a number of thin solid tori (bagels) formed by the orbits of the field and into a remainder of small volume. Then deform each solid torus (preserving its volume) such that it becomes fat and small, with the hole decreasing almost to zero. (Such deformations must violate the axial symmetry of the field, since any axisymmetric diffeomorphism sends the rotation field to itself and hence preserves the total energy.) Now the field energy in the solid tori is decreased (since the field lines are shortened). The whole

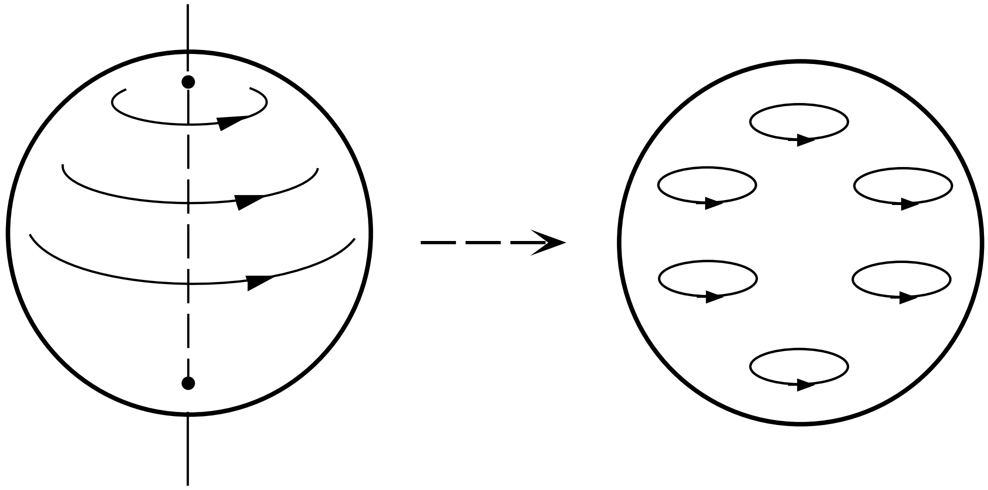


FIGURE 27. A rotation field in a 3-dimensional ball can dissipate its energy almost completely.

construction can be carried out in such a way that the field energy in the remaining small volume is not increased by too much. As a result, the total energy remains arbitrarily small.

This consideration was placed on a rigorous foundation by M. Freedman. We outline the main ideas of his proof below.

Let B^3 be a ball in three-dimensional Euclidean space and ξ the vector field generated by infinitesimal rotation about the vertical axis. The trajectories of this field are horizontal pairwise unlinked circles (and their limits, the points on the vertical axis).

THEOREM 3.3 [Fr2]. *There exists a family of volume-preserving diffeomorphisms $\varphi_t : B^3 \to B^3$, $1 \le t \le \infty$, such that it starts at the identity diffeomorphism ($\varphi_1 = \mathrm{Id}$), it is steady on the boundary ($\varphi_t \mid_{\partial B^3} = \mathrm{Id}$) for all t, and the family of the transformed vector fields $\xi_t := \varphi_{t*}\xi$ (being the image of the rotation field ξ under the φ_t-action) fulfills the following conditions as $t \to \infty$:*

1) *the energy of the field ξ_t decays as $E(\xi_t) := \|\xi_t\|^2_{L^2} = \mathcal{O}(1/t)$,*
2) *the supremum norm is unbounded: $\|\xi_t\|_{L^\infty} = \mathcal{O}(t)$, yet*
3) *for all $k, p < \infty$ the Sobolev norms decay: $\|\xi_t\|_{L^{k,p}} \to 0$ (here the norm $\|\eta\|_{L^{k,p}}$ is the L^p-norm in the space of η's derivatives of orders $0, \dots, k$).*

REMARK 3.4 [Fr2]. For this family of diffeomorphisms, the limit of $\xi_t = \varphi_{t*}\xi$ at infinity $t \to \infty$ does not exist, but for large t the regions of large norm $\|\xi_t\|$ constitute a "topological froth" $\mathcal{F}_t$ with trivial relative topology. The froth $\mathcal{F}_t$ is a "time-fractal" (the facet size drops abruptly in a sequence of catastrophes as t

increases) and becomes dense as $t \to \infty$.

PROOF SKETCH. The following lemma is a modification of Moser's result [Mos1] on the existence of volume-preserving diffeomorphisms between diffeomorphic manifolds of equal volume.

LEMMA 3.5. *Let D and D' be domains of equal volume in $\mathbb{R}^m$ and $f : D \to D'$ a diffeomorphism. Then f is isotopic to a volume-preserving diffeomorphism f_0 between the domains.*

Moreover, if f preserves orientation and a function ρ is the "excess density" $\rho = 1 - \det\,(f_*)$, then there exist constants $C_{k,p}$ depending only on the domain D such that

$$\|f - f_0\|_{L^{k+1,p}} \leq C_{k,p}\|\rho\|_{L^{k,p}} \quad \textit{for any } k, p < \infty.$$

PROOF OF LEMMA 3.5. Pull back the D'-volume form $\mu_{D'}$ to D. The density function ρ manifests the excess of the volume $f^*(\mu_{D'})$ over μ_D. The mean value of ρ is zero due to the volume equality condition.

Let ψ be a solution of the Neumann problem on D for ρ, i.e., $\Delta\psi = \rho$ on D and $\frac{\partial}{\partial n}\psi = 0$ on the boundary ∂D (where $\partial/\partial n$ indicates differentiation in the direction of the exterior normals; see Lemma 3.7 below on solvability of the Neumann problem).

Rewrite this system in the form $\operatorname{div}\,(\nabla\psi) = \rho$, $\nabla\psi \parallel \partial D$. Then the gradient vector field $\nabla\,\psi$ is tangent to the boundary ∂D and defines infinitesimally an isotopy of D moving the volume element μ_D into $f^*(\mu_{D'})$. The isotopy itself is now the phase flow of the dynamical system on D defined by the instant field $\nabla\,\psi$.

Finally, the required estimate is a consequence of the inequality $|\lambda_1| \cdot \|\psi\|_{L^2} \leq \|\rho\|_{L^2}$, where λ_1 is the closest to 0 (from the left) eigenvalue of the Neumann problem. Taking the gradient $\nabla\,\psi$, we lose one order in the Sobolev norm. □

REMARK 3.6. For application to the case where D is a spherical shell, note that the constants $C_{k,p}$ may be chosen independent of the thickness. It follows from the fact that the closest to 0 eigenvalue λ_1 of the Neumann problem on the shell tends to the smallest Laplace–Beltrami eigenvalue on the sphere S^2 as the shell thickness goes to zero.

Indeed, the eigenvalues of the Laplace operator on such a shell are sums of those on the sphere and of the eigenvalues of the radial component

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

of the Laplacian. One immediately sees that all but the first eigenfunctions of the latter operator with the Neumann boundary conditions highly oscillate on a short segment. Hence, all but the first corresponding eigenvalues tend to infinity, while the first one goes to zero as the segment shrinks to a point. This very first eigenvalue is the only eigenvalue that contributes to the eigenvalue λ_1 of the Neumann problem on the shell, and its contribution vanishes as the shell thickness goes to zero.

LEMMA 3.7. *The Neumann problem $\Delta\psi = \rho$ on D and $\frac{\partial}{\partial n}\psi = 0$ on the boundary ∂D has solution for any function ρ with zero mean (i.e., for ρ that is L^2-orthogonal to constants on D).*

PROOF OF LEMMA 3.7. The image of an operator is the orthogonal complement to the kernel of the corresponding coadjoint operator. To apply it to the Neumann operator we first find the set of functions h orthogonal to all $\Delta\psi$ with $\frac{\partial}{\partial n}\psi = 0$:

$$0 = \int_D (\Delta\psi)h = -\int_D \nabla\psi\nabla h + \int_{\partial D} (\frac{\partial}{\partial n}\psi)h = \int_D \psi\Delta h - \int_{\partial D} \psi(\frac{\partial}{\partial n}h).$$

Taking as test functions those ψ's that vanish on the boundary ∂D, we obtain that $\Delta h = 0$. Then for a generic ψ, the boundary term is equal to zero, and hence $\frac{\partial}{\partial n}h = 0$ on ∂D. Thus only the constant functions g are orthogonal to the image of the Neumann operator $\Delta\psi$ (with the boundary condition $\frac{\partial}{\partial n}\psi = 0$), and any function orthogonal to constants is in the image of this operator. □

Main construction. We first cut the ball B in two parts by splitting out a spherical shell Sh of thickness s from a subball B_s (Fig.28). We will fix s later. The internal subball can be stretched in the vertical direction and squeezed into a thin "snake" by a volume-preserving diffeomorphism.

Such a stretching transformation shrinks all the ξ-orbits (located in the horizontal planes in the internal subball B_s) by an arbitrarily large prescribed factor, and hence it reduces the field energy in the (transformed) subball to an arbitrarily small positive level.

Then we put the snake into the original ball, keeping the volume preserved. Allow the composition map of the subball B_s into a snake inside the ball B to be accompanied with a map of the shell Sh into the snake complement. One may do it first without control of the volume elements but providing smoothness of the transformation $(B_s \cup Sh) \to B$ (see Fig.29). Then the accompanying map of the shell Sh can be made volume-preserving by applying the isotopy of Lemma 3.5.

The total energy of the field ξ after the diffeomorphism action is composed by the energy in the subball image and in the shell image $E = E_{subball} + E_{shell}$. The stretching procedure above allows one to handle the first term completely: Given positive ε, the energy $E_{subball}$ can be suppressed to the level $E_{subball} \approx \varepsilon$

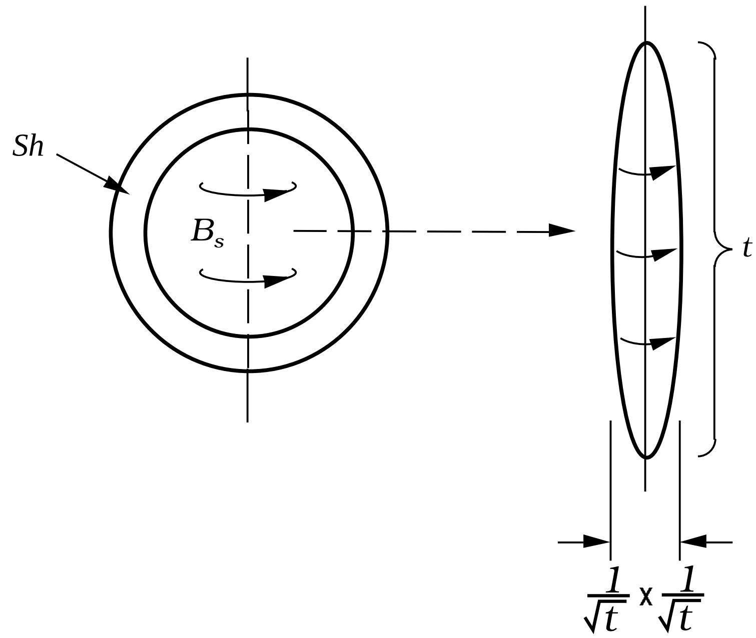


FIGURE 28. Stretching a subball into a snake reduces its energy.

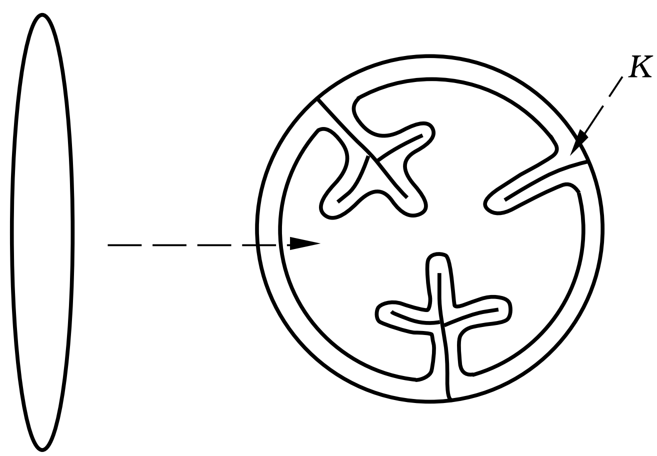


FIGURE 29. The complement of the snake in the ball is a neighborhood of a 2-complex K.

by considering an appropriately long snake. The embedding of the snake into the original ball does not essentially increase its energy, since the bending occurs in the directions orthogonal to the trajectories of the magnetic field, and hence it does not stretch the vectors.

Now we have to estimate the field energy in the shell image. Note that the image is concentrated near a 2-complex K "complementary" to the snake in B. Using Lemma 3.5 and Remark 3.6 in order to control the maximal stretching of orbits in the shell, it is sufficient to provide boundedness of the stretching of the volume element for an arbitrarily thin shell. The latter is achieved by considering a one-parameter family of diffeomorphisms (plotted in Fig.30):

a) first expand a thin shell (of thickness s) to that of a fixed thickness,
b) then map it to a neighborhood of K defined by a given snake embedding,
c) and finally, squeeze this neighborhood to K.

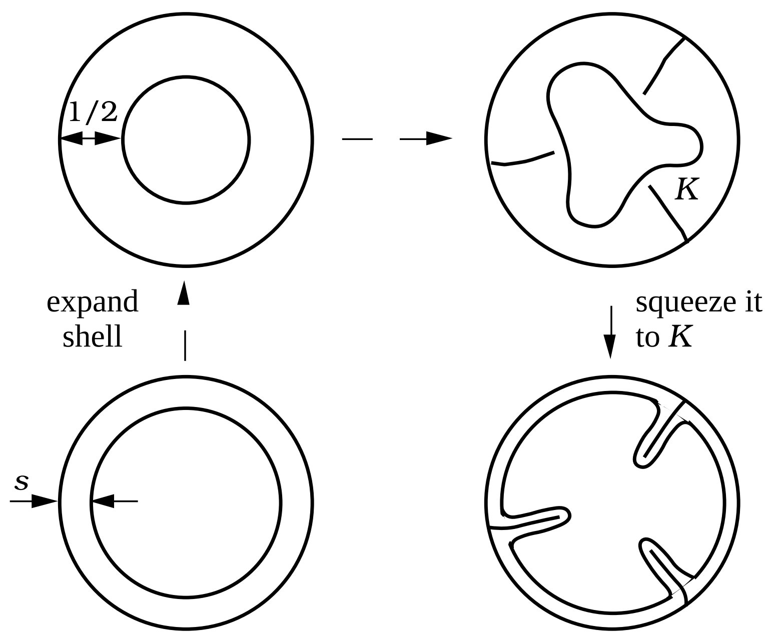


FIGURE 30. Family of maps of a shell into a neighborhood of K.

The energy E_{shell} tends to zero as the thickness $s \to 0$, since the energy integrand is bounded independently of s, while the volume of the integration domain, the shell volume, goes to zero. Thus, having chosen s sufficiently small, one can obtain $E_{shell} \approx \varepsilon$.

Scale estimate. To organize the family φ_t of diffeomorphisms, we will define the initial stretching of the subball into a snake of length t. Then the area of every horizontal section is squeezed by the factor of t, and vectors themselves are squeezed by the factor of $\sqrt{t}$; see Fig.28.

This reduces the total energy to $E(\varphi_t^*\xi) \approx \frac{1}{t}$. However, some orbits in the shell stretch to the "full length" $\approx t$. Hence, the supremum norm $\|\varphi_t^*\xi\|_{L^\infty} = \max\|\xi_t\| = \mathcal{O}(t)$.

Once a length scale ℓ is selected, the energy cannot be squeezed to $< \frac{1}{\ell}$ by using the smooth one-parameter family. To proceed further, one has to renew the original stretching of the subball into the snake. This produces the next collapse at a finer scale. The corresponding 2-complex froth $K = \mathcal{F}_t$ "blossoms and branches" [Fr2].

The topology of K remains trivial (the froth is contractible to the boundary ∂B), since the complement to K is homeomorphic to a ball. □

§4. Asymptotic linking number

The classical Hopf invariant for $S^3 \to S^2$-mappings has two definitions: a topological one (as the linking number of the preimages of two arbitrary points of S^2), and an integral one (as the value of $\int \omega \wedge d^{-1}\omega$ for any two-form ω on S^3 that is a pullback of a normalized area form on S^2); see Example 1.19.

The helicity of an arbitrary divergence-free vector field on a three-dimensional simply connected manifold is a straightforward generalization of the integral definition of the Hopf invariant. The topological counterpart is more subtle and leads to the notions of asymptotic and average linking numbers of field trajectories [Arn9], which replace the linking of the closed curves of the classical definition.

This section deals with such an ergodic interpretation of the helicity.

4.A. Asymptotic linking number of a pair of trajectories. Let M be a three-dimensional closed simply connected manifold with volume element μ. Let ξ be a divergence-free field on M and $\{g^t : M \to M\}$ its phase flow.

Consider a pair of points x_1, x_2 in M. We will associate to this pair of points a number that characterizes the "asymptotic linking" of the trajectories of the flow $\{g^t\}$ issuing from these points. For this purpose, we first connect any two points x and y of M by a "short path" $\Delta(x,y)$. The conditions imposed on a system of short paths will be described below and are satisfied for "almost any" choice of the system.

We select two large numbers T_1 and T_2, and close the segments $g^t x_1 (0 \le t \le T_1)$ and $g^t x_2 (0 \le t \le T_2)$ of the trajectories issuing from x_1 and x_2 by the short paths $\Delta(g^{T_k} x_k, x_k)$ $(k = 1, 2)$. We obtain two closed curves, $\Gamma_1 = \Gamma_{T_1}(x_1)$ and $\Gamma_2 = \Gamma_{T_2}(x_2)$; see Fig.31. Assume that these curves do not intersect (which is true for almost all pairs x_1, x_2 and for almost all T_1, T_2). Then the linking number $lk_\xi(x_1, x_2; T_1, T_2) := lk(\Gamma_1, \Gamma_2)$ of the curves Γ_1 and Γ_2 is well-defined.

Definition 4.1. The *asymptotic linking number* of the pair of trajectories $g^t x_1$ and $g^t x_2$ $(x_1, x_2 \in M)$ of the field ξ is defined as the limit

$$\lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{lk_\xi(x_1, x_2; T_1, T_2)}{T_1 \cdot T_2},$$

where T_1 and T_2 are to vary so that Γ_1 and Γ_2 do not intersect.

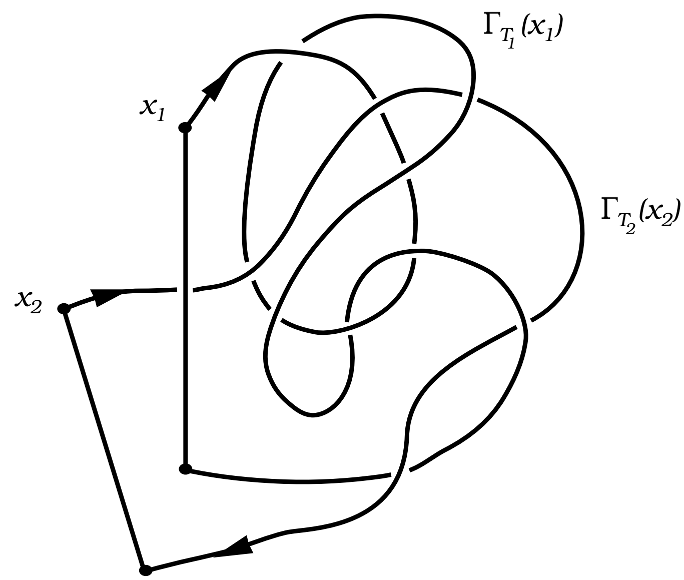


FIGURE 31. The long segments of the trajectories are closed by the "short" paths.

Below we will see that this limit exists almost everywhere and is independent of the system of "short" paths Δ (as an element of the space $L_1(M \times M)$ of the Lebesgue-integrable functions on $M \times M$).

DEFINITION 4.2. The *average (self-)linking number* of a field ξ is the integral over $M \times M$ of the asymptotic linking number $\lambda_\xi(x_1, x_2)$ of the field trajectories:

$$\lambda_\xi = \int_M \int_M \lambda_\xi(x_1, x_2)\ \mu_1 \mu_2. \tag{4.1}$$

REMARK 4.3. The average self-linking number can be defined via an auxiliary step by specifying what the asymptotic linking of field lines with a closed curve is and then by replacing the curve with another orbit. This approach is used in Section 5 to define the average crossing number.

THEOREM 4.4 (HELICITY THEOREM, [Arn9]). *The average self-linking of a divergence-free vector field ξ on a simply connected manifold M with a volume element μ coincides with the field's helicity:*

$$\lambda_\xi = \mathcal{H}(\xi). \tag{4.2}$$

EXAMPLE 4.5. For the Hopf vector field $\mathbf{v}(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$ on the unit sphere $S^3 \subset \mathbb{R}^3$, the linking number of every two orbits (great circles) is equal to 1. All the orbits are periodic with the same period 2π. Hence, the value of

$\lambda_{\mathbf{v}}(x_1, x_2)$, being the asymptotic linking of two trajectories per time unit, is $1/4\pi^2$. The average self-linking number of the Hopf field is

$$\lambda_{\mathbf{v}} = \int\limits_{S^3}\int\limits_{S^3} \lambda_{\mathbf{v}}(x_1, x_2)\ \mu_1\mu_2 = \int\limits_{S^3}\int\limits_{S^3} \frac{1}{4\pi^2}\ \mu_1\mu_2 = \frac{(\mathrm{vol}(S^3))^2}{4\pi^2} = \frac{(2\pi^2)^2}{4\pi^2} = \pi^2,$$

which coincides with the mean helicity $\mathcal{H}(\mathbf{v})$ of the field $\mathbf{v}$; see Example 1.19.

