CHAPTER II

TOPOLOGY OF STEADY FLUID FLOWS

Cold and warm ocean currents (for instance, the Gulf Stream) determine the climate of continents beyond the reach of human intervention. The power of the currents influence is due to their permanentness and stability. In this chapter we are going to study the corresponding idealized model of steady flows of an incompressible fluid. Such flows are stationary solutions of the Euler equation, and they have very peculiar topology and existence conditions. They often turn out to be "attractors" in phase space of the viscous Navier–Stokes equation. In this case the structure of such flows might give an "approximate picture" of an arbitrary fluid motion after a long period of time.

§1. Classification of three-dimensional steady flows

1.A. Stationary Euler solutions and Bernoulli functions. In this chapter we will be dealing with solutions of the Euler equation that do not depend on time.

DEFINITION 1.1. An ideal *steady* (or *stationary*) incompressible fluid flow $v(x)$ in a domain $M \subset \mathbb{R}^n$ is a divergence-free solution ($\operatorname{div} v = 0$) of the *stationary Euler equation*

$$0 = -(v, \nabla)v - \nabla p \ ,$$

for some pressure function p on M.

The same equation in the form $-\nabla_v v - \nabla p = 0$ for a velocity field satisfying $L_v \mu = 0$ is valid for an arbitrary n-dimensional Riemannian manifold M with measure μ.

For the three-dimensional case ($n = 3$), a virtually complete description of analytic stationary flows is given by the following theorem:

THEOREM 1.2 [Arn3,4,16]. *Assume that the region $M \subset \mathbb{R}^3$ is bounded by a compact analytic surface, and that the field of velocities is analytic and not everywhere collinear with its curl. Then the region of the flow can be partitioned by an analytic submanifold into a finite number of cells, in each of which the flow is constructed in a standard way. Namely, the cells are of two types: those fibered into tori invariant under the flow and those fibered into surfaces invariant under the flow, diffeomorphic to the annulus $\mathbb{R} \times S^1$*

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(see Fig.9). On each of these tori the flow lines are either all closed or all dense, and on each annulus all flow lines are closed.

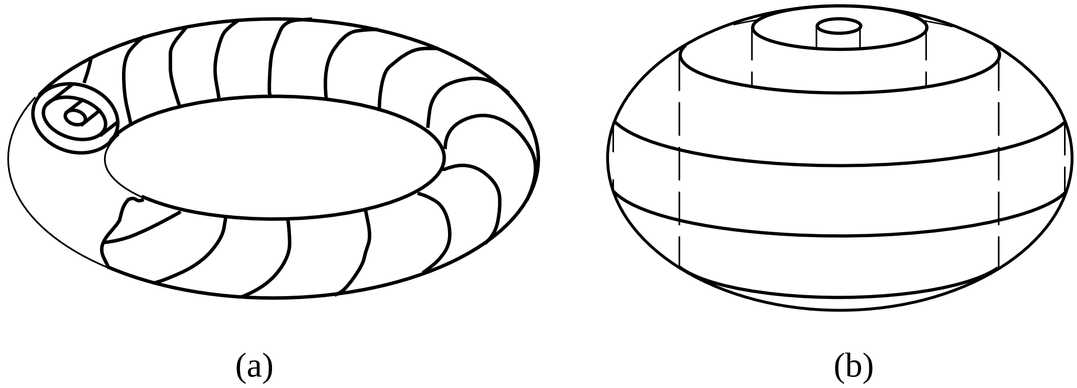

FIGURE 9. Regions of a steady flow fibered (a) into tori and (b) into annuli.

The stationary Euler equation $(v, \nabla)v = -\nabla p$ in $M \subset \mathbb{R}^3$ can be rewritten as

$$v \times \text{ curl } v = \nabla\alpha$$

for the function $\alpha = p + \frac{\|v\|^2}{2}$.

DEFINITION 1.3. The function $\alpha : M \to \mathbb{R}$ defined by the relation $v \times \text{ curl } v = \nabla\alpha$ (modulo an additive constant) is called the *Bernoulli function* of the steady flow v.

By the very definition, the velocity field v, as well as the vorticity field curl v, is tangent to the level surfaces of the Bernoulli function α. In other words, α is the first integral of the flow defined by the field v in the domain M.