REMARK 4.6. The result can be literally generalized to the case of two different divergence-free fields ξ and η on a simply connected M. The linking number $\lambda_{\xi,\eta}(x, y)$ in the latter case measures the asymptotic linkage of the trajectories $g_\xi^t x$ and $g_\eta^t y$ issuing from x and y respectively. The helicity is replaced by the bilinear form $\mathcal{H}(\xi, \eta)$; see Remark 1.22. The Helicity Theorem states in this case that

$$\int\limits_M\int\limits_M \lambda_{\xi,\eta}(x, y)\mu_x\mu_y = \int\limits_M \omega_\xi \wedge (d^{-1}\omega_\eta),$$

where the 2-forms are defined by $\omega_\xi = i_\xi\mu$, $\omega_\eta = i_\eta\mu$, and $d^{-1}\omega_\eta$ denotes an arbitrary potential 1-form α such that $d\alpha = \omega_\eta$.

In the case of a manifold M with boundary, all the vector fields involved are supposed to be tangent to the boundary.

REMARK 4.7. The identity of the two classical definitions of the Hopf invariant (being a nonergodic version of the Helicity Theorem; see Example 1.19) is a manifestation of Poincaré duality.

Assume that we deal with singular forms (of δ-type) supported on compact submanifolds. Replace the differential forms by their supports. Then the operations d^{-1} and $\wedge$ correspond to the passage from the support submanifolds to the film bounded by them and to their intersections, respectively. Finally, the integration $\int_M$ is summation of the intersection points with the corresponding signs. The intersection of a submanifold with a film bounded by another submanifold is the linking number of these two submanifolds.

The consideration of smooth differential forms instead of singular ones leads to the averaging of appropriate linking characteristics. The asymptotic version of the linking number can be regarded in the context of *asymptotic cycles* [SchS, DeR, GPS, Sul].

A counterpart of the homotopy invariance of the classical Hopf invariant is unknown for the asymptotic linking number:

PROBLEM 4.8. Is the average self-linking number of a divergence-free vector field invariant under the action of homeomorphisms preserving the measure on the manifold? Here, a measure-preserving homeomorphism is supposed to transform the flow of one smooth divergence-free vector field into the flow of the other, both fields having well-defined average self-linking numbers.

A partial (affirmative) answer to this question was given in [G-G], where the average linking number for a field in a solitorus was related to the topological invariants of Ruelle [Rue] and Calabi [Ca] for disk diffeomorphisms (see also Sections III.7.A and IV.8.B).

We will give two versions of the proof of the Helicity Theorem. The first one makes explicit use of the Gauss linking formula and of the Biot-Savart integral in $\mathbb{R}^3$. The second, coordinate-free, version reveals the reason for the helicity–linking coincidence on an arbitrary simply connected manifold.

Various generalizations of asymptotic linking are discussed in subsequent sections.

4.B. Digression on the Gauss formula. To state the formula given by Gauss for the linking number of two closed curves in three-dimensional Euclidean space, we introduce the following notation.

Let $\gamma_1 : S^1_1 \to \mathbb{R}^3$ and $\gamma_2 : S^1_2 \to \mathbb{R}^3$ be smooth mappings of two circumferences to $\mathbb{R}^3$ with disjoint images. Let $t_1 (\mathrm{mod}\ T_1)$ and $t_2 (\mathrm{mod}\ T_2)$ be coordinates on the first and second circumferences. We denote by $\dot\gamma_i = \dot\gamma_i(t_i)$, $i = 1, 2$, the corresponding velocity vectors in the images (Fig.32).

Assume that the circumferences are oriented by the choice of the parameters t_1 and t_2, and fix an orientation for $\mathbb{R}^3$. Then we can define vector products and triple scalar products in $\mathbb{R}^3$.

THEOREM 4.9 (GAUSS THEOREM). *The linking number of the closed curves $\gamma_1(S^1)$ and $\gamma_2(S^1)$ in $\mathbb{R}^3$ is equal to*

$$lk(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_0^{T_1} \int_0^{T_2} \frac{(\dot\gamma_1, \dot\gamma_2, \gamma_1 - \gamma_2)}{||\gamma_1 - \gamma_2||^3}\ dt_1 dt_2.$$

PROOF. Consider the mapping

$$f : \ T^2 \to S^2$$

from the torus to the sphere sending a pair of points on our circumferences to the vector of unit length directed from $\gamma_2(t_2)$ to $\gamma_1(t_1)$: $f = F/\|F\|$, where $F(t_1, t_2) = \gamma_1(t_1) - \gamma_2(t_2)$; see Fig.32.

We orient the sphere by the inner normal and the torus by the coordinates t_1, t_2.

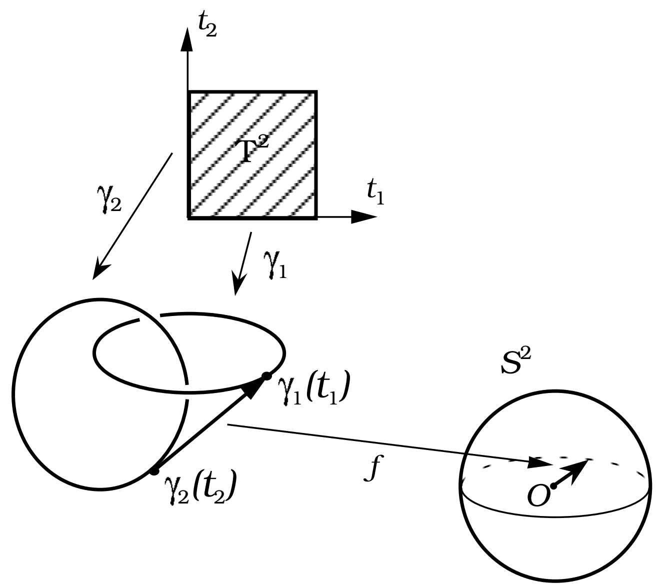


FIGURE 32. Two parametrized linked curves in space define the Gauss map $T^2 \to S^2$.

LEMMA 4.10. *The degree of the mapping f is equal to the linking number $lk(\gamma_1, \gamma_2)$.*

Indeed, this is true for small circumferences situated far away from each other: Both the linking number and the degree of the mapping f are 0; cf. Fig.32. Neither of these quantities changes in the course of any deformation that leaves the curves disjoint. Furthermore, it is easy to verify that under any deformation of the pair of curves containing a passage of one curve through another, both the linking number and the degree of the mapping change by 1 with the same sign. Therefore, the equality $lk(\gamma_1, \gamma_2) = \deg\ f$ follows, in view of the connectedness of the set of smooth mappings $S^1 \to \mathbb{R}^3$.

Now the Gauss Theorem is a consequence of the following lemma.

LEMMA 4.11. *The degree of the mapping $f:\ T^2 \to S^2$ is given by the Gauss integral formula.*

PROOF OF LEMMA 4.11. By definition of the degree,

$$\deg\ f = \frac{1}{4\pi} \iint_{T^2} f^* \nu^2,$$

where the 2-form ν^2 is the area element on the unit sphere. Now, by definition of f, the value of the form $f^*\nu^2$ on a pair of vectors a_1, a_2 tangent to the torus at $t = (t_1, t_2) \in T^2$ is equal to their mixed product with the vector $-f := -f(t)$ (we oriented the sphere by means of the *inner* normal):

$$f^*\nu^2(a_1, a_2) = \nu^2(f_* a_1, f_* a_2) = (f_* a_1, f_* a_2, -f).$$

By differentiating f, we obtain $f_* a = F_* a / \|F\| + c(a,f) f$ (here $c(a,f)$ is a scalar factor), and therefore

$$\nu^2(f_* a_1, f_* a_2) = (F_* a_1, F_* a_2, -F)/\|F\|^3.$$

Recalling that $F = x_1 - x_2$, we obtain the expression

$$f^* \omega^2 = (\dot{x}_1, \dot{x}_2, x_1 - x_2)\|x_1 - x_2\|^{-3} dt_1 \wedge dt_2$$

for an element of the spherical image of the torus, as was to be shown. □

The higher-dimensional version of the Gauss linking formula, developed in [Poh, Wh], is based on the same observation about equivalence of the linking and the degree of the Gauss map.

4.C. Another definition of the asymptotic linking number. Let $\{g^t\}$ be the phase flow defined by a divergence-free field ξ in a three-dimensional compact Euclidean domain $M \subset \mathbb{R}^3$. The field is assumed to be tangent to the boundary ∂M.

Define the Gauss linking of the ξ-trajectories as

$$\Lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{4\pi \cdot T_1 T_2} \int_0^{T_2} \int_0^{T_1} \frac{(\dot{x}_1(t_1), \dot{x}_2(t_2), x_1(t_1) - x_2(t_2))}{\|x_1(t_1) - x_2(t_2)\|^3} \, dt_1 dt_2,$$

where $x_i(t_i) = g^{t_i}(x_i)$ is the trajectory of the point x_i, and $\dot{x}_i(t_i) = \frac{d}{dt_i} g^{t_i} x_i$ is the corresponding velocity vector.

Lemma 4.12.

1) *The limit $\Lambda_\xi(x_1, x_2)$ exists almost everywhere on $M \times M$.*
2) *The value $\Lambda_\xi(x_1, x_2)$ coincides with the number $\lambda_\xi(x_1, x_2)$ defined above for almost all x_1, x_2.*

Proof. To prove the first statement, it is enough to verify that Λ is the "time average" of an integrable function on the manifold $M \times M$, on which the abelian group $\{g^{t_1}\} \times \{g^{t_2}\}$ acts. The integrand is the function

$$G(x_1, x_2) = \frac{(a_1, a_2, x_1 - x_2)}{\|x_1 - x_2\|^3},$$

where $a_k = \frac{d}{dt_k}\big|_{t_k = 0} g^{t_k} x_k = \xi(x_k)$. The function G has a singularity on the diagonal of $M \times M$: It grows at most like r^{-2}, where r is the distance to the diagonal. Since the codimension of the diagonal is 3, the function G belongs to the space $L^1(M \times M)$, as required.

To compare Λ_ξ with λ_ξ, we represent the linking coefficient of the curves $\Gamma_1 = \Gamma_{T_1}(x_1)$ and $\Gamma_2 = \Gamma_{T_2}(x_2)$ by the Gauss integral with $0 \le t_1 \le T_1 + 1$, $0 \le t_2 \le T_2 + 1$, by using the value of the parameter t_k from T_k to $T_k + 1$ for parametrizing the "short path" $\Delta(g^{T_k} x_k, x_k)$ that joins $g^{T_k} x_k$ to x_k.

DEFINITION 4.13. A *system of short paths* joining every two points in M is a system of paths depending in a measurable way on the points x and y in M and obeying the following condition. The integrals of Gauss type for every pair of nonintersecting paths of the system, and also for every nonintersecting pair (a path of the system, a segment of the phase curve $g^t x$, $0 \le t \le 1$), are bounded independently of the pair by a constant C.

REMARK 4.14. One can verify that systems of short paths exist for nowhere vanishing vector fields or even for generic vector fields (with isolated zeroes). It is useful to keep in mind that an integral of Gauss type for a pair of straight-line segments remains bounded when these segments approach each other. The phenomenon one has to avoid is the winding of a trajectory around a path of the system, which implies unboundedness of the integral. However, a small perturbation of the short path system leads to a system satisfying the condition above.

Indeed, the phenomenon of winding does not occur in systems where there is $N \in \mathbb{Z}_+$ such that at any point of the manifold M at least one of the derivatives of Δ (along the paths) of order less than N does not coincide with that of g^t (along the flow). Given a vector field ξ (or equivalently, given flow g^t), the systems of short paths Δ subject to the latter constraint form an ample set (cf. the strong transversality theorem [AVG]).

The fields with nonisolated zeroes constitute a set of infinite codimension in the space of all vector fields. For such vector fields, the existence question of systems of short paths is more subtle, and there still are some unresolved issues related to it.[1] It would be very interesting to complete the proof of existence in full generality.

Now, the difference $\int_0^{T_2+1}\int_0^{T_1+1} - \int_0^{T_2}\int_0^{T_1}$ of Gauss-type integrals can be estimated by the sum of at most $[T_1] + [T_2] + 1$ terms, none of which exceeds C. Therefore,

$$\lambda_\xi(x_1, x_2) - \Lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{4\pi \cdot T_1 T_2} \left(\int_0^{T_2+1}\int_0^{T_1+1} - \int_0^{T_2}\int_0^{T_1} \right) = 0$$

[1] We are grateful to P. Laurence, who noted that an existence proof would require some kind of "global approach," considering vector fields on the whole manifold, while the transversality theorem is "local" in nature.

(where T_1 and T_2 tend to infinity over any sequence for which the curves $\Gamma_1 = \Gamma_{T_1}(x_1)$ and $\Gamma_2 = \Gamma_{T_2}(x_2)$ do not meet). □

Now we complete the proof of the Helicity Theorem on the equivalence of the ergodic and integral definitions of the helicity of a divergence-free vector field defined in a domain $M \subset \mathbb{R}^3$ (and tangent to the boundary ∂M).

Consider the Biot–Savart integral

$$A(x_2) = -\frac{1}{4\pi}\int_M \frac{\xi(x_1)\times(x_1 - x_2)}{\|x_1 - x_2\|^3}\ \mu(x_1)$$

(where $\times$ denotes the cross product) that defines a vector-potential $A = \operatorname{curl}^{-1}\xi$ in $\mathbb{R}^3$. It allows one to obtain the integral representation of the helicity

$$\mathcal{H}(\xi) = \langle \xi, \operatorname{curl}^{-1}\xi\rangle = \langle \xi, A\rangle = \frac{1}{4\pi}\iint\limits_{M\times M} \frac{(\xi(x_1), \xi(x_2), x_1 - x_2)}{\|x_1 - x_2\|^3}\ \mu(x_1)\ \mu(x_2).$$

The Helicity Theorem follows from this formula and from the Birkhoff ergodic theorem applied to the integrable function $(\xi(x_1), \xi(x_2), x_1 - x_2)/(4\pi\|x_1 - x_2\|^3)$ on $M \times M$. The space average

$$\frac{\iint\limits_{M\times M} \Lambda_\xi(x_1, x_2)\ \mu(x_1)\ \mu(x_2)}{(\operatorname{vol}(M))^2} = \frac{\lambda_\xi}{(\operatorname{vol}(M))^2}$$

of the time average Λ_ξ along the trajectories of the measure-preserving flow of ξ coincides with the space average $\mathcal{H}(\xi)/(\operatorname{vol}(M))^2$ of the function. □

REMARK 4.15. Note that for an ergodic field (ξ, ξ) on $M \times M$ the function $\Lambda_\xi(x_1, x_2)$ is constant almost everywhere: The asymptotic linking numbers for almost all pairs of ξ-trajectories are equal to each other.

4.D. Linking forms on manifolds. Here we show how the preceding arguments can be adjusted to the case of an arbitrary simply connected manifold, where the Gauss-type integral of De Rham's "double form" [DeR] cannot be written as explicitly as in $\mathbb{R}^3$ (see [KhC]).

THEOREM 4.16 (=4.6′). *The average linking number of two divergence-free vector fields ξ and η coincides with $\mathcal{H}(\xi, \eta)$:*

$$\iint\limits_{M\times M} \lambda_{\xi,\eta}(x, y)\mu_x\mu_y = \int_M i_\xi\mu \wedge d^{-1}(i_\eta\mu).$$

PROOF. We start by recalling some facts about double bundles and linking forms. Denote by $\Omega^k(M)$ the space of differential k-forms on a manifold M.

DEFINITION 4.17. A differential 2-form $G \in \Omega^2(M \times M)$ is called a *Gauss–De Rham linking form* on a simply connected manifold M if for an arbitrary pair of nonintersecting closed curves Γ_1 and Γ_2 the integral of this form over $\Gamma_1 \times \Gamma_2$ equals the corresponding linking number:

$$\iint\limits_{\Gamma_1\times\Gamma_2\subset M\times M} G = lk(\Gamma_1, \Gamma_2).$$

Here $\Gamma_1 \times \Gamma_2 = \{(x, y) \in M \times M \mid x \in \Gamma_1,\ y \in \Gamma_2\}$. The existence of such a form will be established later.

DEFINITION 4.18. Each differential form $K(x, y) \in \Omega^*(M \times M)$ determines an *operator* $\tilde{K} : \Omega^*(M) \to \Omega^*(M)$ on the space of differential forms $\Omega^*(M)$ on M that sends a differential form $\varphi(y)$ into the differential form

$$(\tilde{K}\varphi)(x) = \int\limits_{\pi^{-1}(x)} K(x, y) \wedge \varphi(y),$$

where $\pi : M\times M \to M$ is the projection on the first component, and the integration is carried out over the fibers of this projection; see Fig.33. The value of the form $\tilde{K}\varphi$ at a point $x \in M$ is the integral over the fiber $\pi^{-1}(x) \subset M \times M$ of the wedge product $K(x, y) \wedge \varphi(y)$. If the product $K(x, y) \wedge \varphi(y)$ is an n-form in y, then by definition, $(\tilde{K}\varphi)(x) = 0$.

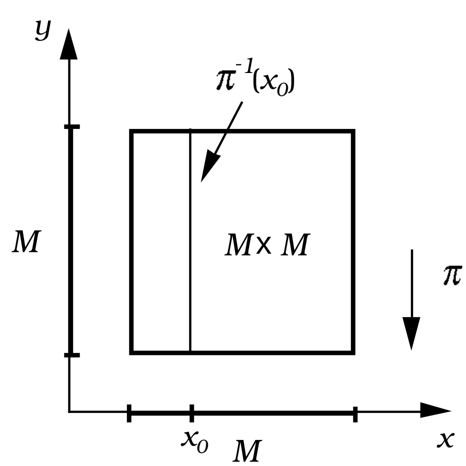


FIGURE 33. Any form on $M \times M$ defines an operator on $\Omega^*(M)$.

PROPOSITION 4.19. *The operator $\widetilde{G}$ corresponding to the linking form is the Green operator inverse to the exterior derivative of 1-forms: If $\psi = d\varphi$ and $\varphi \in \Omega^1(M)$, then*

$$\varphi = \widetilde{G}(\psi) + dh$$

for a certain function h.

The term dh materializes the fact that a potential 1-form φ can be reconstructed from an exact 2-form ψ modulo a full differential only.

PROOF OF PROPOSITION. Let $\mathbf{d} = d_x + d_y$ be the operator of the exterior derivative on $\Omega^*(M \times M)$.

LEMMA 4.20. $\widetilde{d_x K} = d \circ \widetilde{K}$.

Indeed, $[d_x K(x,y)] \wedge \varphi(y) = d_x[K(x,y) \wedge \varphi(y)]$, and hence

$$\int\limits_{\pi^{-1}(x)} [d_x K(x,y)] \wedge \varphi(y) = d\left(\int\limits_{\pi^{-1}(x)} K(x,y) \wedge \varphi(y)\right).$$

LEMMA 4.21. *If K is a 1-form in the variable y, then $\widetilde{d_y K} = \widetilde{K} \circ d$.*

This follows from the identity

$$\int\limits_{\pi^{-1}(x)} [d_y K(x,y)] \wedge \varphi(y) = \int\limits_{\pi^{-1}(x)} K(x,y) \wedge d\varphi(y).$$

LEMMA 4.22. *The exterior derivative of a Gauss–De Rham form G on $M \times M$ is the sum $\mathbf{d}G = \delta + \beta$ of*

- *the δ-form on the diagonal $\Delta \subset M \times M$ (the integral of the δ-form over any 3-chain in $M \times M$ is equal to the algebraic number of intersection points of the chain with the diagonal Δ), and of*
- *some form $\beta \in \Omega^3(M \times M)$ that is a linear combination of forms from $\Omega^k(M) \otimes \Omega^{3-k}(M)$-forms with each factor being closed.*

PROOF.

$$lk(\Gamma_1, \Gamma_2) = \iint\limits_{\Gamma_1 \times \Gamma_2} G = \iiint\limits_{\partial^{-1}(\Gamma_1 \times \Gamma_2)} \mathbf{d}G = \iiint\limits_{(\partial^{-1}\Gamma_1) \times \Gamma_2} \mathbf{d}G.$$

On the other hand, since the linking number is the intersection number of the cycle Γ_2 and a surface $\partial^{-1}\Gamma_1$ (whose boundary is Γ_1), it can be represented as the integral of the δ-form over the chain $(\partial^{-1}\Gamma_1) \times \Gamma_2$:

$$lk(\Gamma_1, \Gamma_2) = \iiint\limits_{(\partial^{-1}\Gamma_1) \times \Gamma_2} \delta.$$

Now the statement follows from the fact that all those β's are closed, and each β is characterized by the conditions

$$\iiint_{(\partial^{-1}\Gamma_1)\times\Gamma_2} \beta = 0 \quad\text{and}\quad \iiint_{\Gamma_1\times(\partial^{-1}\Gamma_2)} \beta = 0.$$

□

REMARK 4.23. The form β can be chosen in such a way that the cohomology class of $\delta+\beta$ in $H^3(M\times M)$ is trivial. Indeed, though the class of δ in $H^3(M\times M)=\sum_k H^k(M)\otimes H^{3-k}(M)$ is nontrivial (the diagonal in $M\times M$ is not a boundary), adding an appropriate β we can get rid of the $H^0(M)$- and $H^3(M)$-terms. Hence, the class of $\delta+\beta$ vanishes due to the simple-connectedness of M ($H^1(M)=H^2(M)=\ 0$).

This proves the existence of a Gauss–De Rham linking form G as a solution of the equation $[\mathbf{d}G]=0\in H^3(M\times M)$, where $[*]$ denotes the cohomology class of a differential form.

To complete the proof of Proposition 4.19, we pass from the equation on forms $\mathbf{d}G=\delta+\beta$ to the relation on the corresponding operators: $\widetilde{\mathbf{d}G}=\tilde\delta+\tilde\beta$, or

$$\widetilde{d_xG}+\widetilde{d_yG}=\tilde\delta+\tilde\beta.$$

At this point we notice that

a) the δ-form corresponds to the identity operator $\tilde\delta=\mathrm{Id}$, and
b) the image of the operator $\tilde\beta$ in $\Omega^*(M)$ belongs to the subspace of closed forms (see Lemma 4.22). In particular, within $\Omega^1(M)$ the image consists of the exact forms.

Combining these facts with Lemmas 4.20-21, we come to the relation

$$d\circ\widetilde{G}+\widetilde{G}\circ d=\mathrm{Id}+d\circ\tilde\gamma$$

for operators on one-forms in M. Having applied the operators of both sides of the relation to a form φ and rearranging the terms, one transforms this relation into

$$d\circ(\widetilde{G}(\varphi)-\tilde\gamma(\varphi))+\widetilde{G}(d\varphi)=\varphi.$$

Finally, for $\psi=d\varphi$, we obtain

$$\varphi=\widetilde{G}(\psi)+dh$$

for some function h. □

LEMMA 4.24. *There exists a Gauss–De Rham linking form $G(x, y)$ with a pole of order 2 on the diagonal of $M \times M$.*

PROOF. The linking number of Γ_1 with Γ_2 by definition coincides with the linking number of $\Gamma_1 \times \Gamma_2$ with the diagonal Δ in $M \times M$. Identify a neighborhood of the diagonal in $M \times M$ with a neighborhood of the zero section in the normal bundle $T^{\perp}\Delta$ over the diagonal via the geodesic exponential map (Fig.34).

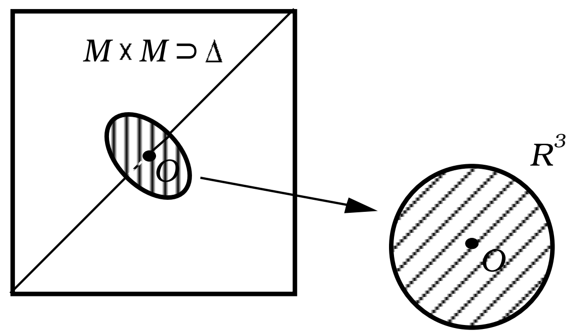


FIGURE 34. For any point of the diagonal $\Delta \subset M \times M$ a neighborhood in the transversal to Δ direction can be identified with a neighborhood in $\mathbb{R}^3$.

Then, in every fiber (being a neighborhood of $0 \in \mathbb{R}^3$), we consider the standard Gauss linking form singular at the origin. The latter is the 2-form obtained by the substitution of the radius vector field $\nabla(1/r)$ into the standard volume element in $\mathbb{R}^3$. It has a pole of order 2 at the origin. Extend the definition of this form from one fiber to the entire neighborhood of Δ in $M \times M$ by prescribing that this form vanishes on vectors parallel to $\Delta \subset T^{\perp}\Delta$. We obtain a linking form in $M \times M$ that has a pole of the desired order 2 on the diagonal. $\square$

COROLLARY 4.25. *The linking form G is integrable: $G \in L^1(M \times M)$, i.e., the value of G evaluated on any two smooth vector fields is an integrable function on $M \times M$.*

Indeed, the codimension of the diagonal in $M \times M$ equals 3, and the growth order of the form G near the diagonal is 2.

REMARK 4.26. All the above arguments on the Gauss–De Rham linking forms hold (with certain evident modifications) for manifolds of arbitrary dimension. Further consideration in this section is essentially three-dimensional.

Let ξ and η be divergence-free fields on M equipped with a volume form μ. Let $g^t_\xi x$ and $g^s_\eta y$ be the segments of the trajectories of these fields starting at x and y for time intervals $0 \le t \le T$ and $0 \le s \le S$. Denote by Δ_x and Δ_y the corresponding "short paths" closing the segments of the trajectories and making them into two piecewise smooth closed curves.

The asymptotic linking number is equal to

$$\lambda_{\xi,\eta}(x,y) = \lim_{T,S\to\infty} \frac{1}{T\cdot S} \iint\limits_{(g_\xi^t x\cup\Delta_x)\times(g_\eta^s y\cup\Delta_y)} G = \lim_{T,S\to\infty} \frac{1}{T\cdot S} \iint\limits_{g_\xi^t x\times g_\eta^t y} G.$$

The last equality of the limits follows from the boundedness of the integrals over the short paths (see Definition 4.13 of a short paths system). Hence,

$$\begin{aligned}\lambda_{\xi,\eta} &= \iint\limits_{M\times M} \lambda_{\xi,\eta}(x,y)\ \mu_x\mu_y = \int_M \mu_x \int_M \mu_y \left(\lim_{T,S\to\infty} \frac{1}{T\cdot S} \iint\limits_{g_\xi^t x\times g_\eta^t y} G\right)\\ &= \int_M \mu_x \int_M \mu_y \left(\lim_{T,S\to\infty} \frac{1}{T\cdot S} \int_0^T \int_0^S (i_\xi i_\eta G)\ dsdt\right),\end{aligned}$$

where $i_\xi i_\eta G$ is regarded as a function on $M\times M$ and $\int_0^T\int_0^S$ denotes the integral of this function over the product of (the pieces of) the trajectories $g_\xi^t x$ and $g_\eta^s y$.

By the Birkhoff ergodic theorem applied to the integrable function $i_\xi i_\eta G$, we can pass from the time averages to the space average:

$$\lambda_{\xi,\eta} = \iint\limits_{M\times M} \lambda_{\xi,\eta}(x,y)\ \mu_x\mu_y = \int\limits_M \mu_x \int\limits_M \mu_y\ (i_\xi i_\eta G).$$

Finally, shift the substitution operators i_ξ and i_η from G to the forms μ_x and μ_y (the operation i_ξ is inner differentiation; see Section 1):

$$\begin{aligned}\lambda_{\xi,\eta} &= \iint\limits_{M\times M} \lambda_{\xi,\eta}(x,y)\ \mu_x\mu_y = \int\limits_M \mu_x \int\limits_M \mu_y\ (i_\xi i_\eta G) = \int\limits_M i_\xi\mu_x \wedge \left(\int\limits_M i_\eta\mu_y \wedge G\right)\\ &= \int\limits_M i_\xi\mu \wedge \widetilde{G}(i_\eta\mu).\end{aligned}$$

By Proposition 4.19 the operator $\widetilde{G}$ is inverse to exterior differentiation: $\widetilde{G}(i_\eta\mu) = d^{-1}(i_\eta\mu)$ modulo an exact form. This completes the proof of Theorem 4.16:

$$\lambda_{\xi,\eta} = \int\limits_M i_\xi\mu \wedge d^{-1}(i_\eta\mu) = \mathcal{H}(\xi,\eta).$$

$\square$

§5. Asymptotic crossing number

The helicity approach to magnetic energy minoration in terms of the topology of magnetic lines was generalized by Freedman and He [FH1,2] by introducing the notion of asymptotic crossing number. They determined the complexity of a knotted orbit by the "minimal number of crossings" in its projections. It replaces the linking number, where the crossings are counted with appropriate signs. In the presentation below we mostly follow the paper [FH2].

5.A. Energy minoration for generic vector fields.

DEFINITION 5.1. For two closed curves γ_1 and γ_2 in $\mathbb{R}^3$ the *crossing number* $c(\gamma_1, \gamma_2)$ is equal to the integral of the absolute value of the Gauss integrand for their linking number:

$$c(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_0^{T_1} \int_0^{T_2} \frac{|(\dot{\gamma}_1, \dot{\gamma}_2, \gamma_1 - \gamma_2)|}{||\gamma_1 - \gamma_2||^3} \, dt_1 dt_2. \tag{5.1}$$

This quantity is no longer invariant under a curve isotopy. However, all the notions and definitions regarding the corresponding asymptotic version can be literally transferred to this situation.

For a vector field ξ defined in a domain $M \subset \mathbb{R}^3$ (and tangent to the boundary ∂M), we use the same definition of a "system of short paths" as above (see Definition 4.13 and subsequent Remark 4.14). Denote by $\Gamma_T(x)$ the piece of the ξ-orbit of $x \in M$ run in the time period $[0, T]$ and closed by a short path.

DEFINITION 5.2. The *asymptotic crossing number* of the field lines of a divergence-free vector field ξ with a closed curve γ in a simply connected manifold M^3 is the limit

$$c_\xi(x, \gamma) = \lim \sup_{T \to \infty} \frac{1}{T} \, c(\Gamma_T(x), \gamma).$$

This limit exists, belongs to $L^1(M)$, and is well defined in $L^1(M)$ in spite of the ambiguity in the choice of the system of short curves.

Similarly, the *average crossing number* of the field lines of ξ with the curve γ is given by the integral

$$c_\xi(\gamma) = \int_M c_\xi(x, \gamma) \, \mu_x,$$

where μ is a volume form on M.

Finally, given two divergence-free vector fields ξ and η, their *asymptotic crossing number* $Cr(\xi,\eta)$ is defined as the space integral of the crossing number of one of the fields with the trajectories of the other:

$$Cr(\xi,\eta) = \int_M (\limsup_{T\to\infty} \frac{1}{T} c_\xi(\Gamma_T(y))\ \mu_y,$$

where $\Gamma_T(y)$ is the piece $0 \le t \le T$ of the ξ-field line issuing from the point y and closed by a short path. This crossing number admits the integral representation

$$Cr(\xi,\eta) = \frac{1}{4\pi} \iint\limits_{M\times M} \frac{|(\xi(x),\eta(y),x-y)|}{||x-y||^3}\ \mu_x\mu_y. \tag{5.2}$$

The asymptotic crossing number yields the following lower bound for the $E_{3/2}$-*energy* $E_{3/2}(\xi) := \int_M \|\xi\|^{3/2}\ \mu$.

THEOREM 5.3 [FH2]. *For any divergence-free vector field ξ in M*

$$E_{3/2}(\xi) \ge \left(\frac{16}{\pi}\right)^{1/4} Cr(\xi,\xi)^{3/4}. \tag{5.3}$$

REMARKS 5.4 [FH2]. A) The $L^{3/2}$-norm used in the definition of the $E_{3/2}$-energy is justified by the "conformal nature" of the problem. Any lower bound for the $E_{3/2}$-energy implies a lower bound for the standard E_2-energy $E_2(\xi) := \int_M \|\xi\|^2\ \mu$ due to a straightforward application of the Hölder inequality:

$$E_2(\xi) \ge \frac{(E_{3/2}(\xi))^{4/3}}{(\mathrm{vol}(M))^{1/3}} \ge \left(\frac{16}{\pi\cdot \mathrm{vol}(M)}\right)^{1/3} Cr(\xi,\xi), \tag{5.4}$$

or, in a more recognizable form, $\int(\|\xi\|^{3/2}\cdot 1) \le (\int \|\xi\|^2)^{3/4}\cdot(\mathrm{vol}(M))^{1/4}$.

B) Similarly, for any two divergence-free vector fields ξ and η in M,

$$Cr(\xi,\eta) \le \left(\frac{16}{\pi}\right)^{1/4} \left(E_{3/2}(\xi)\right)^{2/3}\cdot\left(E_{3/2}(\eta)\right)^{2/3}.$$

C) Both sides of the inequality have geometric nature (they rely on a particular choice of metric) and are not topologically invariant. On the other hand, the energy estimate in terms of the helicity gives a topological bound for a geometric quantity.

One can make the right-hand side of the inequality (5.3) topological by brute force, defining the *topological crossing number*

$$Cr_{top}(\xi,\eta) = \inf_{h\in \mathrm{Diff}(\mathbb{R}^3)} Cr(h_*\xi, h_*\eta).$$

Then

$$E_{3/2}(h_*\xi) \geq (\frac{\pi}{16})^{1/4} Cr_{top}(\xi,\xi)^{3/4},$$

for any $h \in \text{Diff}(\mathbb{R}^3)$.

D) Theorem 5.3 holds for vector fields with an arbitrary divergence, provided that $Cr(\xi,\xi)$ is defined by the integral formula (5.2) and not ergodically. Having used the integral definition of the helicity as well (see Definition 1.3), one obtains

$$E_{3/2}(h_*\xi) \geq \left(\frac{\pi}{16}\right)^{1/4} |\mathcal{H}(\xi)|^{3/4}, \quad \text{for any } h \in \text{Diff}(M \subset \mathbb{R}^3),$$

by virtue of the evident inequality $Cr(\xi,\xi) \geq |\mathcal{H}(\xi)|$.

REMARK 5.5. A two-dimensional version of the asymptotic crossing number has been developed and applied to energy estimates of the braided magnetic tubes in [Be2]. In this case the energy lower bound appears to be quadratic in the total crossing number of a braided field, while the energy of a knotted field in three-dimensional space is bounded by an expression linear in Cr (see the estimate (5.4) for the E_2-energy above).

PROOF OF THEOREM 5.3. The integral form of the asymptotic crossing number yields the following upper bound:

$$\begin{aligned} Cr(\xi,\xi) &= \frac{1}{4\pi} \int_M \int_M \frac{|(\xi(x),\xi(y),x-y)|}{||x-y||^3}\ \mu_x\mu_y \\ &\leq \frac{1}{4\pi} \int_M \int_M ||\xi(y)|| \left(\frac{||\xi(x)||}{||x-y||^2}\right)\ \mu_x\mu_y = \int_M ||\xi(y)||\ \rho(y)\ \mu_y, \end{aligned}$$

where the density $\rho : \mathbb{R}^3 \to \mathbb{R}^+$ is defined as

$$\rho(y) = \frac{1}{4\pi} \int_M \frac{||\xi(x)||}{||x-y||^2}\ \mu_x.$$

By the Hardy–Littlewood–Sobolev inequality [Sob1, Lieb] in potential theory, one obtains

$$||\rho||_{L^3} = \left(\int_M \rho^3\ \mu\right)^{1/3} \leq \left(\frac{\pi}{16}\right)^{1/3} \left(\int_M ||\xi||^{3/2}\ \mu\right)^{2/3}.$$

After combining it with Hölder's inequality one sees that

$$\begin{aligned} Cr(\xi,\xi) &\leq \int_M ||\xi(y)||\ \rho(y)\ \mu_y \\ &\leq ||\xi||_{L^{3/2}} \cdot ||\rho||_{L^3} \leq \left(\frac{\pi}{16}\right)^{1/3} (||\xi||_{L^{3/2}})^2, \end{aligned}$$

and the theorem follows. □

5.B. Asymptotic crossing number of knots and links. Apparently, any reasonably sharp estimates of Cr_{top} for a fairly generic field ξ are beyond reach. However, much more can be done under the (already exploited) assumption that the vector field has some linked or knotted invariant tori.

DEFINITIONS 5.6. The *crossing number* $cn(K)$ (or $cn(L)$) of a knot K (or link L) in $\mathbb{R}^3$ is the minimum number of crossings of all plane diagrams representing the knot (or the link).

Consider some tubular neighborhood $\mathbf{T}$ of the (oriented) knot K. An arbitrary closed oriented curve confined to the neighborhood is said to be of *degree* p if it can be isotoped within $\mathbf{T}$ to the curve that is K covered p times.

A two-component link (P, Q) in $\mathbb{R}^3$ is called a *degree* (p, q) *satellite link* of K (p and q are positive integers) if (P, Q) can be (simultaneously) isotoped to a pair of curves $(P', Q') \subset \mathbf{T}$ with $degree(P') = p$ and $degree(Q') = q$. The *over-crossing number* $cn(P, Q)$ of the link (P, Q) is defined to be the minimum number of overcrossings of P over Q among all planar knot diagrams representing (P, Q); see Fig.35.

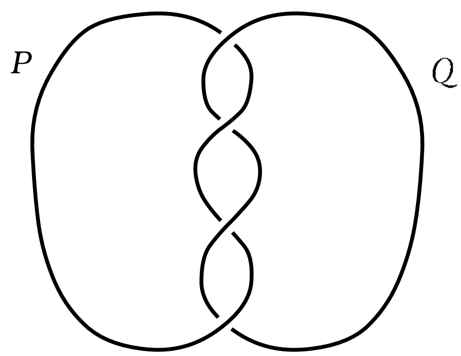


FIGURE 35. The crossing number of this link $L = P \cup Q$ is $cn(L) = 4$. The over-crossing number is $cn(P, Q) = 2$.

Let $cn_{p,q}(K)$ be the minimum of $cn(P, Q)$ over all degree (p, q) satellite links (P, Q) of K. Define the *asymptotic crossing number* of the knot K to be

$$ac(K) = \lim \inf_{p,q \to \infty} cn_{p,q}(K)/pq = \inf\{cn_{p,q}(K)/pq \mid p, q \geq 1\}. \tag{5.5}$$

REMARK 5.7. The equivalence of the two definitions of $ac(K)$ follows from the construction of an analogue of a k-fold alternate diagram for a degree (p, q) satellite that represents a (kp, kq) satellite. The number of crossings of the (smartly chosen) degree (kp, kq) satellite differs from that of the degree (p, q) satellite by the factor k^2; see [FH2].

Obviously, $ac(K) \leq cn(K)$, since $cn(P,Q) \leq cn(K)$ for P and Q taken to be copies of a minimal knot diagram.

CONJECTURE 5.8 [FH2]. $ac(K) = cn(K)$.

THEOREM 5.9 [FH2]. *For a divergence-free field ξ defined in the solid torus $\mathbf{T}$ of knot type K and parallel to the boundary $\partial\mathbf{T}$ one has the inequality*

$$Cr(\xi,\xi) \geq |Flux(\xi)|^2 ac(K).$$

COROLLARY 5.10. $Cr_{top}(\xi,\xi) \geq |Flux(\xi)|^2 ac(K)$.

COROLLARY 5.11. *The $E_{3/2}$-energy of such a field ξ yields the following lower bound:*

$$E_{3/2}(\xi) \geq \left(\frac{16}{\pi}\right)^{1/4} |Flux(\xi)|^{3/2} \left(ac(K)\right)^{3/4}.$$

PROOF. Combine the above with Theorem 5.3. □

Notice that the right-hand side of the energy inequality is now topologically invariant.

The estimate can be specified even further in terms of knot invariants (we refer to [FH2] for the details and the proofs). A *Seifert surface* of a knot $K \in \mathbb{R}^3$ is an arbitrary surface embedded in $\mathbb{R}^3$ whose boundary is the knot K. The *genus* of a knot is the minimal genus (number of handles pasted to a disk) of an oriented Seifert surface. By the very definition, the genus is at least 1 for nontrivial knots (an unknot bounds a genuine embedded disk).

THEOREM 5.12 [FH2]. *For any knot K the asymptotic crossing number $ac(K)$ satisfies: $ac(K) \geq 2 \cdot genus(K) - 1$. In particular, $ac(K) \geq 1$ for a nontrivial knot.*

DEFINITION 5.13. For a link $L = (L_1, \ldots, L_k)$, one first chooses a neighborhood consisting of k solitori $\mathbf{T}_1, \ldots, \mathbf{T}_k$ disjointly embedded in $\mathbb{R}^3$. Introduce quantities $cn_{p,q}(L_i; L)$, $i \in \{1, \ldots, k\}$ to be the minimal number of times a curve of degree p in $\mathbf{T}_i$ must pass over (when projected into a plane) a k component link created by choosing degree one curves in $\mathbf{T}_1, \ldots, \mathbf{T}_k$. Similarly, one defines the *asymptotic crossing number* $ac(L_i, L)$ of L_i over L by formula (5.5), with the replacement of $cn_{p,q}(K)$ by $cn_{p,q}(L_i; L)$.