Note that the stationary three-dimensional Navier–Stokes equation (describing a viscous incompressible fluid) generically does not admit any nontrivial first integrals [Ko3].

REMARK 1.4. In invariant terms the stationary Euler equation

$$L_v u = -dp$$

is equivalent to $i_v du + di_v u = -dp$, or to the equation

$$i_v du = -d\alpha \qquad \text{for } \alpha = p + i_v u.$$

The invariance of α (i.e., $L_v\alpha = 0$) follows from the relation $i_v d\alpha = 0$.

Note that the condition $v \times$ curl $v = \nabla\alpha$ in $\mathbb{R}^3$ can be reformulated in a form valid for any manifold M: The vector fields v and curl v commute ($\{v, \text{curl } v\} \equiv 0$). To verify this for a three-dimensional Riemannian manifold M, one employs the following formula of vector calculus:

$$\text{curl } (\eta \times \xi) = \{\xi, \eta\} + \eta \ (\text{div } \xi) - \xi \ (\text{div } \eta) \tag{1.1}$$

on any three-dimensional Riemannian manifold. (Here $(\eta \times \xi)$ is the vector field dual to the 1-form $i_\xi i_\eta \mu$ on M : $(i_\xi i_\eta \mu)\zeta = \mu(\eta, \xi, \zeta) = (\eta \times \xi, \zeta)$.) By taking vorticity of both sides of $v \times$ curl $v =$ grad α, we obtain $\{v, \text{curl } v\} \equiv 0$.

The classification theorem above relies on the following observation about the structure of α-level surfaces for a three-dimensional manifold M.

PROPOSITION 1.5. *Every noncritical level surface of α that does not intersect the boundary of M^3 is diffeomorphic to a torus. For appropriate variables $(\varphi_1, \varphi_2 \mid \text{ mod } 2\pi)$ and z in a neighborhood of such a torus both fields v and $\xi =$ curl v have constant components*

$$v = v_1(z)\frac{\partial}{\partial \varphi_1} + v_2(z)\frac{\partial}{\partial \varphi_2}, \qquad \xi = \textit{curl } v = \xi_1(z)\frac{\partial}{\partial \varphi_1} + \xi_2(z)\frac{\partial}{\partial \varphi_2},$$

along the torus with angular coordinates (φ_1, φ_2), while z indexes the tori.

The coordinates φ_1, φ_2, z are analogues of the action-angle variables of classical mechanics. The theorem means, in particular, that the field lines of both v and curl v lie on the tori $\alpha =$ const. These lines on a given torus are either closed (if the ratio of the frequencies v_2/v_1 for the field v, resp. ξ_2/ξ_1 for the field $\xi =$ curl v, is rational) or dense. The proof is given in Section 1.B.

REMARK 1.6. In the case of $\alpha \equiv$ *const* (all α-levels are critical), the fields v and curl v are collinear at each point ($v \times$ curl $v \equiv 0$). Such fields are called *force-free* fields in magnetohydrodynamics.

If a force-free field v is nowhere zero, then curl $v = \varkappa \cdot v$, where the "ratio" $\varkappa : M \to \mathbb{R}$ is a smooth function. The function $\varkappa$ is a first integral of the field v (as well as of the field curl v). Indeed, $0 \equiv$ div (curl v) $=$ div $\varkappa \cdot v =$ (grad $\varkappa, v$). Hence, every connected component of a nonsingular level surface of $\varkappa$ is a torus, since such a surface is oriented and it admits a nonvanishing tangent vector field v (the same reasoning is used in the proof of Proposition 1.5, see Section 1.B). The field lines of v are windings on these tori (in the corresponding coordinates φ_1, φ_2, z, the frequency ratios $\dot\varphi_1/\dot\varphi_2 = \kappa(z)$ will be constant along the field lines of v). Therefore, even in the case of a force-free field the field lines lie on two-dimensional tori, provided that the field does not have zeros and the function $\varkappa$ is not constant.

A force-free field v with curl $v = \lambda v$, where λ is a constant (i.e., an eigenfield v of the curl operator), can have a much more complicated topology.