Then for a divergence-free field ξ leaving invariant the link of solid tori,

$$Cr_{top}(\xi,\xi) \geq \left(\sum_{i=1}^{k} ac(L_i, L) \cdot |Flux(\xi|_{\mathbf{T}_i})|\right) \cdot \min_{1 \leq j \leq k}\{|Flux(\xi|_{\mathbf{T}_j})|\}.$$

In particular, for a two-component link of solid tori $(\mathbf{T}_1, \mathbf{T}_2)$, one can deduce that

$$Cr_{top}(\xi, \xi) \geq 2|lk(L_1, L_2) \cdot Flux(\xi|_{\mathbf{T}_1}) \cdot Flux(\xi|_{\mathbf{T}_2})|.$$

Thus certain energy minorations can be obtained from the solution of a purely topological problem of the calculation of the quantities $ac(K)$ and $ac(L_i, L)$ for given types of knots and links of vortex tubes [FH2].

REMARK 5.14. These invariants are finer than the linking numbers, due to the following immediate corollary of the plane projection method of computation of linking numbers:

$$ac(L_i, L) \geq \sum_{i \neq j} |lk(L_i, L_j)|, \ 1 \leq i \leq k.$$

This estimate is useless for configurations with vanishing linking numbers (as the Borromean rings; Fig.26b). A statement similar to Theorem 5.12 provides a lower bound for $ac(L_i, L)$ in terms of the so-called Thurston norm of certain surfaces associated to a link L (see [FH2]). In particular, if L_i is not a trivial component split away from the rest of the link L (say, L_i is one of components of the Borromean rings), then the asymptotic crossing number $ac(L_i, L)$ is minorized by 1.

PROOF OF THEOREM 5.9. Define the *degree* of a (multivalued) function $f : \mathbf{T} \to S^1 = \mathbb{R}/\mathbb{Z}$ to be its homological degree, i.e., the winding number of the function on the solitorus.

LEMMA 5.15. *For a vector field ξ parallel to the boundary $\partial\mathbf{T}$ of a solitorus $\mathbf{T}$*

$$Flux(\xi) = \int_{\mathbf{T}} (\xi, \nabla f) \ \mu, \tag{5.6}$$

for any degree 1 function $f : \mathbf{T} \to \mathbb{R}/\mathbb{Z}$.

PROOF OF LEMMA. Cut the solid torus $\mathbf{T}$ along any surface Σ (representing the generator of $H_2(\mathbf{T}, \partial\mathbf{T})$, Fig.36) to form a cylinder F.

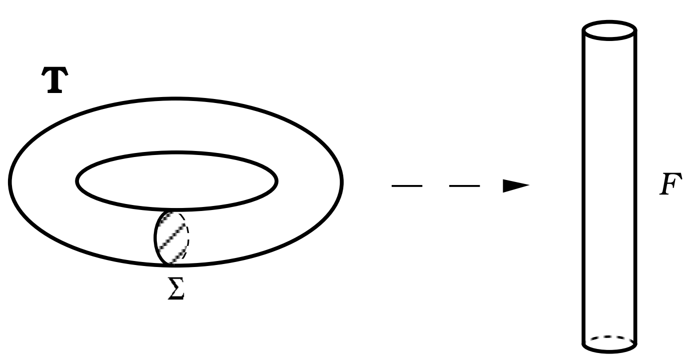


FIGURE 36. Cut the solid torus along Σ to obtain a cylinder.

The function f on $\mathbf{T}$ gives rise to a function $\tilde f : F \to \mathbb{R}$ on the cylinder. The values of $\tilde f$ at the corresponding points of the cylinder top $\partial_+ F$ and bottom $\partial_- F$ differ by 1. Denote by dA the area element on the section Σ. Then

$$\begin{aligned}\int_{\mathbf{T}} (\xi, \nabla f)\ \mu &= \int_F (\xi, \nabla \tilde f)\ \mu = \int_F \operatorname{div}(\tilde f \xi)\ \mu \\ &= \int_{\partial_+ F} (\tilde f \xi, n)\ dA + \int_{\partial_- F} (\tilde f \xi, n)\ dA \\ &= \int_\Sigma ([\tilde f(top(x)) - \tilde f(bottom(x)]\ \xi, n)\ dA(x) = \int_\Sigma (\xi, n)\ dA.\end{aligned}$$

The lemma is proved. $\square$

To prove the theorem, we assume that $Flux(\xi) = 1$, and g^t is the phase flow of ξ. Then, for a fixed C^1-mapping $f : \mathbf{T} \to \mathbb{R}/\mathbb{Z}$ of degree 1 and its lift $\tilde f : \tilde{\mathbf{T}} \to \mathbb{R}$,

$$\begin{aligned}(5.7) \qquad \int_{\mathbf{T}} (\tilde f(g^\tau(x)) - \tilde f(x))\ \mu &= \int_0^\tau \int_{\mathbf{T}} (\nabla f(g^t(x)), \xi(g^t(x)))\ \mu_x dt \\ &= \int_0^\tau Flux(\xi)\ dt = \tau.\end{aligned}$$

Recall that $\Gamma_\tau(x)$ is the curve $g^t(x), 0 \le t \le \tau$, joined to the "short curve" $\Delta(g^\tau(x), x)$ for any $x \in \mathbf{T}$. Then

$$|degree(\Gamma_\tau(x)) - (\tilde f(g^\tau(x)) - \tilde f(x))| \le C,$$

since the lengths of the short paths are uniformly bounded and the function $\tilde f$ is continuously differentiable.

On the other hand, by definition of the asymptotic crossing number,

$$(5.8) \qquad c(\Gamma_\tau(x), \gamma) \ge ac(K) \cdot degree(\Gamma_\tau(x)) \cdot degree(\gamma)$$

for any closed curve γ in the solitorus $\mathbf{T}$. Therefore,

$$c(\Gamma_\tau(x), \gamma) \ge ac(K) \cdot degree(\gamma) \cdot [(\tilde f(g^\tau(x)) - \tilde f(x)) - C].$$

Combining this inequality with formula (5.7), we obtain

$$\frac{1}{\tau} \int_{\mathbf{T}} c(\Gamma_\tau(x), \gamma)\ \mu_x \ge ac(K) \cdot degree(\gamma) \left(1 - \frac{C \cdot \mathrm{vol}(\mathbf{T})}{\tau}\right).$$

Finally, as $\tau \to \infty$ it bounds below the average crossing number $c_\xi(\gamma)$:

$$c_\xi(\gamma) \geq ac(K) \cdot degree(\gamma).$$

Similarly, letting $\gamma = \Gamma_\tau(y),\ y \in \mathbf{T}$, and utilizing formula (5.8) and the definition of the asymptotic crossing number, we deduce the required inequality

$$Cr(\xi, \xi) \geq ac(K).$$

□

5.C. Conformal modulus of a torus. Some energy bounds for vector fields possessing invariant tori can be formulated in terms of the conformal modulus of a solid torus.

Let $\mathbf{T}$ be a solitorus endowed with some Riemannian metric. Homotopically $\mathbf{T}$ is equivalent to the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

DEFINITION 5.16. The *conformal modulus* of a solitorus $\mathbf{T}$ with a metric on it is

$$m(\mathbf{T}) = \inf_f \int_{\mathbf{T}} ||\nabla f||^3 \ \mu,$$

where $f : \mathbf{T} \to \mathbb{R}/\mathbb{Z}$ is taken to be any degree one, C^1-function.

REMARK 5.17. The modulus may be thought of as a measure of the "electrical conductivity" for currents along $\mathbf{T}$: A "fat" torus will have a large modulus, while a very thin one will have a modulus close to zero. The modulus $m(\mathbf{T})$ is a conformal invariant: It is preserved under a conformal change of metric, since ∇ scales as length^{-1}.

THEOREM 5.18 [FH2]. *For any divergence-free vector field ξ leaving a solid torus $\mathbf{T}$ invariant,*

$$E_{3/2}(\xi) = \int_{\mathbf{T}} ||\xi||^{3/2} \ \mu \geq \frac{|Flux(\xi)|^{3/2}}{m(\mathbf{T})^{1/2}},$$

where $Flux(\xi)$ is the flux of the field ξ through any surface Σ representing the generator of $H_2(\mathbf{T}, \partial\mathbf{T})$; see Fig.36.

PROOF. The theorem follows immediately from Lemma 5.15. Indeed, the Hölder inequality applied to (5.6) gives $|Flux(\xi)| \leq ||\xi||_{L^{3/2}} ||\nabla f||_{L^3}$; therefore

$$E_{3/2}(\xi) = (||\xi||_{L^{3/2}})^{3/2} \geq \frac{|Flux(\xi)|^{3/2}}{(||\nabla f||_{L^3})^{3/2}} \ .$$

The minimization over degree 1 functions f turns the L^3-norm in the denominator into the conformal modulus. □

REMARK 5.19. An incompressible diffeomorphism action preserves $Flux(\xi)$, and therefore it leaves the energy of the field $h_*\xi$ bounded from below once the modulus of the torus has an upper bound. In turn, the modulus $m(\mathbf{T})$ can be bounded by purely topological quantities associated to the knot (or link) type of the torus (or of the collection of tori).

THEOREM 5.20 [FH2]. *For any solid torus* $\mathbf{T}$ *of knot type* K *embedded in Euclidean three-space* $\mathbb{R}^3$,

$$m(\mathbf{T}) \leq \frac{\sqrt{\pi}}{4(ac(K))^{3/2}}.$$

We refer to [FH2] for the proofs and for other interesting inequalities relating energy, linking, and moduli of solid tori. Conjecturally, for a nontrivial link of solid tori, $\min\{m(\mathbf{T}_1), \dots, m(\mathbf{T}_k)\}$ is majorized by a universal constant independent of k (the upper bound obtained in [FH2] is $\leq \sqrt{\pi} k^{1/2}/4$).

§6. Energy of a knot

The relaxation process of magnetic tubes to a state with minimal energy raises a question on optimal embeddings of curves, or of more general submanifolds, into the space. Is there a natural way to associate such an "energy" to a submanifold so that the energy is infinite for immersions that are not embeddings, and so that the gradient flow of the energy would preserve isotopy type and evolve the submanifold to the "optimal" state?

6.A. Energy of a charged loop. Imagine an infinitesimal relative of a magnetic tube, a charged loop of string. Among various possible potential energies for a loop in 3-space, the one recently suggested by O'Hare [OH1] is of special interest because of its nice invariance properties (see [BFHW, FHW]).

Let $\gamma = \gamma(u)$ be a rectifiable curve embedded in $\mathbb{R}^3$, where u belongs to the circle S^1. For any pair of points $\gamma(u), \gamma(v)$ we denote by $\mathrm{dist}(\gamma(u), \gamma(v))$ the distance between them along the curve, i.e., the minimum of the lengths of the two subarcs of γ with endpoints at $\gamma(u)$ and $\gamma(v)$.

DEFINITION 6.1 [OH1]. The *energy of the curve* γ is the following integral:

$$E(\gamma) = \iint_{S^1 \times S^1} \left\{ \frac{1}{||\gamma(v) - \gamma(u)||^2} - \frac{1}{|\mathrm{dist}(\gamma(v), \gamma(u))|^2} \right\} \cdot ||\dot{\gamma}(u)|| \cdot ||\dot{\gamma}(v)|| \, dudv.$$

The invariance of the energy under reparametrizations and dilations of the space is immediate.

REMARK 6.2. The energy E is defined on the space of embeddings $S^1 \to \mathbb{R}^3$. It tends to infinity when the embedding becomes singular. It is a regularization of $1/r^2$-potential energy of a charged curve, while the Newton–Coulomb potential in $\mathbb{R}^3$ is $1/r$. The energies corresponding to the exponents smaller than or equal to -2 (in particular, to the case at hand) blow up as two distinct arcs of a curve get closer to each other and the curve acquires a double point. It creates an infinite barrier against any change of the knot topology. Indeed, the unregularized energy for two pieces of straight lines intersecting transversally (say, for segments of the x- and y-axes, respectively) is given by the integral

$$\iint\limits_{y\,x} \frac{dxdy}{(\sqrt{x^2+y^2})^2} = \iint\limits_{\theta\,r} \frac{r}{r^2} dr \, d\theta,$$

which diverges at the origin.

A remarkable property of $E(\gamma)$ is a form of Möbius invariance. Recall that a *Möbius transformation* in $\mathbb{R}^3$ is a composition of a Euclidean motion, a dilation, and an inversion with respect to a sphere. Adding one point at infinity, one makes the Möbius transforms into bijections of the 3-sphere $\mathbb{R}^3 \cup \{\infty\}$.

THEOREM 6.3 [BFHW, FHW]. *Let γ be a simple closed curve in $\mathbb{R}^3$ and let MT be a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$. The following statements hold:*

(i) If $MT \circ \gamma \subset \mathbb{R}^3$, then $E(MT \circ \gamma) = E(\gamma)$.

(ii) If $MT \circ \gamma$ passes through ∞, then $E((MT \circ \gamma) \cap \mathbb{R}^3) = E(\gamma) - 4$.

O'Hara [OH1] proved that there exist only finitely many knot types among the curves with a given simultaneous upper bound on energy, length, and L^2-norm of the curvature. The conditions on the length and the L^2-norm of the curvature can be dropped, as the following theorem shows.

THEOREM 6.4 [FHW]. *Let γ be a simple closed curve in $\mathbb{R}^3$ and let $cn(\gamma)$ (respectively, $c(\gamma, \gamma)$) denote the topological (respectively, average self-) crossing number of the knot type of γ (respectively, of the curve γ itself). Then*

$$2\pi \cdot cn(\gamma) + 4 \leq E(\gamma),$$

$$12\pi \cdot c(\gamma, \gamma) \leq 11E(\gamma) + 12.$$

Notice that the average self-crossing number $c(\gamma, \gamma)$ given by the Gauss-type integral (5.1) is bounded, since the numerator undergoes a double degeneracy on the diagonal of $S^1 \times S^1$.

The energy of any round circle is $E(circle) = 4$, being the minimum of the energy for closed curves in $\mathbb{R}^3$. The theorem implies that if a closed curve satisfies the inequality $E(\gamma) < 6\pi + 4 \approx 22.849$, then γ is unknotted (the number of crossings $cn(\gamma) \geq 3$ for any essential knot γ).

Using the exponential upper bound of the number of distinct knots with a given bound for the number of crossings, one obtains the following

COROLLARY 6.5 [FHW]. *The number of (the isomorphism classes of) knots that can be represented by curves whose energy E does not exceed N is bounded by*

$$2 \cdot (24^{-4/2\pi}) \cdot (24^{1/2\pi})^N \approx (0.264)(1.658)^N.$$

Milnor [Mil1] showed that for the *total curvature*

$$TK(\gamma) = \int \left| \left(\frac{\dot{\gamma}(u)}{||\dot{\gamma}(u)||} \right)' \right| \, du$$

(where $'$ and $\dot{}$ stand for the derivative in u), the inequality $TK(\gamma) \leq 4\pi$ implies that γ is unknotted ($TK(circle) = 2\pi$). However, for any given $\epsilon > 0$ there exist infinitely many knot types having representatives of total curvature $TK \leq 4\pi + \epsilon$.

REMARKS 6.6. The total energy can be similarly assigned to a link $(\gamma_1, \ldots, \gamma_k)$, which consists of k disjoint embeddings of S^1 to $\mathbb{R}^3$:

$$TE(\gamma_1, \ldots, \gamma_k) = \sum_{i=1}^{k} E(\gamma_i, \gamma_i) + \frac{1}{2} \sum_{i,j=1, i \neq j}^{k} E(\gamma_i, \gamma_j),$$

where $E(\gamma_i, \gamma_i) = E(\gamma_i)$, and for $i \neq j$,

$$E(\gamma_i, \gamma_j) = \iint\limits_{S^1 \times S^1} \frac{\|\dot{\gamma}_i(u)\| \cdot \|\dot{\gamma}_j(u)\|}{\|\gamma_i(u) - \gamma_j(u)\|^2} \, du dv.$$

Given $N > 0$ there are finitely many link types that have representatives with $TE \leq N$ (see [FHW]).

REMARKS 6.7. For a divergence-free field confined to nontrivially knotted or linked tubes there is a lower bound of the magnetic energy, as discussed in Section 2.B. Moffatt [Mof5] suggested using these lower bounds of the energy as the invariants of (the tubular neighborhoods of) knots and links.

Namely, for any knot, consider a satellite flux-tube of volume vol carrying an "untwisted" vector field ξ of flux $Flux$ (across any meridian section of the tube) and look at the associated energy of this vector field. This energy can be decreased by a diffeomorphism action, preserving both vol and $Flux$, to a topological accessible minimum. On dimensional grounds, this minimal energy $E(\xi) = m \cdot (Flux)^2 (vol)^{-1/3}$, where $m = m(Flux, vol)$ is a positive real number depending on the knot topology. If for a given knot, different local minima of the energy exist, then the sequence $\{m_0, m_1, \dots, m_r\}$ of possible values could be reasonably described as the energy spectrum of the knot (neighborhood). The lowest number m_0 provides a possible natural measure of the knot complexity (see also [C-M]).

It would be interesting to relate the final positions of the vortex magnetic tubes under the E_2-energy relaxation to the shape of the curves, realizing the minimum of an appropriate energy function on curves. The critical points of such an energy would correspond to the equilibrium states for the Moffatt spectrum.

6.B. Generalizations of the knot energy. There is a variety of Möbius invariant generalizations of the knot energy (see, e.g., [D-S, AuS, KuS]). Imagine a charge uniformly spread over a k-dimensional submanifold $M \subset \mathbb{R}^n$.

DEFINITION 6.8. Given a function f, define the *f-energy* to be

$$E_f(M) = \iint_{M \times M} \frac{f(x,y)}{||x-y||^{2k}} \, d\mathrm{vol}_M(x) \, d\mathrm{vol}_M(y).$$

Regard the function f on $M \times M$ as a function of three arguments, $f = f(M, x, y)$.

DEFINITION 6.9. A function $f(M, x, y)$ is *g-invariant* under the action of a map $g : M \to M$ if $f(g_* M, g(x), g(y)) = f(M, x, y)$.

THEOREM 6.10 [D-S, AuS, KuS]. *Any scale and Möbius invariant factor f gives rise to the energy E_f invariant with respect to the Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$.*

The scale invariance of the integrand justifies the choice of the power $2k$ rather than the physically meaningful $n-2$ in the denominator of the energy in $\mathbb{R}^n$. The submanifold $M \subset \mathbb{R}^n \cup \{\infty\}$ can be viewed as a submanifold of $S^n \subset \mathbb{R}^{n+1}$ via the stereographic projection. Such a projection extends to a Möbius transformation of $\mathbb{R}^{n+1}$, while the energy formula does not depend on the ambient dimension.

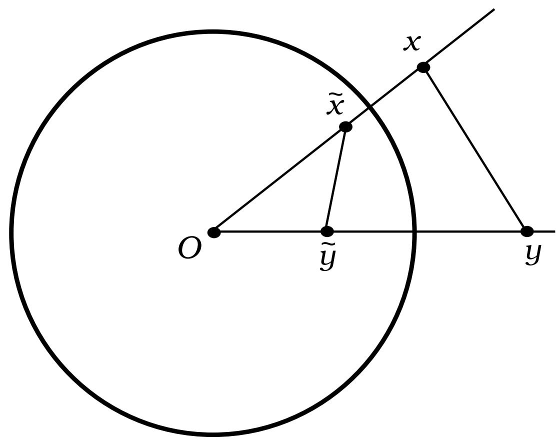


FIGURE 37. Triangles Oxy and $O\tilde{x}\tilde{y}$ are similar (where $\tilde{x}, \tilde{y}$ are the inverses of x, y).

PROOF. The statement follows from the Möbius invariance of the integrand with $f \equiv 1$. For the latter case the scale invariance is evident, while the invariance under inversion $r \mapsto \tilde{r} := r/||r||^2$ follows from Fig.37.

The similar triangles Oxy and $O\tilde{x}\tilde{y}$ provide the identity

$$\frac{||\tilde{x}|| \cdot ||\tilde{y}||}{||\tilde{x} - \tilde{y}||^2} = \frac{||x|| \cdot ||y||}{||x - y||^2}.$$

On the other hand, the inversion transforming M into $\tilde{M}$ expands conformally the lengths at x by the factor $||\tilde{x}||/||x||$. The corresponding change of the volume element is

$$d\text{vol}_{\tilde{M}}(\tilde{x}) = (||\tilde{x}||/||x||)^k \ d\text{vol}_M(x).$$

This shows that the integrand, as a whole, remains invariant under inversion, and hence under an arbitrary Möbius transformation. □

If $f \equiv 1$, the integrand blows up as x approaches y, and therefore the energy is infinite for any M. The regularizing factor f is designed to compensate the singularity, and so it vanishes as $x \to y$.

The list of properties desired from a particular regularization usually includes the infinite barrier against self-crossings, the Möbius invariance, and boundedness of the energy from below. More restrictive is the property of approximate additivity for the connected sum of two remote knots, and the requirement that the energy contribution of any two disjoint arcs would be independent of whether they are in the same component of the link (see the discussion in [AuS]).

To give an example, return to the case of a knot $\gamma \in \mathbb{R}^3$. Define a specific regularization $f_0 : \gamma \times \gamma \to \mathbb{R}$ by the following construction.