DEFINITION 1.7. The eigenfields of the operator "curl" are called *Beltrami* fields.

COROLLARY 1.8. *If a steady analytic flow has a trajectory that is not contained in any analytic (singular) surface, then the flow is defined by a Beltrami field.*

Indeed, non-Beltrami flows enjoy a first integral (either the Bernoulli function α or the ratio function $\varkappa$).

EXAMPLE 1.9. On the three-dimensional torus $\{(x, y, z) \mid \mod 2\pi\}$, a family of Beltrami fields is given by the so-called ABC flows

$$\begin{cases} v_x = A\sin z + C\cos y, \\ v_y = B\sin x + A\cos z, \\ v_z = C\sin y + B\cos x. \end{cases}$$

The divergence-free vector fields of this three-parameter family are eigen for the vorticity operator curl $v = v$. The ABC flows have been discovered by Gromeka in 1881, rediscovered by Beltrami in 1889, and proposed for study in the present context in [Arn4, Chi1] (see the references and details in [VasO]).

When one of the parameters A, B, or C vanishes, the flow is integrable (Fig.10). Perturbation techniques used in the near-integrable cases allows one to predict strong resonances (see discussion and results of numerical simulations in [Dom]). For such perturbations some tori filled out by field lines (magnetic surfaces) persist (see, e.g., [AKN]), whereas others are disrupted, leading to regions with chaotic behavior of trajectories. There is numerical evidence that certain trajectories densely fill three-dimensional domains (Fig.11). In particular, the search for integrable cases, carried out in [Dom] by studying complex-time singularities of field trajectories, showed (numerically) the absence of integrability for $ABC \neq 0$. For the case $A = \sqrt{3}, B = \sqrt{2}, C = \sqrt{1}$ see [Hen], while the more general situation was treated in [Dom]. The absence of meromorphic integrals for generic ABC flows with $A = B$ and for the ABC flows with $0 \neq A \neq B \neq 0$ and small $C \neq 0$ has been proven by Ziglin [Zig2].

A similar study of field symmetries and of the mutual location of stagnation points for an analogue of the ABC flow in a three-dimensional ball can be found in [Zhel].

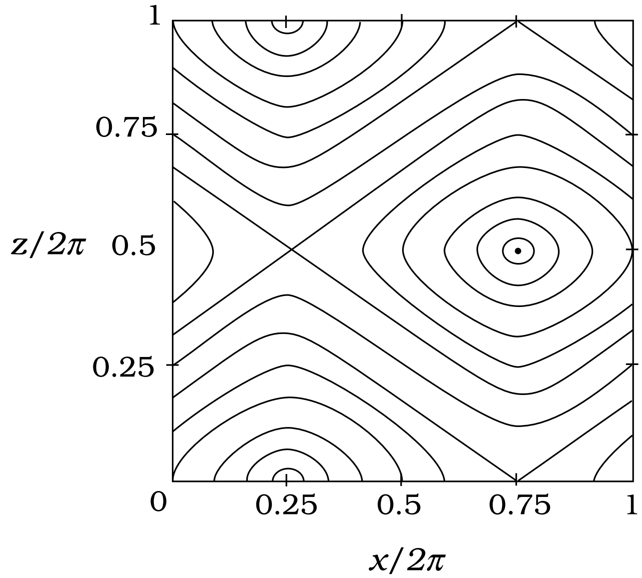

FIGURE 10. The projection of the streamlines on the (x, z)-plane in the integrable case $C = 0$ (see [Dom]). The field components do not depend on y.

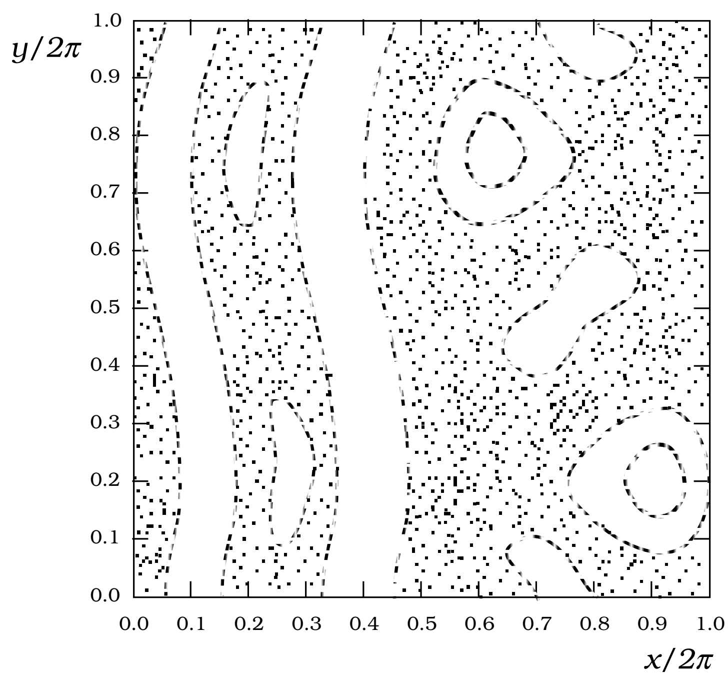

FIGURE 11. A typical Poincaré section for the ABC flows ($A^2 = 1$, $B^2 = \frac{2}{3}$, and $C^2 = \frac{1}{3}$). Some field lines seem to fill three-dimensional regions ([Hen] or [Dom]).

Note that if the field v satisfying $\operatorname{curl} v = \varkappa \cdot v$ is not divergence free, then the topological properties of its trajectories are different from those discussed here: The flow is generically nonintegrable even for a nonconstant function $\varkappa : M \to \mathbb{R}$ (see [MYZ]).

1.B. Structural theorems. We first prove a smooth analogue of (real-analytic) Theorem 1.2 for a closed manifold.

Let α be the Bernoulli function for a steady flow v on an orientable 3-dimensional manifold M without boundary. Denote by $\Gamma \subset M$ the preimage of the critical values of α.

THEOREM 1.10 (=1.5′). *Every connected component of the set $M \setminus \Gamma$ is fibered into two-dimensional tori invariant under the flow of v. The motion on each torus is quasiperiodic (the field lines are either all closed or all dense).*

PROOF. The function α is the first integral for the vector fields v and $\xi :=$ curl v. Since these fields commute, their flows give rise to an $\mathbb{R}^2$-action on every level surface of α. Each noncritical α-level is a smooth closed surface, and hence it is a torus or a Klein bottle. (In other words, the Euler characteristic of any noncritical α-level is zero: If $\nabla\alpha \neq 0$, then the velocity field v provides an example of a tangent vector field nonvanishing on the surface.) Furthermore, this surface is cooriented by $\nabla\alpha$. As a result, we see that the surface is orientable; i.e., it is a torus.

On each α-level the flow of ξ acts transitively on integral curves of v, and thus the latter are either all closed or all dense in the level surface. In the coordinates on a torus in which the $\mathbb{R}^2$-action is given by linear translations, the fields v and curl v become the vector fields with constant coefficients. $\square$

We now turn to the real-analytic theorem (we follow the exposition in [GK2]).

DEFINITION 1.11. A subset of a real analytic manifold is called *semianalytic* if locally it may be defined by a finite number of real-analytic equations and inequalities.

We will need certain properties of such sets summarized in the following

LEMMA 1.12. *Let M and N be compact connected real-analytic manifolds (possibly with boundary) and $f : M \to N$ a real-analytic map. Then*

(i) *Any semianalytic subset X of M divides M into a finite number of connected components.*
(ii) *The image $f(X)$ is a semianalytic subset of N, provided that* $\dim N \leq 2$.
(iii) *Assume that the rank of f is equal to* $\dim N$ *at at least one point of M, and Y is a nowhere dense semianalytic subset of N. Then the preimage $f^{-1}(Y)$ is semianalytic and nowhere dense in M.*

PROOF. Assertions (i) and (ii) are classical results due to Lojasiewicz [Loj]. To prove (iii) consider the set K of critical points of f. The set $f^{-1}(Y) \cap (M \setminus K)$ is nowhere dense because the restriction of f to $M \setminus K$ is a submersion. Since $\operatorname{rank} f = \dim N$ somewhere on M, the set K is, in turn, nowhere dense in M. Thus $f^{-1}(Y)$ is nowhere dense, for it is the union of two sets, each of which is nowhere dense. It is clear by definition that $f^{-1}(Y)$ is semianalytic. □

PROOF OF THEOREM 1.2. Suppose first that M is a connected manifold without boundary ($\partial M = \emptyset$). Assume also that all the data (the volume form, the metric, and the velocity field v) are real-analytic. In this case one claims that $U = M \setminus \Gamma$ has a finite number of connected components, and each of them is fibered into two-dimensional tori invariant under the flow.