DEFINITION 6.11 [D-S]. Given a point $x \in \gamma$ and any other point $p \in \mathbb{R}^3$ there is a unique circumference (or straight line) $S_x(p)$ tangent to γ and passing through p. Thus given two points x and y of γ, we have two oriented circumferences $S_x(y)$ and $S_y(x)$ that meet at equal angles at x and y. Let α be the *angle* at which these two circles meet in $\mathbb{R}^3$. These circles and, in particular, the angle α are defined in a Möbius invariant manner. Set the *special weight* f_0 to be the function $f_0 := 1 - \cos\alpha$. (The angle α can also be defined in the case of an arbitrary k-dimensional submanifold in $\mathbb{R}^n$ by replacing the circumferences $S_x(y)$ by k-spheres [KuS]).

PROPOSITION 6.12 [D-S]. *The knot energy E_{f_0} defined by the special weight f_0 is equivalent to O'Hare's energy E modulo a constant: If γ is a closed curve in $\mathbb{R}^3$, then*

$$E_{f_0}(\gamma) = E(\gamma) - 4.$$

It was shown in [FHW] that for an irreducible knot there is a representative having minimal energy among all simple loops of the same knot type. A criterion describing the "optimal" (minimizing the energy) states was obtained in [OH2]. We also refer to the paper [KuS] for nice stereo-pairs of optimal links (with the number of crossings $cn \leq 8$) that allow one to visualize the three-dimensional picture.

REMARK 6.13. Note that the number of critical points of a function on the space of embeddings $S^1 \to \mathbb{R}^3$ can be minorized by Morse theory and by Vassiliev's calculation of the Betti numbers of the space of embeddings [VasV].

Unlike knots, plane curves (immersions $S^1 \to \mathbb{R}^2$) generically have self-intersection points. The simplest singular plane curves (forming the discriminant hypersurface in the space of maps $S^1 \to \mathbb{R}^2$) have either a triple point or a point of self-tangency (see Fig.38). A treatment of the corresponding theory of Vassiliev-type invariants for the plane and Legendrian curves can be found in [Arn21, Aic, L-W, Vir, Pl1,2, PlV, Shm, Tab2, Gor, FuT].

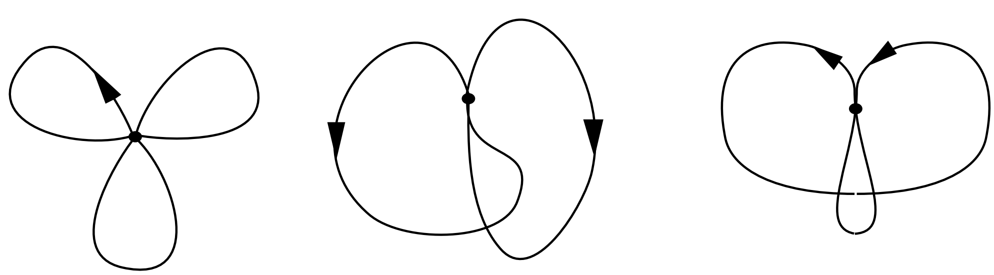


FIGURE 38. Plane curves with triple points and self-tangencies.

PROBLEMS 6.14. A) Is there an energy functional on the space of immersions that is infinite on the discriminant and possesses the property of Möbius invariance (and/or other properties from the discussion above)? Conjecturally, there will be only finitely many homotopy classes of immersed curves whose would-be energy is bounded from above.

B) Are there asymptotic generalizations of invariants of plane curves similar to those discussed above for the linking of space curves? We refer to [Aic, L-W] for very suggestive integral formulas of the invariants.

REMARK 6.15 (D. Kazhdan). The growth rate of the number of types of immersions into the plane as a function of the crossing number suggests the existence of a negative curvature metric in the corresponding spaces of immersions.

§7. Generalized helicities and linking numbers

This section describes various generalizations of the helicity integral to manifolds with boundary, to the non-simply connected and higher-dimensional cases, as well as to magnetic tubes forming links detected by certain higher-order link invariants.

7.A. Relative helicity. The helicity of a vector field in a simply connected manifold with boundary (say, in a domain of $\mathbb{R}^3$) is well-defined, provided only that the field is tangent to the boundary. A vector field crossing the boundary possesses neither the ergodic version of the definition (some of its trajectories leave the region, and therefore their asymptotic linking cannot be specified) nor the integral one (the formula has to include a boundary term). However, the vector fields identical outside the region can be compared by means of the *relative linking* of their trajectories in the interior [Ful, B-F].

The definition of the relative linking number for nonclosed curves rests on the introduction of "reference arcs" with the same endpoints and closing up the curves, Fig.39 (see [Ful], where this construction is applied to the study of DNA knottedness).

The continuous version is as follows [B-F]. Suppose that a domain in the space $\mathbb{R}^3$ (or a closed simply connected manifold M^3) is split into two simply connected regions A and B separated by a boundary surface S. Assume further that two divergence-free vector fields ξ and η in A coincide on the boundary S and have the same extension ζ into the region B. Call the extended fields in M respectively $\tilde{\xi}$ and $\tilde{\eta}$. Abusing notation we will denote them as the sums $\tilde{\xi} = \xi + \zeta$ and $\tilde{\eta} = \eta + \zeta$

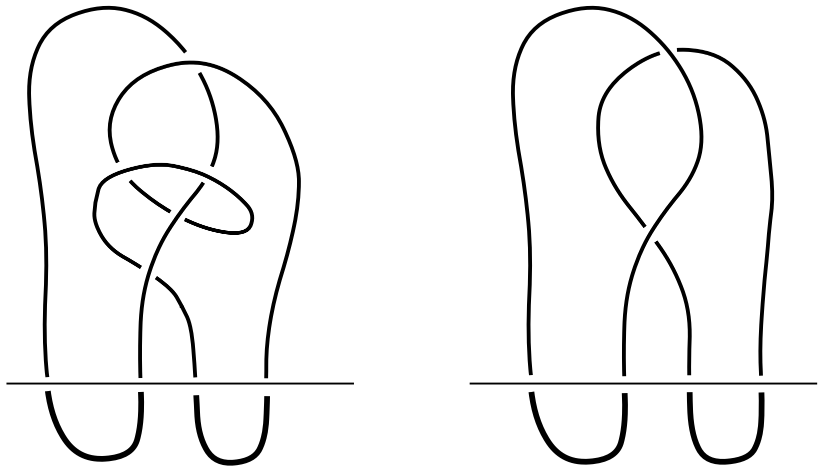


FIGURE 39. Nonclosed curves have relative linking with respect to arcs outside the region.

(where ξ, η, ζ are regarded as the (discontinuous) vector fields in the entire manifold M with supports $\operatorname{supp}\xi$, $\operatorname{supp}\eta \subset A$, and $\operatorname{supp}\zeta \subset B$).

LEMMA–DEFINITION 7.1. *The difference of the helicities of the fields $\tilde{\xi}$ and $\tilde{\eta}$*

$$\Delta\mathcal{H} = \mathcal{H}(\tilde{\xi}) - \mathcal{H}(\tilde{\eta})$$

is independent of their common extension ζ in the region B, and hence it measures the relative helicity *of the fields ξ and η in A.*

PROOF. Define the (closed) two-forms α, β, and ω (by substituting the vector fields ξ, η, and ζ with respect to the volume form μ on M: $i_\xi\mu = \alpha$, etc.). Then one has to show that the difference

$$\mathcal{H}(\tilde{\xi}) - \mathcal{H}(\tilde{\eta}) := \int_M (\alpha+\omega) \wedge d^{-1}(\alpha+\omega) - \int_M (\beta+\omega) \wedge d^{-1}(\beta+\omega)$$

does not depend on ω. One readily obtains

$$\Delta\mathcal{H} = \int_M \alpha \wedge d^{-1}\alpha - \int_M \beta \wedge d^{-1}\beta + \int_M (\alpha-\beta) \wedge d^{-1}\omega + \int_M \omega \wedge d^{-1}(\alpha-\beta).$$

Here d^{-1} applied to a discontinuous 2-form is a continuous 1-form (the "form-potential"). The terms in $\Delta\mathcal{H}$ containing ω are $\int_M(\alpha-\beta)\wedge d^{-1}\omega + \int_M \omega\wedge d^{-1}(\alpha-\beta)$, and we want to show that their contribution vanishes.

Integrating by parts one of the terms, we come to $2\int_M(\alpha-\beta)\wedge d^{-1}\omega$, which, in turn, is equal to $2\int_A(\alpha-\beta)\wedge d^{-1}\omega$, since $\operatorname{supp}(\alpha-\beta) \subset A$.

On the other hand, in A the 1-form $d^{-1}\omega$ is the differential of a function: $d^{-1}\omega = dh$. Indeed, it is closed (the differential $d(d^{-1}\omega) = \omega$ vanishes in A due to the condition on supp $\zeta =$ supp $\omega \subset B$), and hence it is exact in the simply connected region. Hence,

$$2\int_A (\alpha - \beta) \wedge d^{-1}\omega = 2\int_A (\alpha - \beta) \wedge dh = 2\int_S h(\alpha - \beta) = 0,$$

where the last equality is due to the assumption on the identity of the fields ξ and η on the boundary S. This proves that $\Delta\mathcal{H}$ is not affected by the choice of the extension ζ. □

The relative helicity of a field transversal somewhere to the boundary of M is no longer invariant under the action of volume-preserving diffeomorphisms of M.

REMARK 7.2. The phenomenon of the same type holds for divergence-free vector fields on non-simply connected manifolds. A true linking number does not exist for such a case, but two "homologically equivalent" fields can be compared with each other. A nice application to the *linking numbers for cascades* can be found in [GST].

Choose a nonsingular C^1 vector field inside a solid torus such that the flow lines are transversal to its 2-disks, as in Fig. 40a. In this setting one can define the following version of the linking number. We fix the direct product structure $S^1 \times D^2$ in the solid torus trivializing its fibration over S^1.

The *topological linking* of two long pieces of orbits is the algebraic number of times one trajectory winds around the other. Namely, the projections of the orbits to the disk form a moving pair of points in the same 2-disk. The linking number is the rotation number of one point around the other. This definition extends to the case of the cascades of periodic orbits in a solid torus. A cascade flow in the solid torus cyclically interchanges smaller invariant disks in the transverse section and repeats itself inside these disks (Fig.40b).

On the other hand, to the piece $0 \le t \le T$ of a single orbit of a C^1-flow one can associate the *(infinitesimal) self-linking number* by counting how many times a tangent vector in the disk direction turns around the orbit. For almost all points, the infinitesimal self-linking number has a limit as $T \to \infty$, and this limit can be described by a spatial integral of the appropriate derivative [Rue].

Gambaudo, Sullivan, and Tresser showed in [GST] that the sequence of the topologically defined average linking numbers between successive orbits in the cascade converges to the average self-linking number of the invariant set. They also described the sequences of rational numbers (in a sense, counterparts of the rotation

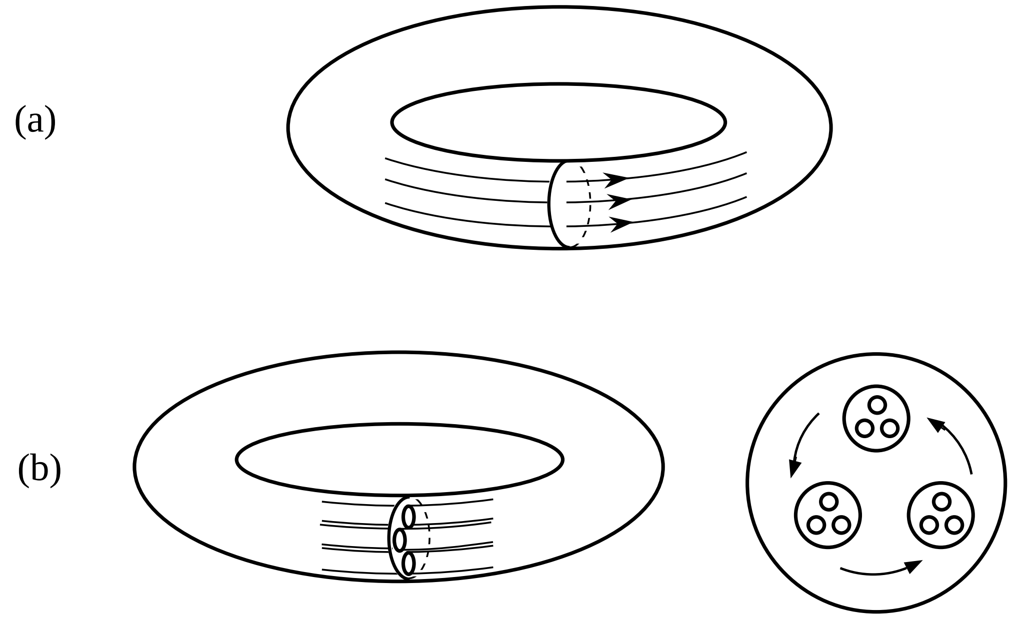


FIGURE 40. a) A solid torus with a vector field transversal to the 2-disk D^2. b) Cascade of embedded solitori.

numbers of maps of a circle into itself) that can appear as the average linking numbers in a cascade of iterated torus knots.

7.B. Ergodic meaning of higher-dimensional helicity integrals. The higher-dimensional integrals generalizing the helicity of a vector field in $\mathbb{R}^3$ were introduced by Novikov [Nov1]. His idea was to extend to closed differential forms on higher-dimensional spheres (which are not necessarily the pullbacks of the forms from the spheres of smaller dimension) the Whitehead operations in the homotopy groups of the spheres (simulating the approach, transforming the Hopf invariant on the homotopy group $\pi_3(S^2)$ into the helicity of divergence-free vector fields on S^3).

An ergodic interpretation of Novikov's constructions encounters the following difficulty. Unlike the three-dimensional case, where the asymptotic linking number is defined for almost every pair of trajectories, the field lines are not linked if the dimension of the ambient manifold is greater than 3. Thus, instead of the curves, one should consider the submanifolds of higher dimensions. But for nonclosed submanifolds of dimension ≥ 2 one lacks a satisfactory generalization of the system of short paths.

We consider the geometric meaning of the invariants of closed two-forms on manifolds of arbitrary dimension. For odd-dimensional manifolds quantities like $\int d^{-1}\alpha \wedge \beta \wedge \cdots \wedge \omega$ arise as first integrals in the theory of an ideal or barotropic fluid (Sections I.9, VI.2) or in the Chern–Simons theory (Section 8.A). Here the asymptotic linking number of every pair of field lines is replaced by the linking of a trajectory with a foliation of codimension 2. For even-dimesional manifolds the Novikov invariants are described as the average *nongeneric* linkings [Kh1]. The interpretation presented here is an ergodic counterpart of the Poincaré duality that translates facts on the differential forms into a description of the intersections of their kernel foliations (cf. Remark 4.7).

Let M^n be a compact connected manifold without boundary and $H_1(M, \mathbb{R}) = H_2(M, \mathbb{R}) = 0$. Denote closed (and hence, exact) two-forms on M by $\alpha, \beta, \dots \in \Omega^2(M)$, while $d^{-1}\alpha, d^{-1}\beta, \dots \in \Omega^1(M)$ are arbitrary primitive one-forms (form-potentials) for the corresponding two-forms. We start with the following simple observations:

PROPOSITION 7.3. *i) For an odd-dimensional manifold M^{2m+1} and arbitrary $m+1$ closed two-forms $\alpha, \beta, \dots \omega$, the Hopf-type integral $I(\alpha, \beta, \dots \omega) = \int_M d^{-1}\alpha \wedge \beta \wedge \dots \wedge \omega$ is symmetric under the permutations of $\alpha, \dots, \omega$ and does not depend on the choice of the primitive $d^{-1}\alpha$.*

ii) [Nov1] *On a four-dimensional manifold M^4 for any two 2-forms α and β that obey the conditions $\alpha \wedge \alpha = \beta \wedge \beta = \alpha \wedge \beta = 0$, the integrals*

$$J_1(\alpha, \beta) = \int_M d^{-1}\alpha \wedge \alpha \wedge d^{-1}\beta \qquad \textit{and}$$
$$J_2(\alpha, \beta) = \int_M d^{-1}\alpha \wedge \beta \wedge d^{-1}\beta$$

do not depend on the choices of $d^{-1}\alpha$ and $d^{-1}\beta$.

In [Nov1], Novikov defined a set of invariants on manifolds of an arbitrary dimension, and we consider the case of M^4 for illustration. We are going to represent these integrals as the generalized linking numbers of certain foliations associated to the differential forms.

DEFINITION 7.4. A closed 2-form α of rank ≤ 2 on a manifold M^n determines a (singular) foliation (called a *kernel foliation*) of codimension 2 in M: the tangent plane to this foliation at any point of M is spanned by the $(n-2)$-vector being the kernel of α at that point.

If the manifold is equipped with a volume form μ, then this foliation is generated by the field of $(n-2)$-vectors $\mathcal{A}$ whose substitution $i_{\mathcal{A}}$ into the volume form gives α (i.e., $i_{\mathcal{A}}\, \mu = \alpha$).

PROPOSITION 7.5. *The kernel field of a closed two-form is completely integrable. In particular, for the form of rank ≤ 2, it spans a foliation of codimension ≥ 2.*

PROOF is an application of the Frobenius integrability criterion. □

REMARK 7.6. Without the restriction on rank of the two-form α the corresponding kernel $(n-2)$-vector field $\mathcal{A}$ is generically indecomposable. The conditions $\alpha \wedge \alpha = \beta \wedge \beta = 0$ on the pair α, β in $ii)$ in the above proposition are exactly the limitations on the ranks: $rk(\alpha), rk(\beta) \leq 2$. The third condition $\alpha \wedge \beta = 0$ ensures that the kernel foliations (of dimension 2 in M^4) determined by the forms α and β (near a point where $rk(\alpha) = rk(\beta) = 2$) are allocated in the following peculiar way. The intersections of their leaves form a 1-dimensional foliation, provided that α and β are not proportional. Moreover, the distribution spanned by the kernels of α and β determines in this case a 3-dimensional foliation [Arn9].

DEFINITION 7.7. The *average linking* of a curve Γ with the foliation $\mathcal{A}$ is the flux of the two-form $\alpha = i_{\mathcal{A}}\,\mu$ through an arbitrary surface $\partial^{-1}\Gamma$ bounded by Γ:

$$lk(\Gamma, \mathcal{A}) = \int_{\partial^{-1}\Gamma} \alpha = \int_{\Gamma} d^{-1}\alpha.$$

The following proposition motivates the definition of $lk(\Gamma, \mathcal{A})$.

PROPOSITION 7.8. *The number $lk(\Gamma, \mathcal{A})$ coincides with the average linking number (evaluated with the help of the linking form $G \in \Omega^{n-2}(M) \times \Omega^1(M)$) of the leaves of foliation $\mathcal{A}$ with the curve Γ.*

PROOF. By definition of the form G the linking number of two submanifolds P and Q in M is given by the integral $\iint_{P\times Q \subset M\times M} G$, see Section 4. Therefore,

$$\iint\limits_{\mathcal{A}\times\Gamma} G = \iint\limits_{M\times\Gamma} i_{\mathcal{A}}G \wedge \mu = \iint\limits_{M\times\Gamma} G \wedge i_{\mathcal{A}}\mu = \iint\limits_{M\times\Gamma} G \wedge \alpha = \int\limits_{\Gamma} d^{-1}\alpha.$$

(Here the first identity is the definition of $\iint_{\mathcal{A}\times\Gamma} G$, the last one is the main property of G: the operator corresponding to the linking form acts on the exact differential 2-forms just like the operator d^{-1}; see Section 4.D.) $\square$

By analogy with the three-dimensional case, we can now define an *asymptotic linking* $lk_{\xi}(x, \mathcal{A})$ of the trajectory of a vector field ξ passing through a point $x \in M$ with the foliation $\mathcal{A}$. It is the time-average of the linking number with $\mathcal{A}$ of the curve $\Gamma_T(x)$ consisting of the long segment (for time $0 \leq t \leq T$) of the ξ-trajectory $g_{\xi}^t x$ starting at $x \in M$ and of a short closing path:

$$lk_{\xi}(x, \mathcal{A}) = \lim_{T\to\infty} \frac{1}{T} lk(\Gamma_T(x), \mathcal{A}).$$

DEFINITION 7.9. The *average linking number* of the vector field ξ with the foliation $\mathcal{A}$ defined on the manifold M equipped with the volume form μ is

$$lk_\xi(\mathcal{A}) = \int_M lk_\xi(x, \mathcal{A})\mu.$$

THEOREM 7.10. *Let $\alpha, \beta, \dots, \omega$ be a set of $m+1$ closed two-forms on M^{2m+1}. Assume that the rank of one of the forms (for example, α) is at most 2. Then the Hopf-type integral $I(\alpha, \beta, \dots, \omega) = \int_M d^{-1}\alpha \wedge \dots \wedge \omega$ coincides with the average linking number of the vector field ξ with the foliation $\mathcal{A}$:*

$$I(\alpha, \dots, \omega) = lk_\xi(\mathcal{A}),$$

where the fields ξ and $\mathcal{A}$ are defined by $i_\xi \mu = \beta \wedge \dots \wedge \omega$ and $i_{\mathcal{A}}\mu = \alpha$.