Indeed, under the hypothesis of the theorem, the map $\alpha : M \to \mathbb{R}$ is analytic, and we can take $f = \alpha$. As above, let K be the critical set of α. Then $\alpha(K)$ is semianalytic by (ii) and nowhere dense by the Sard lemma. Therefore by (iii), $\Gamma = \alpha^{-1}(\alpha(K))$ is semianalytic and nowhere dense in M. Applying (i) to $X = \Gamma$, we see that U is dense in M, and U has a finite number of connected components.

To complete the proof for M without boundary, it suffices to apply Theorem 1.10.

Consider now the case of M with boundary ($\partial M \neq \emptyset$). Again, let K be the critical set of α and C the critical set of $\alpha\,|_{\partial M}$. As above, the union Y of the sets $\alpha(K)$ and $\alpha(C)$ is a semianalytic set nowhere dense in $\mathbb{R}^2$. Therefore, $\Gamma = \alpha^{-1}(Y)$ is nowhere dense, semianalytic, and invariant with respect to the flow.

Although we may not have an $\mathbb{R}^2$-action now, since M is a manifold with boundary, we do have a local $\mathbb{R}^2$-action on $M \setminus \partial M$. Furthermore, the maps $\alpha\,|_U$ and $\alpha\,|_{\partial M \cap U}$ are still proper submersions onto their images. Consider the orbit $\mathcal{O}_x$ through a point $x \in U$ of the local $\mathbb{R}^2$-action. The same argument as in the proof of Theorem 1.10 shows that $\mathcal{O}_x$ is either a torus or an annulus. In the former case the integral curves of v are all closed or all dense on $\mathcal{O}_x$. Observe that $L = \mathcal{O}_x \cap \partial M$ is invariant under the flow of v, and thus $\mathcal{O}_x$ is an annulus if and only if it meets ∂M. By the definition of U, the field ξ is transversal to ∂M along L. This implies that L is the union of two closed integral curves of v. Since we have a locally well-defined $\mathbb{R}^2$-action, all the v-streamlines on $\mathcal{O}_x$ must be closed.

Let U_0 be a connected component of U. The orbits $\mathcal{O}_x, x \in U_0$, are either all tori or all annuli. Indeed, for all $x \in U$ the levels $F_x = \alpha^{-1}(\alpha(x))$ are transversal to ∂M, and hence the connected components $\mathcal{O}_x$ of F_x are diffeomorphic to each other for all $x \in U_0$. Theorem 1.2 is proved. □

§2. Variational principles for steady solutions and applications to two-dimensional flows

2.A. Minimization of the energy. Consider the following variational problem (which *a priori* is not related to the stationary Euler solutions). Let M be a three-dimensional closed Riemannian manifold equipped with a volume form μ, and ξ a divergence-free vector field on M. The *energy* of the field is the integral

$$E = \frac{1}{2}\langle \xi, \xi \rangle = \frac{1}{2}\int_M (\xi, \xi)\ \mu.$$

PROBLEM 2.1. Find the minimum energy and the extremals among all fields obtained from a given field ξ by the action of volume-preserving diffeomorphisms of the manifold M.

Here the action of a volume-preserving diffeomorphism $g : M \to M$ associates to a divergence-free field ξ on M another divergence-free field $g_*\xi$ such that the flux of the field ξ across any surface σ is equal to the flux of $g_*\xi$ across $g(\sigma)$. In other words, the field is frozen into an incompressible fluid filling M: The vector field can be thought of as drawn on the elements of the fluid and expanding as these elements expand.

In the case of the manifold M with boundary ∂M, the field ξ is assumed to be tangent to ∂M, and the diffeomorphisms send the boundary ∂M into itself.