PROOF is a straightforward application of the Birkhoff ergodic theorem. □

The rank of α is essential merely to define the foliation $\mathcal{A}$. In the general case, we would consider a linking with an abstract $(n-2)$-vector field instead of an $(n-2)$-dimensional foliation. If, conversely, all these forms have rank ≤ 2 (of course, this is seldom the case), then one can interpret the number $I(\alpha, \dots, \omega)$ as the multilinking of all the corresponding foliations.

Namely, the usual linking number is a bilinear form on the space of disjoint submanifolds of appropriate dimensions: It is defined for a pair of submanifolds P^k and Q^l in M^n, subject to the conditions $k + l = n - 1$ and $P \cap Q = \emptyset$. Similarly, we define the multilinking number as a multilinear form on the space of r-tuples of submanifolds $(P_1, \dots, P_r)$ such that

$$\sum_{i=1}^{r} \text{codim } P_i = n + 1 \tag{7.1}$$

and

$$\bigcap_{i=1}^{r} P_i = \emptyset. \tag{7.2}$$

DEFINITION 7.11. The *mutual linking number* of r oriented closed submanifolds $P_1, \dots, P_r$ in $M = \mathbb{R}^n$ (or S^n) satisfying the condition above is the signed number of the intersection points of a manifold $F \subset M$ bounded by one of these surfaces $P_i = \partial F$ with the intersection of all the other submanifolds.

If these submanifolds are equipped with some transversal orientations, then so are all the manifolds bounded by them and all their intersections, and hence the

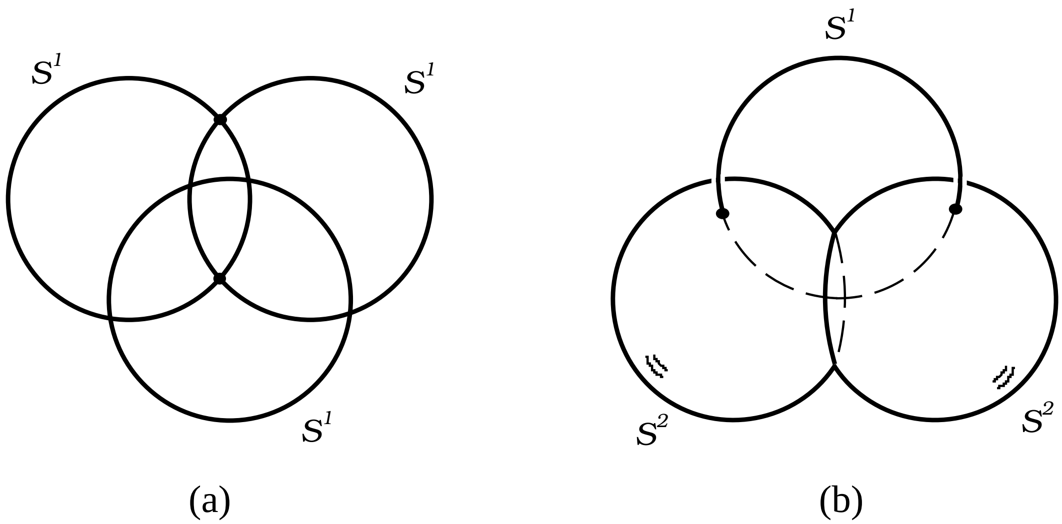


FIGURE 41. Links of a) three circles in the plane; b) two spheres and a circle in space.

signs of the intersection points are well-defined. For example, it is possible to link three circles in the plane or two spheres and one circle in 3-space (Fig.41).

Note that the mutual linking number of a collection $P_1, \ldots, P_r \subset M$ is the usual linking number of the submanifold $P_1 \times \cdots \times P_r \subset M \times \cdots \times M$ with the diagonal $\Delta = \{(x, \ldots, x) \mid x \in M\} \subset M \times \cdots \times M$.

We recall that every closed 2-form of rank ≤ 2 determines a foliation of codimension 2. If the leaves were compact, one could consider the mutual linking of these leaves for $(m+1)$ two-forms in M^{2m+1} due to

$$\sum_{i=1}^{m+1} \text{(codimension of leaves)} = 2m + 2 = \dim M + 1.$$

So in these terms, Theorem 7.10 above reads as

THEOREM 7.10′. *The Hopf-type invariant is equal to the average asymptotic multilinking number of the leaves determined by the given 2-forms.*

To describe the ergodic meaning of the Novikov integrals J_1 and J_2, we shall extend the concept of multilinking. We are going to drop the codimension condition (7.1) if it is compensated in (7.2) by an assumption on the nongeneric intersection of the submanifolds. For example, two circles S^1 and a sphere S^2 cannot be linked in $\mathbb{R}^3$ (one can untie any configuration of them not passing through any triple point, Fig.42a). However, if these two circles are two meridians of the same ball (and so their intersection S^0 consists of two points), the linking may be nontrivial (Fig.42b).

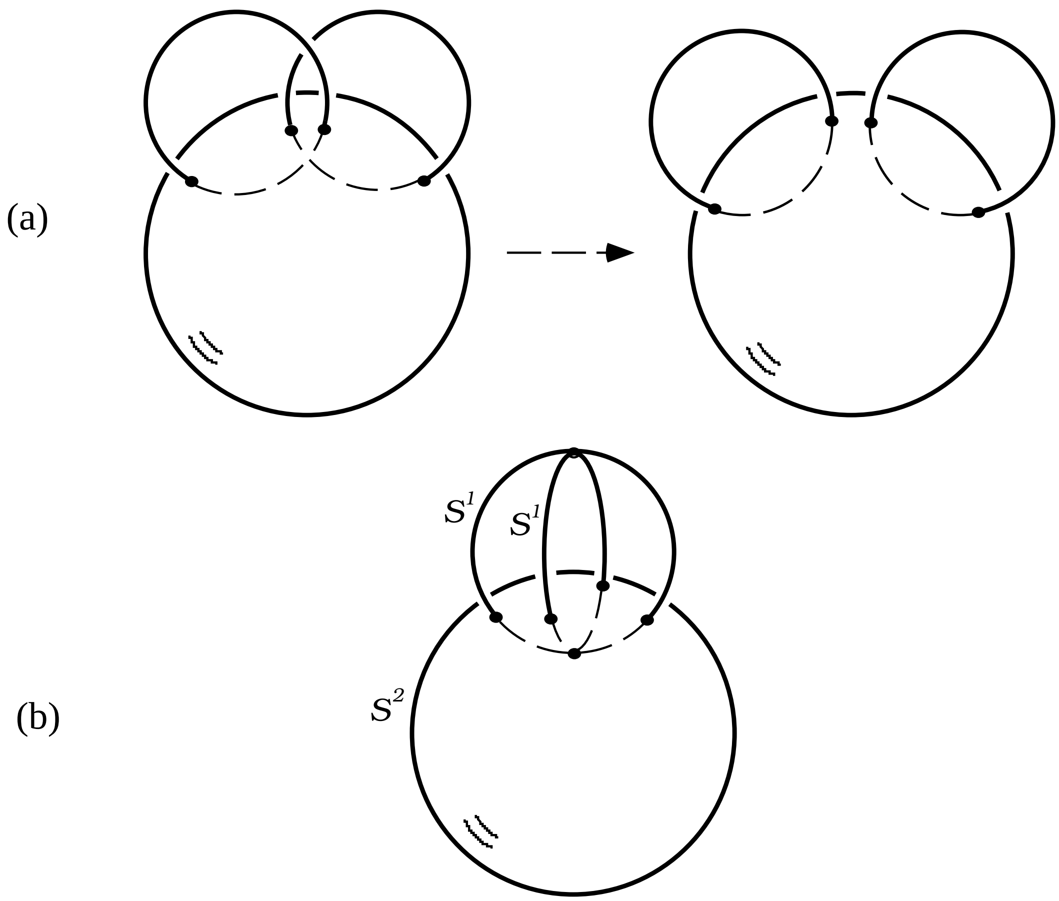


FIGURE 42. a) Generic and b) nongeneric linkings of two circles and a sphere.

Namely, one cannot remove S^2 far from the two meridians unless it passes through an intersection point of these two meridians.

In the definition of invariants J_i, the $(\alpha \wedge \beta = 0)$-type conditions provide the nongeneric intersections of the corresponding leaves.

THEOREM 7.12. *The invariant $J_1(\alpha, \beta)$ (respectively, $J_2(\alpha, \beta)$) coincides with the average linking number of the foliation $\mathcal{A}$ of the 2-form α (respectively, of the foliation $\mathcal{B}$ of β) with the vector field ξ satisfying $i_\xi \mu = d(d^{-1}\alpha \wedge d^{-1}\beta)$.*

Roughly speaking, each of these two amounts is the average linking number of the 1-dimensional foliation formed by the intersections of $\mathcal{A}$ and $\mathcal{B}$ with the foliation $\mathcal{A}$ or $\mathcal{B}$ (determined, respecively, by α or by β).

REMARK 7.13. The Hopf-type invariants arise in [Nov1] in a context of quantum anomalies. Consider the space $\mathcal{L}$ of smooth mappings $f : S^q \to M^n$ homotopic to zero. To a closed $(q + 1)$-form θ on M one naturally associates a multivalued function $F_\theta(f)$ (or a closed 1-form δF_θ) on the space $\mathcal{L}$:

$$F_\theta(f) = \int\limits_{f(D^{q+1})} \theta.$$

Here $f : S^q \to M^n$ is extended to a mapping $D^{q+1} \to M^n$ of the ball D^{q+1} bounded by the sphere. Closedness of the $(q+1)$-form θ implies that δF depends just on $f\mid_{\partial D^{q+1}=S^q}$.

The differential δF of a multivalued functional $F(f)$ on the space $\mathcal{L}$ is said to be *local* if it depends on f and on a finite number of its derivatives. For $n \geq q+1$ all multivalued functionals $F(f)$ with local differentials are the sums of a local univalued functional and $F_\theta(f)$ [Nov2]. A construction of multivalued functionals for $n \leq q+1$ that conjecturally describes all functionals with local differentials is given in [Nov1].

There is an integer lattice inside the space of θ's consisting of homotopy invariant elements. The meaning of this lattice is exactly equivalent to the role of the usual integer-valued Hopf invariant of mappings $S^3 \to S^2$ among all asymptotic linking invariants for arbitrary divergence-free vector fields on S^3. It is natural to call the appearance of the integral lattice a quantization condition [Nov1].

7.C. Higher-order linking integrals. The Gauss linking integral fails to detect the entanglements of curves in $\mathbb{R}^3$ with an equal number of "oppositely signed crossings." The Whitehead link and the Borromean rings are examples of this kind (see Fig.43). In this section we consider the higher-order invariants called Massey numbers (see [Mas]) that generalize the linking number of two curves and allow one to detect more general curve configurations.

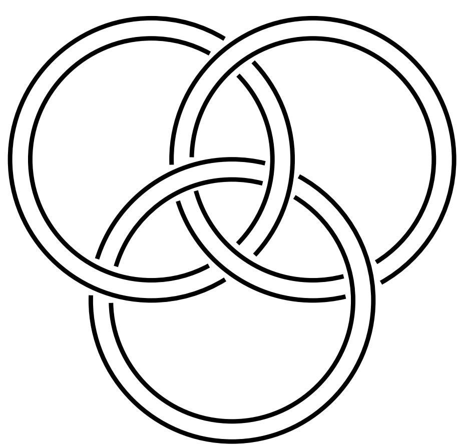


FIGURE 43. Three solid tori form the Borromean rings.

The formalism of differential forms for the hierarchy of higher link invariants was developed in [Mas] (see also [MRe]). This notion was introduced in a magnetohydrodynamical setting in the paper [MSa] and rediscovered in [Be1, E-B], to which we refer for more detail (cf. [LS2]). The topological obstruction rules for the links in nematics and in certain superfluids can be found in [MRe].

The helicity of field tubes is quadratic in the magnetic fluxes (see formula (2.1)), and therefore it describes a *second-order* invariant. For the Borromean rings the Gauss integral taken over any two rings vanishes and so does the helicity of the entire tube configuration. The Borromean rings can be distinguished from the three totally unlinked rings by means of a *third-order* linking invariant, cubic in the fluxes.

We start with the three closed curves forming the Borromean rings and encased in toroidal volumes $\mathbf{T}_k,\ k=1,2,3$. The field ξ_k is concentrated in the tube $\mathbf{T}_k$, vanishes outside, and has unit flux in $\mathbf{T}_k$. Denote by A_k a vector-potential for ξ_k and by ϕ_k the associated 1-form-potential. (In invariant terms, one first finds a closed two-form $\alpha_k = i_{\xi_k}\,\mu$, which is the substitution of the field ξ_k into the volume form μ, and then $\phi_k = d^{-1}\alpha_k$ is any primitive one-form such that $d\phi_k = \alpha_k$.)

Having defined the two-forms $\omega_{ij} = \phi_i \wedge \phi_j = -\phi_j \wedge \phi_i$ for $i \neq j$ (note: $d\omega_{ij} = 0$ outside of $\mathbf{T}_i \cup \mathbf{T}_j$), the helicity integral becomes

$$\mathcal{H}_{ij} := \mathcal{H}(\xi_i, \xi_j) = \int\limits_{\mathbf{T}_i \cup \mathbf{T}_j} \alpha_i \wedge d^{-1}\alpha_j = \int\limits_{\mathbf{T}_i} \alpha_i \wedge \phi_j = \int\limits_{\mathbf{T}_i} d\omega_{ij}$$

due to supp $\alpha_i \subset \mathbf{T}_i$. By virtue of the Stokes formula, the latter integral is equal to $\mathcal{H}_{ij} = \int_{\partial \mathbf{T}_i} \omega_{ij}$. All the quantities $\mathcal{H}_{12}, \mathcal{H}_{23}, \mathcal{H}_{31}$ vanish for the Borromean rings.

One can modify the form ω_{ij} inside the tubes to make it closed everywhere. Namely, one has to add the 2-form $h_{(j)i} \cdot \alpha_j$ to ω_{ij} inside $\mathbf{T}_j$ and to subtract $h_{(i)j} \cdot \alpha_i$ from ω_{ij} inside $\mathbf{T}_i$, where $h_{(i)j}$ is a scalar potential satisfying $\phi_j = dh_{(i)j}$. The function $h_{(i)j}$ exists in the tube $\mathbf{T}_i$ (but not globally), since the magnetic field ξ_j (and the corresponding two-form $\alpha_j = d\phi_j$) is zero there, and because $\mathbf{T}_i$ is not linked with $\mathbf{T}_j$. The Poincaré lemma applied to the new ω_{ij} guarantees that there is a one-form θ_{ij} such that $\omega_{ij} = d\theta_{ij}$.

DEFINITION 7.14. The *third-order linking integral* is

$$\mathcal{H}_{ijk} = \int\limits_{\partial \mathbf{T}_i} \omega_{ijk} = \int\limits_{\mathbf{T}_i} d\omega_{ijk} = -\int\limits_{\partial \mathbf{T}_k} \omega_{ijk}$$

for distinct i, j, k, where ω_{ijk} is the *Massey triple product*

$$\omega_{ijk} = \phi_i \wedge \theta_{jk} + \theta_{ij} \wedge \phi_k.$$

As a matter of fact, the Massey product is a map defined on cohomology classes. This implies both the gauge invariance of the third-order linking integral $\mathcal{H}_{ijk}$ and its invariance under deformations of the three curves. It vanishes for three unlinked circuits but is equal to ± 1 for the Borromean rings.

REMARK 7.15. In the language of vector calculus the Massey product becomes

$$\Omega_{ijk} = A_i \times \mathrm{curl}^{-1}\Omega_{jk} + \mathrm{curl}^{-1}\Omega_{ij} \times A_k,$$

where Ω_{ij} is the vector field $A_i \times A_j$ modified inside the tubes to make it divergence free and hence to provide the existence of a potential $\mathrm{curl}^{-1}\Omega_{ij}$ (see [E-B]).

The purely cohomological description of the numbers $\mathcal{H}_{ijk}$ is as follows (see, e.g., [MRe]). Let the curves Γ_k, $k = 1, 2, 3$, constitute the "axes" of the Borromean rings $\mathbf{T}_k$ in S^3. A closed 1-form ϕ_k is the Alexander dual of the circle Γ_k. It is defined in $S^3 \setminus \Gamma_k$ and can be regarded as a linking form: For any closed curve γ in this complement $\int_\gamma \phi_k = lk(\Gamma_k, \gamma)$.

The condition $lk(\Gamma_i, \Gamma_j) = 0$ allows one to find a 1-form ω_{ij} on $S^3 \setminus (\Gamma_i \cup \Gamma_j)$ such that $d\omega_{ij} = \phi_i \wedge \phi_j$. Now $\omega_{123} = \omega_{12} \wedge \phi_3 + \phi_1 \wedge \omega_{23}$ is defined on S^3 with the three circles removed, and it can be integrated over the boundary $\partial \mathbf{T}_1$.

REMARK 7.16. This is the starting point for a hierarchy of the invariants. (The invariants of order n can be defined for configurations whose invariants of order $\leq n-1$ vanish.)

A fourth-order linking invariant capturing the Whitehead link was suggested in [A-R]. Consider Seifert surfaces corresponding to two closed disjoint curves. For each of the curves such a surface can be chosen not to intersect the other curve, provided that the linking number of the pair vanishes. Then, generically, the intersection of the two Siefert surfaces is a closed curve equipped with a framing. The self-linking number of the framed curve is a topological invariant; and it is independent of the choice of the surfaces [Sat]. By making the curves into thin solid tori, one can obtain an integral form of the invariant [A-R].

REMARK 7.17. Another way to generalize the linking number to more complicated links was suggested by Milnor [Mil2]. For all necessary definitions of *higher-order Milnor coefficients* and for their relation to the higher-order Massey linking numbers see [Mil2, Tu1, Por, MRe].

REMARK 7.18. In all the constructions of this section, the magnetic field is assumed to be highly degenerate: It is concentrated in toroidal tubes with all the trajectories closed inside the tubes. Such fields form a slim set of infinite codimension in the space of all divergence-free vector fields. No asymptotic version of these constructions is known.

The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order $\leq k$ have zero value for a given field and there exists a nonzero invariant of order $k+1$, this nonzero invariant provides a lower bound for the field energy.

REMARK 7.19. It should be mentioned that the total helicity is approximately preserved even if the magnetic field is not frozen into the media but undergoes a small-scale turbulence [Tay]. In this case the fast reconnections of the field trajectories drastically change the local topological characteristics of the field. However, averaged over the entire domain, the helicity persists for large time intervals.

This phenomenon is based on the fact that small-scale components of the field (the components with wave vectors of large length k) contribute to the total helicity the amount of order $(amplitude)^2/k$, while their contribution to the energy is of order $(amplitude)^2$. Hence, a change of the higher harmonics of the field affects the helicity approximately k times more weakly than it affects the energy.

Analytically, an evolution of the magnetic field $\mathbf{B}$ ($\operatorname{div} \mathbf{B} = 0$) in the presence of diffusion is described by the equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\{v, \mathbf{B}\} + \eta \Delta \mathbf{B}.$$

The helicity dissipation over a fixed time δt is

$$\delta \mathcal{H} = 2 \int_M (\operatorname{curl}^{-1}(\eta \Delta \mathbf{B}), \mathbf{B})\ \mu = -2\eta \int_M (\mathbf{j}, \mathbf{B})\ \mu,$$

whereas the energy $E = \int_M (\mathbf{B}, \mathbf{B})\ \mu$ dissipates as

$$\delta E = 2 \int_M (\eta \Delta \mathbf{B}, \mathbf{B})\ \mu = 2\eta \int_M (\mathbf{j}, \mathbf{j})\ \mu$$

(here $\mathbf{j} = \operatorname{curl} \mathbf{B}$ is the current density). The Schwarz inequality gives the upper bound for $\delta\mathcal{H}$ of order $\eta^{1/2}$: $|\delta \mathcal{H}| \leq |\eta\ (\delta E)\ E|^{1/2}$.

The combinatorial arguments of [FrB] show that there are "reconnection pathways" that remove other invariants while changing the helicity only at a rate η^2. Neither of the linking invariants of higher order (≥ 3) defined above for tubes of closed trajectories persist under the reconnection deformations [MSa, FrB].

The reconnection of magnetic lines under magnetic diffusion is similar to the vortex reconnection in a viscid incompressible fluid. We refer to [KiT] for a survey on vortex reconnection and to [Ryl] for other topological properties of various vortex flows.

7.D. Calugareanu invariant and self-linking number. Let a narrow tube around a curve γ in $\mathbb{R}^3$ be filled by the trajectories of a vector field ξ. Suppose that all the ξ-trajectories in the tube are closed and that one of them is the curve γ itself.

The helicity of the field inside the tube is proportional to the linking number lk of any two trajectories inside the pencil:

$$\mathcal{H}(\xi) = lk \cdot Q^2,$$

where Q is the flux of ξ across any section of the tube. A straightforward application of the helicity formulas (4.1-4.2) for a field filling an arbitrary volume, this formula can also be visualized by presenting the tube as consisting of many slim solitori and by counting their mutual helicity (see formula (2.1)).

On the other hand, the linking number lk between the curve γ and a neighboring curve γ' is a quantity assigned to a ribbon bounded by γ and γ'. Precisely, the linking number is the sum

$$lk = Wr + Tw$$

of the writhing number Wr and the total twisting number Tw defined as follows.

Definitions 7.20. The *writhing number* is the algebraic number of crossovers of the curve $\gamma \subset \mathbb{R}^3$ averaged over all the projection directions:

$$Wr = -\frac{1}{4\pi} \int\limits_{S^1} \int\limits_{S^1} \frac{(\dot{\gamma}(t_1), \dot{\gamma}(t_2), \gamma(t_1) - \gamma(t_2))}{||\gamma(t_1) - \gamma(t_2)||^3} \, dt_1 dt_2,$$

where the curve $\gamma = \gamma(t)$ is parametrized by $t \in S^1$ (see, e.g., [Ful]). Just as it is for the average self-crossing number $c(\gamma, \gamma)$ (see Theorem 6.4), the integral above is bounded. Its value is not supposed to be an integer, and it is not a topological invariant. For instance, for a plane (or spherical) curve the writhing number is zero.

The *twist number* is not defined for a curve, but it can be defined for a ribbon. It specifies the total rotation number of the edge $\tilde{\gamma}$ revolving about the "axis" curve γ:

$$Tw = \frac{1}{2\pi} \int\limits_{S^1} (\frac{dn(t)}{dt}, n(t), \gamma(t)) \, dt,$$

where $\gamma(t)$ is an arc-parametrization of the curve γ, and the family $n(t)$ consists of the unit normals attached along γ and pointing in the direction of $\tilde{\gamma}$.

The formula $lk = Wr + Tw$ is illustrated in Fig.44. This relation, due to Calugareanu [Cal], was extensively studied along with its numerous applications (e.g.,

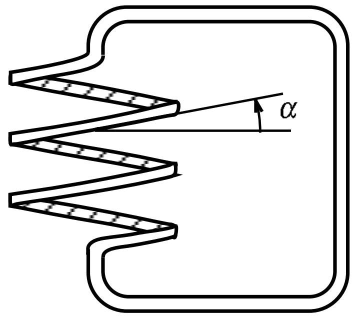


FIGURE 44. The formula $lk = Wr + Tw$ for a helical ribbon (see [Ful]). Here $lk = n$, $Tw = n \sin \alpha$, $Wr = n(1 - \sin \alpha)$, where α is the pitch angle of a helix, and n is the number of turns.

the helical DNA structure) by Fuller [Ful], Pohl [Poh], White [Wh], and in the hydrodynamical context by Berger and Field [B-F], and Moffatt and Ricca [MoR, RiM]. We refer to [MoR] for a derivation of the Calugareanu invariant from basic hydrodynamical principles, as well as for the invariant history and extensive bibliography. The decomposition $lk = Wr + Tw$ corresponds to the writhe and twist contributions to the helicity of a bundle of field lines, which is a substitution for a ribbon in the hydrodynamical setting.

We also refer to the paper by Bott and Taubes [B-T] for a purely topological notion of the self-linking number of a knot, which has been conceived in the context of the Chern–Simons topological quantum field theory and then decoupled from the group structure involved (see the references therein for the earlier papers by D. Bar-Natan, by A. Guadaguini, M. Martinelli, and M. Mintchev, and by M. Kontsevich). In the next section we describe the relation of the linking numbers to the Chern–Simons functional.

7.E. Holomorphic linking number. Many real notions in mathematics have their complex counterparts. The analogies can be as "straightforward" as the correspondence of real and complex manifolds, or of the groups of orthogonal and unitary matrices ($O(n)$, $U(n)$), or much more elaborate, say, the Stiefel–Whitney and Chern characteristic classes of vector bundles. Another nontrivial example is the duality of the homotopy groups π_0 (in the real setting) and π_1 (in the complex setting). It can be understood as follows: The number of connected components (π_0) is a measure of complexity of the complement to a hypersurface in a real manifold. On the other hand, a complex hypersurface does not split a complex manifold, and it can be bypassed. The fundamental group (π_1) measures the complexity of the complement in the latter case. We refer to [Arn23, Kh2] for other examples of informal complexification.

Here we discuss a complex counterpart of the notion of linking number (following the ideas of [At]; see [KhR, FKT, Ger, F-K]). Instead of linking two smooth closed curves in a simply connected real three-manifold, we will deal with an invariant associated to a pair of closed complex curves (Riemann surfaces) in a complex three-dimensional (i.e., of real dimension 6) manifold. In the scketch below we always assume that the described manifolds and forms exist, and we briefly mention the necessary existence conditions.

REMARK 7.21. The classical linking number lk is an integer topological invariant equal to the algebraic number of crossings of one curve in $\mathbb{R}^3$ with a two-dimensional surface bounded by the other curve (Fig.25). The topological invariance of lk and its independence of the choice of surface follow from the fact that the algebraic number of intersections of a closed curve and a closed surface is equal to zero.

The latter invariance can also be viewed as the Stokes formula for δ-type forms supported on closed curves and surfaces (cf. Remark 4.7). The Stokes formula, and more generally, the De Rham theory of smooth differential forms, has a genuine real flavor: One considers real manifolds with boundary and an appropriate orientation, the $\mathbb{Z}/2\mathbb{Z}$-valued invariant.

One argues in [F-K, Kh2] that the Leray theory of meromorphic forms on complex manifolds is an informal complexification of De Rham theory. The Leray residue formula is a higher-dimensional generalization of the Cauchy formula, which gives the value of a contour integral of a meromorphic 1-form via the form's residue at the pole. It "replaces" the Stokes formula in the complexification. Instead of restricting a form to the boundary, one takes the residue of a meromorphic form at the polar set.

To define the Leray residue, let ω be a closed meromorphic k-form on a compact complex n-dimensional manifold M with poles on a nonsingular complex hypersurface $N \subset M$. All poles here and below are supposed to be of the first order. Let ψ be a function defining N in a neighborhood of some point $p \in N$. Then locally, in a certain neighborhood $U(p)$, the k-form ω can be decomposed into the sum

$$\omega = \frac{d\psi}{\psi} \wedge \alpha + \beta, \tag{7.3}$$

where α and β are holomorphic in $U(p)$. One can show that the restriction $\alpha\big|_N$ is a well-defined (i.e., independent of ψ) holomorphic $(k-1)$-form (see [Ler]).

DEFINITION 7.22. The *form-residue* res ω of the closed meromorphic k-form ω is the holomorphic $(k-1)$-form on N such that, in any neighborhood $U(p)$ of an arbitrary point $p \in N$, it coincides with the form $\alpha\,|_N$ of the decomposition (7.3):

$$\mathrm{res}\ \omega = \alpha\,|_N\ .$$

Similarly, one defines the residue in the case of polar sets consisting of several complex hypersurfaces in a general position in M.

REMARK 7.23. For a complex manifold M with $h^{n,1}(M) := \dim H^1(M, \Omega^n) = 0$, every holomorphic $(n-1)$-form on N is the residue of some meromorphic n-form on M with poles on N of the first order; see, e.g., [Chr]. This meromorphic n-form on M is defined by its residue on N uniquely up to a holomorphic n-form on M. Note that the condition $h^{n,1}(M) = 0$ in the complex setting can be thought of as an analogue of simple-connectedness of a real manifold.

Now let $C_1, C_2 \subset M$ be two complex closed nonintersecting curves in a complex closed three-fold M: $C_1 \cap C_2 = \emptyset$. Fix some holomorphic differentials α_1 and α_2 on the curves C_1 and C_2, respectively, and a meromorphic 3-form η on M, satisfying the following condition: The zero locus of η intersects neither of the curves C_1 and C_2 (e.g., if M is a Calabi–Yau manifold, it possesses a nonvanishing holomorphic 3-form η, unique up to a factor). The number we are going to assign to this pair of curves depends linearly on α_1, α_2, and η^{-1}.

Suppose that there exists a complex surface S_1 in M that contains the complex curve C_1. Denote by β_1 any meromorphic 2-form on S_1 with a polar set on the curve C_1, and such that the residue of this 2-form β_1 is equal to α_1: res $\beta_1|_{C_1} = \alpha_1$. By virtue of the remark above, such a 2-form β_1 exists as soon as there is a complex surface $S_1 \subset M$ containing the curve C_1 and such that $H^1(S_1, \Omega^2) = 0$.

DEFINITION 7.24 [KhR, FKT]. The *holomorphic linking number* $lk_{\mathbb{C}}$ of the pair of complex curves C_j with chosen holomorphic differentials α_j on them (in the manifold M with the meromorphic form η) is the following sum over all intersection points of the surface S_1 and the curve C_2:

$$lk_{\mathbb{C}}\left((C_1, \alpha_1), (C_2, \alpha_2)\right) := \sum_{S_1 \cap C_2} \frac{\beta_1 \wedge \alpha_2}{\eta}. \tag{7.4}$$

Note that the 3-form $\beta_1 \wedge \alpha_2$ is well-defined at the points of intersection $S_1 \cap C_2$, and the ratio on the right-hand side measures its proportionality coefficient with the 3-form η at the same points.

Unlike the real case, the holomorphic linking number is not integer valued, and it is not an isotopy invariant. Its value can be any complex number, and it depends on the mutual location of the complex curves C_1 and C_2 in M, as well as on the differential forms α_1, α_2, and η involved. However, it will be the same for all additional choices.

PROPOSITION 7.25.

i) The holomorphic linking number $lk_{\mathbb{C}}$ *is well-defined; i.e., it does not depend on the choice of complex surface* $S_1 \supset C_1$ *or the meromorphic two-form* β_1 *on it, provided that* $res\ \beta_1|_{C_1} = \alpha_1$ *(Fig.45).*

ii) The value $lk_{\mathbb{C}}$ *is a symmetric function of its arguments: One gets the same linking number by embedding the curve* C_2 *into a complex surface* S_2*, taking a meromorphic form* β_2 *such that* $res\ \beta_2|_{C_2} = \alpha_2$*, and forming the sum*

$$lk_{\mathbb{C}}\left((C_1,\alpha_1),(C_2,\alpha_2)\right) = \sum_{C_1\cap S_2} \frac{\alpha_1 \wedge \beta_2}{\eta}. = lk_{\mathbb{C}}\left((C_2,\alpha_2),(C_1,\alpha_1)\right).$$

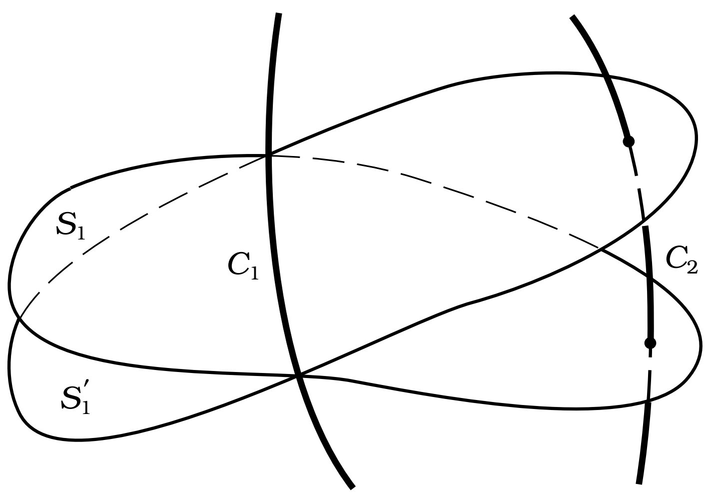


FIGURE 45. The holomorphic linking number of complex curves C_1 and C_2 counts the contributions of the intersections of the curve C_2 with a surface $S_1 \supset C_1$, or equivalently with another surface $S_1' \supset C_1$.

PROOF. Assume that the complex curve C_1 is a transversal intersection of two complex surfaces S_1 and S_1', and each of the surfaces is equipped with a meromorphic 2-form (respectively, β_1 and β_1') whose residues on C_1 are α_1. Define a meromorphic 3-form γ_1 on M with poles (of the first order) on S_1 and S_1' and residues β_1 and $-\beta_1'$, respectively. These conditions on the form γ_1 are consistent.

Indeed, on the intersection of two surfaces the form (second) residue depends on the order in which the repeated residue is taken: It differs by the sign. For example, according to the order, the form $dx \wedge dy/xy$ has the second residue 1 or -1 at the origin:

$$\begin{aligned} \mathrm{res}|_{y=0}\mathrm{res}|_{x=0}\frac{dx \wedge dy}{xy} &= \mathrm{res}|_{y=0}\frac{dy}{y} = 1, \text{ while} \\ \mathrm{res}|_{x=0}\mathrm{res}|_{y=0}\frac{dx \wedge dy}{xy} &= -\mathrm{res}|_{x=0}\frac{dx}{x} = -1. \end{aligned}$$

Similarly, the second residue of the 3-form γ_1 on the curve $C_1 = S_1 \cap S_1'$ is the 1-form α_1 or $-\alpha_1$. For instance,

$$\mathrm{res}|_{C_1}\mathrm{res}|_{S_1}\gamma_1 = \mathrm{res}|_{C_1}\beta_1 = \alpha_1.$$

Then, by the definition of the holomorphic linking number (7.4),

$$lk_{\mathbb{C}}\left((C_1, \alpha_1), (C_2, \alpha_2)\right) = \sum_{S_1 \cap C_2} \frac{(\mathrm{res}\ \gamma_1) \wedge \alpha_2}{\eta},$$

since $\mathrm{res}\ \gamma_1|_{S_1} = \beta_1$. The latter ratio at every point of $S_1 \cap C_2$ is equal to

$$\mathrm{res}\ (\frac{\gamma_1}{\eta} \wedge \alpha_2),$$

where $\frac{\gamma_1}{\eta}$ is a meromorphic function on M, and $\frac{\gamma_1}{\eta} \wedge \alpha_2$ is a meromorphic 1-form defined on C_2. Indeed, one can easily see that the equality

$$\frac{(\mathrm{res}\ \gamma_1) \wedge \alpha_2}{\eta} = \mathrm{res}\ (\frac{\gamma_1}{\eta} \wedge \alpha_2)$$

holds at every point of the intersection $S_1 \cap C_2$ by doing calculations in local coordinates.

Then $lk_{\mathbb{C}}$ is the sum of residues of the meromorphic 1-form $\frac{\gamma_1}{\eta} \wedge \alpha_2$ on the complex curve C_2 at the poles $S_1 \cap C_2$. By using the surface S_1' instead of S_1 for the same calculation, one obtains minus the sum of residues of the same 1-form $\frac{\gamma_1}{\eta} \wedge \alpha_2$ on C_2, where the residues are taken at the poles $S_1' \cap C_2$. The latter follows from the assumption that $\mathrm{res}\ \gamma_1|_{S_1'} = -\beta_1'$.

The Cauchy theorem states that the sum of residues of a meromorphic 1-form on a complex curve is equal to zero. We apply it to the meromorphic 1-form $\frac{\gamma_1}{\eta} \wedge \alpha_2$ on the complex curve C_2. Then the sum of the form's residues at all poles, i.e., at the points of intersection of C_2 with both S_1 and S_1', is equal to zero. This shows that $lk_{\mathbb{C}}$ does not depend on whether we use the surface S_1 or S_1' (statement (i)).

The symmetry of $lk_{\mathbb{C}}$ can be immediatelly seen if we present C_2 as a transversal intersection of two surfaces S_2 and S_2' and associate to it a meromorphic 3-form γ_2 in the same way as above. Then

$$lk_{\mathbb{C}}\left((C_1,\alpha_1),(C_2,\alpha_2)\right) = \sum_{S_1\cap S_2'\cap S_2} \operatorname{res}^3\left(\frac{\gamma_1\wedge\gamma_2}{\eta}\right),$$

where res^3 is the residue of the meromorphic 3-form

$$\frac{\gamma_1\wedge\gamma_2}{\eta}$$

at the triple intersections $S_1\cap S_2'\cap S_2$. The skew symmetry of the wedge product and the sign change when passing to the intersections $S_1\cap S_1'\cap S_2$ complete the proof of (ii). $\square$

REMARK 7.26. The main reason for introducing the complex linking number is that it arises as the "first approximation" of the complex analogue of the Chern–Simons functional (see [FKT, FKR] and Remark 8.9). The standard linking number governs the asymptotics of the classical Chern–Simons functional ([Pol, Wit2], Section 8 below).

REMARK 7.27. In a real three-dimensional manifold M, a knot (or link) invariant is a locally constant function on the space of embeddings of a circle (respectively, a union of circles) into the manifold M. In [VasV], V. Vassiliev defined the jump of an invariant as the function assigned to the immersions of the circle with one point of self-intersection and whose value is equal to the difference of the knot invariant on the embeddings "before" and "after" the self-intersection. (Here the notions of "before" and "after" are determined by the orientation of the circle and of the ambient manifold M.) One can iterate the jumps and define the function on immersions with any finite number of self-intersection points.

By definition, the Vassiliev invariant of order k is a knot or link invariant whose jump function vanishes on all immersions with at least $k+1$ self-intersection points. In particular, one has the following

PROPOSITION 7.28. *The linking number of two curves in $\mathbb{R}^3$ is an invariant of order 1.*

REMARK 7.29. The holomorphic linking number $lk_{\mathbb{C}}$ is not defined if the two complex curves C_1 and C_2 intersect, and it tends to infinity as the curves approach each other. We suggest the following "meromorphic" counterpart of the Vassiliev theory.

Let M be a complex three-dimensional manifold equipped with a nonvanishing holomorphic form η. Denote by $\mathcal{M}_j$ the moduli space of all embedded holomorphic curves of fixed genus g_j $(j = 1, 2)$ in the complex manifold M. The space $\mathcal{M}_j$ is a finite-dimensional complex manifold by itself (and we assume that its dimension is nonzero). The product $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ can be thought of as a complex analog of the space of (real) knots or links.

Similar to the real case, it is natural to call the *discriminant* $\Delta \subset \mathcal{M}$ the subset of all configurations in the moduli space $\mathcal{M}$ such that the curves C_1 and C_2 hit each other. The discriminant Δ is a (singular) complex hypersurface in $\mathcal{M}$, and its regular points Δ_0 correspond to simple intersections of the curves C_1 and C_2. Further degenerations of the discriminant variety $\Delta \supset \Delta_0 \supset \Delta_1 \supset \dots$ are stratified by the number and multiplicity of the intersections.

It would be interesting to define the holomorphic linking number $lk_{\mathbb{C}}$ as a closed differential form on the moduli space $\mathcal{M}$ or on some bundle over it. Since $lk_{\mathbb{C}}$ tends to infinity as the two curves get close to each other, this differential linking form is supposed to have a pole of first order along (the regular part Δ_0 of) the discriminant Δ. In particular, the corresponding residue might be well-defined along Δ_0.

More generally, one can call a *complex link invariant* of complex curves of genera $g_1, g_2, \dots, g_m$ in a complex three-manifold M any closed meromorphic k-form on the appropriate moduli space $\mathcal{M} := \mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_m$ of the holomorphic embeddings in M.

DEFINITION 7.30. A *complex link invariant of order k* is a closed meromorphic form on the moduli space $\mathcal{M}$ whose $(k+1)$-st residue vanishes on all strata Δ_k of the discriminant $\Delta \subset \mathcal{M}$ that correspond to embeddings of complex curves with k points of pairwise intersections.

PROBLEMS 7.31. A) Show that the complex linking form $lk_{\mathbb{C}}$ can be defined as a complex link invariant of order 1. Similarly, one can try to define the complex analogues of Massey products and of other cohomological operations on knots and links.

B) Give an ergodic interpretation of the holomorphic version of the linking number in the spirit of Section 4.

§8. Asymptotic holonomy and applications

8.A. Jones–Witten invariants for vector fields. There is a diversity of subtle invariants for knots and links. For instance, one might consider the knot polynomials (of Alexander, Kauffman, Jones, HOMFLY, Reshetikhin and Turaev, etc.) or the Vassiliev invariants of finite order (see, e.g., [Tu2, VasV]). It is of great interest to extend the domain of such invariants to the case of (divergence-free) vector fields, to "diffuse knots" in the three-space $\mathbb{R}^3$. From this standpoint, a regular knot is understood as a vector field supported on a single closed curve.

The classical (combinatorial) approach to introducing the knot invariants is based on some type of recurrence relation: One starts with an unknot and defines a precise recipe for how the invariant changes under elementary surgeries (for example, the connected sum). This strategy seemed to be nonapplicable to extending the definitions to vector fields.

The situation changed after Witten's generalization [Wit2] of the Jones polynomial to arbitrary closed 3-manifolds in terms of the asymptotics of the Chern–Simons functional on the space of connections over the manifold. The structure group of the connection gives one more parameter to the problem, and the actual Jones polynomial corresponds to the $SU(2)$-connections.