In the next chapter we will be concerned with the energy minimum and explicit estimates on it in terms of the field topology. Here we deal exclusively with the topology of the extremal fields.

THEOREM 2.2 (SEE, E.G., [Arn9]). *The extremals of the problem stated above are the divergence-free vector fields that commute with their vorticities. In particular, they coincide with the steady Euler flows in M.*

PROOF. Let η be any divergence-free field on M. The variation $\delta\xi$ of a field ξ under the infinitesimal diffeomorphism defined by η is given by the Lie bracket $\delta\xi = [\eta, \xi] = \{\xi, \eta\}$ (in the coordinate form the Poisson bracket of the vector fields ξ and η is $\{\xi, \eta\} = (\xi, \nabla)\eta - (\eta, \nabla)\xi$).

Consequently, the variation of the energy is $\delta E = \langle \xi, \delta\xi \rangle = \langle \xi, \{\xi, \eta\}\rangle$. Assume that the vector field ξ is extremal for the energy functional.

By formula (1.1) — $\operatorname{curl}\,(\eta \times \xi) = \{\xi, \eta\} + \eta\ (\operatorname{div}\ \xi) - \xi\ (\operatorname{div}\ \eta)$ — which is valid on any three-dimensional Riemannian manifold, and by the divergence-free property for the fields ξ and η, one can rewrite the energy variation at the extremal field ξ as

$$0 = \delta E = \langle \xi, \operatorname{curl}\,(\eta \times \xi)\rangle = \langle \operatorname{curl}\,\xi, (\eta \times \xi)\rangle = \langle \eta, (\xi \times \operatorname{curl}\,\xi)\rangle.$$

Since η is divergence free, the cross product $\xi \times \operatorname{curl} \xi$ is orthogonal to all divergence-free fields. Therefore, it is a gradient: $\xi \times \operatorname{curl} \xi = \operatorname{grad} \alpha$, whence, by taking the curl of both sides we obtain $\{\xi, \operatorname{curl} \xi\} \equiv 0$, as required. □

REMARK 2.3. In the case of a two-dimensional manifold M, we obtain the equation

$$\nabla u \times \nabla \Delta u \equiv 0$$

on the stream function u of the extremal field $\xi = \operatorname{grad} u$. This equation says that the gradient of the extremal function is collinear with that of its Laplacian (see Section 2.C).

The above result is valid not only for smooth vector fields ξ, but it holds also in a weaker form of the integral identity $\langle \eta, (\xi \times \operatorname{curl} \xi) \rangle = 0$, provided that a minimizer ξ exists. Note that existence of smooth and nonsmooth extremals in this problem is a very subtle question. We refer to [Bur, ATL] (see also Sections 2 and 6 below) for existence theorems (of, generally speaking, nonsmooth minimizers) in the two-dimensional case. For dimension greater than 2, there is no proof that the extremals exist except for certain partial results (cf. [L-A, LS4, Vai, GK2]).

REMARK 2.4. A similar calculation leads to the following expression for the second variation of the energy:

$$\delta^2 E = \langle \{\xi, \eta\}, \{\xi, \eta\} \rangle + \langle \{\xi, \eta\}, ((\operatorname{curl}\ \xi) \times \eta) \rangle,$$

where ξ is an extremal field whose first and second variations are given by the Taylor formula

$$\xi(\varepsilon) = \xi + \varepsilon\{\xi, \eta\} + \frac{\varepsilon^2}{2}\{\{\xi, \eta\}, \eta\} + \cdots, \qquad \varepsilon \to 0, \tag{2.1}$$

in terms of a divergence-free vector field η.

REMARK 2.5. The Taylor series (2.1) for $\xi(\varepsilon)$ is obtained while solving the ordinary differential equation on $\xi(t)$,

$$\frac{d\xi(t)}{dt} = \{\xi(t), \eta\},$$

by substituting the series

$$\xi(t) = \xi + t\xi_1 + \frac{t^2}{2!}\xi_2 + \ldots.$$

The field $\xi(\varepsilon)$ is obtained from ξ by the action of the phase flow transformation of η corresponding to a small time interval ε.