The extension of Witten's approach from links to "diffuse knots" was started by Verjovsky and Freyer in [V-F], and we present below the main steps of that paper. In the abelian case of the $U(1)$- (or $GL(1)$-)connections the asymptotics in question are essentially determined by the helicity invariant of the corresponding divergence-free vector field. The $GL(n)$-version of the asymptotic monodromy along a nonclosed trajectory of a vector field is provided by Oseledets's multiplicative ergodic theorem [Ose1]. However, the extension of the invariants to the nonabelian case encounters serious obstacles arising from the lack of a nonabelian version of the Birkhoff ergodic theorem on the equality of time and space averages.

Let M be a closed compact real three-manifold M and let $L \subset M$ be a link (a disjoint union $L = \cup_{i=1}^{n} C_i$ of smoothly embedded circles C_i). Further, let $P = M \times G$ be the G-principal bundle over M, where the structure group G might be $U(1)$ or $SU(2)$.

Denote by $\mathcal{A}$ the space of all connections in the (trivial) bundle P. It can be identified with the affine space $\Lambda^1(M, \mathfrak{g})$ of 1-forms on M with values in the Lie algebra $\mathfrak{g}$ of G. Finally, let $\tilde{G} = C^\infty(M, G)$ be the current group of fiber-preserving automorphisms of P.

Definition 8.1. The *Jones–Witten invariant* of a link $L \subset M$ is the following function of k:

$$W_L(k) = \int\limits_{\mathcal{A}/\tilde{G}} \left\{ \exp\ (ik \int\limits_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)) \cdot \prod_{C_i \subset L} tr(P \exp \int\limits_{C_i} A) \right\} \cdot DA,$$

where $P\exp$ is the path ordered exponential integral, and DA is "an appropriate measure on the moduli space of the connections." From the mathematical point of view, neither DA nor W has a sound definition.

Witten showed in [Wit2] that for $M = S^3$ and $G = SU(2)$ this corresponds to the Jones polynomial (in k) for the link L. Though justification of the meaning of this integral is still not complete, it looks a lot simpler for an abelian group G, say $U(1)$:

$$W_L(k) = \int\limits_{\mathcal{A}/\tilde{G}} \left\{ \exp\ (ik \int\limits_M A \wedge dA) \cdot \prod_{C_i \subset L} (\exp \int\limits_{C_i} A) \right\} \cdot DA. \tag{8.1}$$

One can think of $\int_M A \wedge dA$ as a quadratic form $Q(A)$ on $\mathcal{A}$, while the line integral $\int_{C_i} A$ is regarded as a linear functional (the so-called *De Rham current*) $I_{C_i}(A)$ evaluated at the 1-form A.

For the abelian case (see [SchA, Pol]), the path integral modulo factors related to a regularization and topology of the manifold M is equal to

(8.2)

$$W_L(k) = \text{const} \cdot \exp\ \left\{ \frac{i}{2k} \sum_{i,j} \langle I_{C_i}, d^{-1} I_{C_j} \rangle \right\} = \text{const} \cdot \exp\ \left\{ \frac{i}{2k} \sum_{i,j} lk(C_i, C_j) \right\}.$$

The regularization is needed to define the linking number for each curve C_i with itself (cf. the definition of self-linking number in Section 7.D). The topological factor, being the value of $W_L(k)$ in the case without any link ($L = \emptyset$), is the Ray–Singer torsion of the manifold M [SchA].

REMARK 8.2. Heuristically, one computes here a quadratic Gaussian integral of the type

$$\int_{\mathbb{R}^n} e^{ik\langle x, Qx\rangle} e^{i\langle b, x\rangle} (\pi^{-n/2})\ dx,$$

which, upon the extraction of a complete square, is equal to

$$\begin{aligned} &e^{\frac{i}{2k}\langle b, Q^{-1}b\rangle} \int_{\mathbb{R}^n} e^{ik\langle x+Q^{-1}b, Q(x+Q^{-1}b)\rangle} (\pi^{-n/2})\ dx \\ =&e^{\frac{i}{2k}\langle b, Q^{-1}b\rangle} (\det Q)^{-1/2} (e^{(\frac{i\pi}{4} \cdot \operatorname{sign} Q)}). \end{aligned}$$

One can apply this formula to the (completion of the) infinite-dimensional space $\mathcal{A} = \Omega^1(M, \mathfrak{g})$ in the case of the quadratic form $Q(A) = \int_M A \wedge dA$. Since the form Q is degenerate, the integration is carried out only along a subspace in the space $\mathcal{A}$ transversal to the kernel of Q. This corresponds to integration over the $\tilde{G}$-quotient of the space $\mathcal{A}$, see [Wit2, V-F]. Although this differs from the above case of a nondegenerate form, here we are interested only in the factor $e^{\frac{i}{2k}\langle b, Q^{-1}b\rangle}$, which has a straightforward analogue.

In our context, this factor turns out to be the linking term:

$$e^{\frac{i}{2k}\langle b, Q^{-1}b\rangle} = \exp\left\{\frac{i}{2k}\sum_{i,j}\langle I_{C_i}, d^{-1}I_{C_j}\rangle\right\}.$$

The ergodic ("diffuse") version of this approach has to do with notions of asymptotic and average holonomy. (One can think of diffusing the knot as the way of its regularization: The neighboring trajectories can be regarded as a framing. In particular, it allows one to determine the knot self-linking as the linking number of the knot with its shift in the direction of the frame.)

DEFINITION 8.3. The *asymptotic holonomy* of a connection A along the trajectory $\Gamma_\xi(p) = \{g_\xi^t \mid t \geq 0\}$ of a vector field ξ issuing from a point $p \in M$ is the following element of the Lie group G:

$$P\exp\int_{\Gamma_\xi(p)} A := \lim_{T\to\infty} P\exp\int_{\{g_\xi^t \mid 0\leq t\leq T\}} (\frac{1}{T}A).$$

The last integral is defined by the limiting procedure $T \to \infty$, due to the trivialization of the bundle $P = M \times G$ (or by means of a system of short paths used for the asymptotic linking number, see Definition 4.13).

It can also be thought of as follows. The (indefinite) integration

$$P\exp\int_{\{g_\xi^t \mid t\geq 0\}} A$$

along the trajectory $\Gamma_\xi(p)$ defines a curve in the Lie group G. Choose the one-parameter subgroup in G approximating this curve as $t \to \infty$. Then the asymptotic holonomy is the point $t = 1$ on the subgroup. The existence of this limit for an arbitrary group G is obscure. However, in some cases the limiting eigenvalues for almost all initial points $p \in M$ are provided by the multiplicative ergodic theorem [Osel].

Though in the nonabelian case no simple answer for the space average of the asymptotic holonomy is known (there is no matrix analogue to the Birkhoff ergodic theorem on the equality of time- and space-averages), we present the would-be definition in "full" generality (see [V-F]).

DEFINITION 8.4. The *average holonomy* $\mathrm{hol}_{\xi,\mu}(A)$ of a connection A on a divergence-free field ξ preserving the measure μ on M is the group exponent of the Lie algebra element $\int_M A(\xi)\ \mu$.

REMARK 8.5. In general, neither the average holonomy $\mathrm{hol}_{\xi,\mu}(A)$ nor its conjugacy class in the group G is gauge invariant (i.e., preserved under the change of the connection A to $A+\epsilon([A,f]+df)$ for an arbitrary $f\in C^\infty(M,\mathfrak{g})$).

In the case of abelian (say, $GL(1)$) connection, the form A is a real-valued 1-form on M, and the ordered exponent $P\exp$ becomes an ordinary exponent. Then the holonomy $\mathrm{hol}_{\xi,\mu}(A)$ is gauge invariant, and the above definitions exactly correspond to the ergodic interpretation of the Hopf (helicity) functional in terms of average linking number considered above:

THEOREM 8.5 (= HELICITY THEOREM 4.4). *For an abelian group G the multiplicative average of the asymptotic holonomy over the entire manifold M coincides with the average holonomy calculated by total integration of the "infinitesimal transforms" $A(\xi)$:*

$$\exp\int_M\left(\int_{\Gamma_\xi(p)}A\right)\ \mu_p = hol_{\xi,\mu}(A).$$

The latter identity of the two invariants suggests the following definition.

DEFINITION 8.6 [V-F]. The *Jones–Witten functional* for a divergence free *vector field* ξ on a closed three-manifold M endowed with a measure μ is the expression

$$W_{\xi,\mu}(k)=\int_{\mathcal{A}/\tilde{G}}\{\exp(ik\int_M tr(A\wedge dA+\frac{2}{3}A\wedge A\wedge A))\cdot tr(\mathrm{hol}_{\xi,\mu}(A))\}\cdot DA,$$

where the average holonomy $\mathrm{hol}_{\xi,\mu}(A)$ is defined above.

REMARK 8.7. Note that the case of an actual knot or link $L=\bigcup_{i=1}^n C_i$ can be understood as a particular case of this definition for a "δ-type" measure μ supported on a finite number of curves $\{C_i\}$.

Assume now that M is a closed three-manifold, μ is a smooth volume form on M, and ξ is a null-homologous trivial vector field on M, i.e., the two-form $i_\xi\mu$ is exact: $i_\xi\mu = d\theta$ for some 1-form θ. The case of the abelian connection can be handled completely:

THEOREM 8.8 [V-F]. *For a topologically trivial linear bundle over M (with $G = U(1)$ or $GL(1)$), the Jones–Witten functional for the vector field ξ reduces to its helicity invariant:*

$$W_{\xi,\mu}(k) = \mathrm{const} \cdot \exp\left(\frac{i}{2k} \int_M d\theta \wedge \theta\right).$$

PROOF SKETCH. The average holonomy in the abelian case is

$$\mathrm{hol}_{\xi,\mu}(A) = \exp \int_M A(\xi) \wedge \mu = \exp \int_M A \wedge d\theta = \exp(I_\xi(A)),$$

where I_ξ is the De Rham current corresponding to the field ξ. (The "diffuse" term

$$\mathrm{hol}_{\xi,\mu}(A) = \exp \int_M A(\xi) \wedge \mu = \exp \int_M \int_{\Gamma_\xi(p)} A \wedge \mu_p$$

replaces the "discrete" counterpart

$$\prod_{C_i \subset L} (\exp \int_{C_i} A) = \exp(\sum_{C_i \subset L} \int_{C_i} A)$$

in (8.1).) Then the expression (8.2) for the abelian case becomes

$$W_{\xi,\mu}(k) = \mathrm{const} \cdot \exp\left\{\frac{i}{2k} \langle I_\xi, d^{-1} I_\xi \rangle\right\} = \mathrm{const} \cdot \exp\left(\frac{i}{2k} \int_M d\theta \wedge \theta\right).$$

□

REMARKS 8.9. The *Chern–Simons functional*

$$\mathrm{CS}(A) = \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

on G-connections $\{A\}$ over real three-dimensional manifolds M has a *complex analogue* for Calabi–Yau manifolds, or, more generally, for any three-dimensional complex manifold N; see [Wit3]:

$$\mathrm{CS}_{\mathbb{C}}(A) = \int_N tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \wedge \eta,$$

where η is a holomorphic (or meromorphic) 3-form on N. In the case of the abelian group $G = GL(1, \mathbb{C})$ and a complex link L, being a disjoint union of *complex curves* C_i with holomorphic differentials α_i on them, the asymptotics of the corresponding complex analogue of the Jones–Witten functional W_L is given by the holomorphic linking number $lk_{\mathbb{C}}((C_i, \alpha_i), (C_j, \alpha_j))$ defined in Section 7.E (see [FKT, FKR, Ger]).

REMARKS 8.10. The higher linking numbers introduced in Section 7.B arise in the calculation of correlators in Chern–Simons theories in dimensions greater than 3 (see [FNRS]).

A higher-dimensional version of the Chern–Simons path integral can be regarded as a nonabelian counterpart of the corresponding hydrodynamical integral. Being an example of so-called topological field theories, by its very definition it does not require a metric to specify the action functional. Hence, all *gauge-invariant* observables in the theory are topologically invariant, provided that the measure in the path integral does not spoil the invariance under diffeomorphisms.

Let $\{A\}$ be the space of $U(1)$-connections on a manifold M^{2m+1}; DA is a shift-invariant integration measure. For a collection of cycles $C_1, \dots, C_r$ of dimensions $\dim C_i = 2d_i + 1,\ i = 1, \dots, r$, define the gauge-invariant functional

$$\Phi_{\{C_1,\dots,C_r\}}(A) := \prod_{i=1}^{r} \exp\left(\int\limits_{C_i} A \wedge (dA)^{d_i} \right).$$

Suppose that the cycles obey the linking condition (7.1): $\sum_{i=1}^{r}(m - d_i) = m + 1$. Then asymptotically for large k the expectation value of the functional Φ, that is,

$$< \Phi_{\{C_1,\dots,C_r\}}(A) >= \int \Phi_{\{C_1,\dots,C_r\}}(A) \cdot \exp\left(\frac{ik}{2m+1} \int_M A \wedge (dA)^m \right) DA,$$

is given by the exponent of the mutual linking number for the collection of cycles: $\exp\left(lk(C_1, \dots, C_r) \,/ k^{r-1}\right)$, where the number $lk(C_1, \dots, C_r)$ is the linking number of, say, C_r with the intersection of all other cycles; see Section 7.B. (To avoid the contribution of the self-linking of the cycles into the integral, one assumes the so-called *normal ordering* of the operators involved.) If the linking condition is not fulfilled, but there are sublinks saturating the condition, then the leading term in the asymptotics is given by the mutual linking numbers of these sublinks.

REMARKS 8.11. The above holonomy functional can be regarded as a counterpart of the Radon transform: given a Lie group G it sends a gauge equivalence class of the G-connections on M to a G-valued functional on the space of loops in M.

The value of the *holonomy functional* on a loop Γ is the holonomy of a connection A around Γ. In the abelian case ($G = \mathbb{R}$) the Radon transform associates to a one-form θ on M the corresponding functional I_θ on the free loop space $\mathcal{L}M$ (the space of smooth maps $S^1 \to M$):

$$I_\theta(\Gamma) = \int_\Gamma \theta.$$

In [Bry2], Brylinski characterizes the range of the Radon transform as the set of functionals on $\mathcal{L}M$ obeying a certain system of second-order linear PDE (called the Radon–John system). The necessary and sufficient conditions are constraints on the partial derivatives $\partial^2 I_\theta/\partial x_k^i \partial x_l^j$, where the coordinates $\{x_k^i\}$ are the Fourier components of small variations of the curve Γ. In dimension 2, this system gives rise to the hypergeometric systems in the spirit of [GGZ]. A nonabelian counterpart of the Radon–John equations involves the bracket iterated integrals (see [Bry2]).

Note that in three dimensions the Radon transform displays the kind of functionals on vector fields that can be defined as fluxes of fields through surfaces bounded by embedded curves (or, the same, as the average linking number of the fields and the curves). Indeed, the embedded nonparametrized curves in $\mathbb{R}^3$ form a subset in the dual $S\mathrm{Vect}(\mathbb{R}^3)^*$ of the Lie algebra of divergence-free vector fields in the space (see Section VI.3). A curve $\Gamma \subset \mathbb{R}^3$ defines the functional whose value at a divergence-free field ξ is the flux of ξ through Γ.

To relate it to the description above, fix a vector field ξ and assume that μ is a volume form in the space. Let θ be a one-form such that $i_\xi\mu = d\theta$ (ξ is the vorticity field for θ). Then $I_\theta(\Gamma) := \int_\Gamma \theta = \{$ flux of ξ through Γ $\}$ can be regarded as the functional on the Γ's. A regular element of the dual space $S\mathrm{Vect}(\mathbb{R}^3)^*$ is a "diffuse" loop Γ, a divergence-free vector field η (see Section I.3), while the pairing is

$$I_\theta(\eta) := \int_{\mathbb{R}^3} \theta(\eta)\ \mu = \mathcal{H}(\xi, \eta).$$

8.B. Interpretation of Godbillon–Vey-type characteristic classes. Let $\mathcal{F}$ be a cooriented foliation of codimension 1 on the oriented closed manifold M, and θ a 1-form determining this foliation. Then $d\theta = \theta \wedge w$ for a certain 1-form w.

PROPOSITION 8.12 (SEE, E.G., [Fuks]). *The form $w \wedge dw$ is closed, and its cohomology class does not depend on the choices of θ and w.*

DEFINITION 8.13. The cohomology class of the form $w \wedge dw$ in $H^3(M, \mathbb{R})$ is called the *Godbillon–Vey class* of the foliation $\mathcal{F}$.

On a three-dimensional manifold this class is defined by its value on the fundamental 3-cycle:

$$GV(\mathcal{F}) = \int_M w \wedge dw.$$

Let $\mathbf{v}$ be an arbitrary vector field with the sole restriction $\theta(\mathbf{v}) = 1$, and let $L_{\mathbf{v}}^k$ denote the kth Lie derivative along $\mathbf{v}$.

THEOREM 8.14 (SEE [Sul, Th1]). $\quad GV(\mathcal{F}) = -\int_M L_{\mathbf{v}}^2\theta \wedge d\theta.$

If M^3 is a manifold equipped with a volume form μ, the class GV admits an ergodic interpretation in terms of the asymptotic Hopf invariant of a special vector field.

Define the vector field ζ by the relation

$$i_\zeta \mu = L^2_{\mathbf{v}} \theta \wedge \theta.$$

COROLLARY 8.15 [Tab1]. *The vector field ζ is null-homologous, and its asymptotic Hopf invariant is equal to the Godbillon–Vey invariant of the foliation $\mathcal{F}$.*

PROOFS. By the homotopy formula (see Section I.7.B)

$$L_{\mathbf{v}} \theta = d i_{\mathbf{v}} \theta + i_{\mathbf{v}} d\theta = i_{\mathbf{v}} \theta \wedge w = w - f\theta,$$

where the function f is $f = w(\mathbf{v})$. This implies that $(L_{\mathbf{v}}\theta) \wedge \theta = w \wedge \theta = -d\theta$, and moreover,

$$(L^2_{\mathbf{v}}\theta) \wedge \theta = L_{\mathbf{v}}((L_{\mathbf{v}}\theta) \wedge \theta) = L_{\mathbf{v}}(-d\theta) = -dL_{\mathbf{v}}\theta = -d(w - f\theta).$$

Hence we can take $w' = w - f\theta$ as a new 1-form w in the definition of the Godbillon–Vey class. Theorem 8.10 readily follows:

$$\int_M L^2_{\mathbf{v}}\theta \wedge d\theta = \int_M L^2_{\mathbf{v}}\theta \wedge \theta \wedge w' = -\int_M dw' \wedge w' = -GV(\mathcal{F}).$$

The null-homologous property for the field ζ also follows from the fact that the 2-form $(L^2_{\mathbf{v}}\theta) \wedge \theta = i_\zeta \mu$ is a complete differential. Furthermore, the asymptotic Hopf invariant of ζ is

$$\mathcal{H}(\zeta) = \int_M i_\zeta \mu \wedge d^{-1}(i_\zeta \mu) = \int_M dw' \wedge w' = GV(\mathcal{F}).$$

□

REMARK 8.16 [Sul, Th1, Tab1]. Having defined an auxiliary vector field ξ (tangent to the leaves of $\mathcal{F}$) by the relation

$$i_\xi \mu = L_{\mathbf{v}}\theta \wedge \theta,$$

one may argue that it measures the rotation of the tangent planes to the foliation in the transversal direction $\mathbf{v}$. Namely, the direction of ξ is the axis of rotation, and the modulus of ξ is the angular velocity of the rotation. Then one may say that ζ measures the acceleration of the rotation, and the above statement reads: $GV(\mathcal{F})$ is the asymptotic Hopf invariant of this rotation acceleration field.

As an element of $H^3(M, \mathbb{R})$, the Godbillon–Vey class on manifolds of higher dimensions is determined by its values on 3-cycles. Any such value coincides with the asymptotic Hopf invariant of the corresponding field ζ, constructed for the induced foliation on the 3-cycle.

Similarly, one can define the asymptotic and integral Bennequin invariants for a null-homologous vector field on a contact simply connected three-manifold (see [Tab1]). These invariants generalize the classical Bennequin definition of the self-linking number of a curve transverse to the contact structure [Ben]. Interesting polynomial invariants of Legendrian curves (and more generally, of framed knots) in a solid torus, generalizing the Bennequin invariant, have been introduced by Aicardi [Aic] (see also [FuT, Fer, Pl2]).

In conclusion, we refer to [DeR, SchL, SchS, GPS, Sul] and references therein for various questions related to structure cycles, asymptotic cycles, approximations of cycles by flows and foliations, and the corresponding smoothness conditions.

PROBLEMS 8.17. A) Give an ergodic interpretation of the global real-valued invariant of three-dimensional CR-manifolds found in [B-E].

Roughly speaking, a CR-structure on a $(2n+1)$-dimensional manifold is defined by choosing an n-dimensional integrable subbundle $T^{1,0}M$ of the complexified tangent bundle of M. In particular, this subbundle determines a distribution of the corresponding contact elements on M. The CR-structure gives rise to a real-valued (Chern–Simons-type) 3-form (defined modulo an exact form) on the manifold.

B) It would be interesting to consider whether similar techniques can be applied to generalize the Casson invariant and the Floer homology of homological 3-spheres to aspherical $(4k-1)$-manifolds with an additional structure (say, to contact manifolds); see [CLM, Arn24].