All the fields that can be obtained from ξ by the action of volume-preserving diffeomorphisms form a submanifold in the vector space of all divergence-free vector fields, that is, the orbit of the point ξ. The tangent affine subspace to this "smooth" submanifold at the point ξ is formed by the vectors $\xi + \{\xi, \eta\}$ with arbitrary divergence-free η's.

To calculate the second differential of a function on a submanifold of a vector space at a point *it is not enough* to calculate the second differential of the restriction of the function to the affine subspace tangent to the submanifold at this point. The genuine second differential of the restriction of the function to the *submanifold* and the second differential of the restriction of the same function to the *affine tangent space* at a critical point (of the function restriction to this submanifold) are two *different* quadratic forms on the tangent space. (Here we consider the tangent space as the vector space centered at the critical point.)

Formula (2.1) defines the mapping of a domain of "small" vector fields $\varepsilon\eta$ to the orbit of the field ξ. The energy of the image field, considered as a function of the field $\varepsilon\eta$, is the functional on the vector space of divergence-free vector fields $\{\ \varepsilon\eta\ \}$.

The first variation of this functional vanishes if ξ is a critical point of the restriction of the energy to the orbit. Its second variation $\delta^2 E$ is given by the above formula (as a quadratic form of $\varepsilon\eta$).

PROPOSITION 2.6. *If ξ is a critical point of the restriction of the energy to the submanifold, the value of the second variation quadratic form depends only on the tangent vector $\zeta = \{\xi, \varepsilon\eta\}$, and it* does not *depend on the particular choice of the field η.*

PROOF. We can replace η by a field $\eta + u$ where $\{\xi, u\} = 0$ (otherwise ζ would change). The contribution of u to the quadratic term in series (2.1) is then $\frac{\varepsilon^2}{2}w$, where $w = \{\{\xi, \eta\}, u\}$. Since $\{\xi, u\} = 0$, we get from the Jacobi identity that $w = -\{\{\eta, u\}, \xi\}$. The latter vector is tangent to the orbit at ξ. Hence the first variation of the energy is vanishing on this vector w. Adding the vector $\frac{\varepsilon^2}{2}w$ to the vector $\xi(\varepsilon)$ (given by (2.1) and being at a distance of order ε from ξ) we change the value of the energy by a quantity of order ε^3. Thus the addition of u to η contributes nothing to the quadratic part of the Taylor series of the energy restriction to the orbit of ξ (provided that the vector field ξ is a critical point). $\square$

2.B. The Dirichlet problem and steady flows. The energy minimization Problem 2.1 acquires the following form of the Dirichlet problem in the two-dimensional case. Let M be a two-dimensional Riemannian manifold (possibly with boundary) with a Riemannian volume form μ.

PROBLEM 2.1′. Find the infimum and the minimizer of the *Dirichlet integral*

$$E(u) = \frac{1}{2}\int_M (\nabla u, \nabla u)\ \mu$$

among all the smooth functions u (on the manifold M) that can be obtained from a given function u_0 by the action of area-preserving diffeomorphisms of M to itself.

In order to see that this is the two-dimensional counterpart of Problem 2.1, one can consider the skew gradient $\operatorname{sgrad} u$ instead of the true gradient ∇u (on which the functional E has, of course, the same value). Then u is regarded as a Hamiltonian function, whose definition is invariant: Any area-preserving change of coordinates for the function u implies the corresponding diffeomorphism action on the field $\operatorname{sgrad} u$.

For instance, let M be the disk $x^2+y^2 \leq 1$, and let u_0 be a function that vanishes at the boundary and has only one critical point(for instance, a maximum) in the disk (Fig.12a).

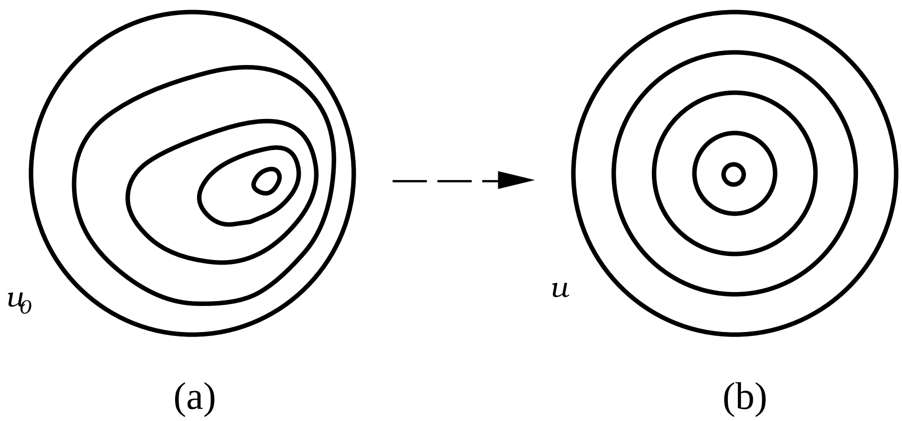

FIGURE 12. Levels of (a) a function u_0 with the only critical point (maximum) inside the disk, and (b) the centrally symmetrical Dirichlet minimizer u among the functions area-preserving rearrangements of u_0.

PROPOSITION 2.7 [Arn9,20]. *The minimum of the Dirichlet functional is attained on the function u that depends only on the distance to the center of the disk and whose sets $\{(x,y) \mid u(x,y) \leq c\}$ of smaller values have the same areas as those of the initial function u_0 (Fig.12b).*

The proof essentially is the application of the isoperimetric and Schwarz inequalities.

□

If the initial function has several critical points (say, two maxima and a saddle point, Fig.13), the situation is far more subtle. Numerical experiments in [Mof4, Baj] suggest various types of minimizers according to the steepness of the initial function u_0, all having "singular" lines. We refer to the extensive surveys [Mof2,4, MoT] (and references therein) for a discussion of the formation of field singularities in a fluid under the relaxation to an extremal state. The obstructions to such relaxation in three dimensions are described in Chapter III.

If instead of the initial function u_0 one prescribes just its boundary conditions, then one may obtain an infinite number of C^∞-steady solutions (or minimizers) for the problem in a rectangle, and a unique solution in the analytic category [Tro].

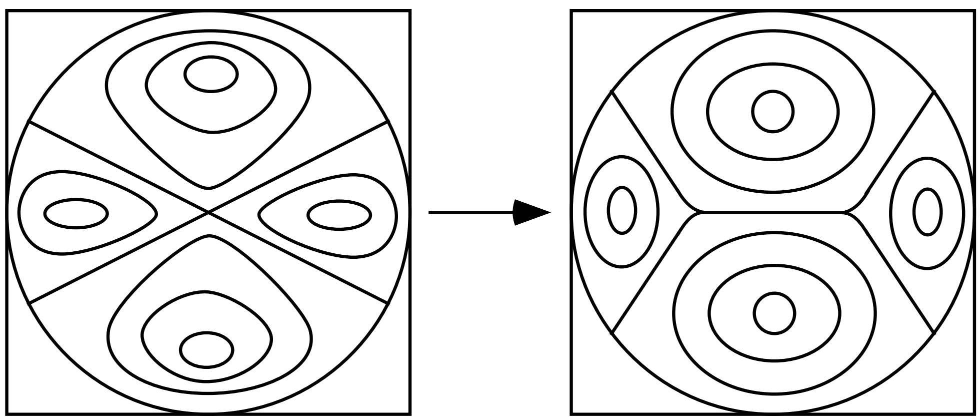

FIGURE 13. A minimizer of the Dirichlet problem for a function with two maxima has a singular line (see [Baj]).

THEOREM 2.8. *A smooth minimizer u of the Dirichlet Problem 2.1′ on a Riemannian manifold M obeys the following condition: The gradients of the functions u and Δu are collinear at every point of M.*

In other words, the extremal functions u have the "same" level curves as their Laplacians: Locally there is a function $F : \mathbb{R} \to \mathbb{R}$ such that $\Delta u = F(u)$. This is just a two-dimensional reformulation of Theorem 2.2. For instance, the axial symmetric function with its only critical point in the disk (Fig.12) not only has the energy minimum among all diffeomorphic fields, but also has the energy maximum among all isovorticed fields [KLe].

The Dirichlet Problem 2.1′ in higher dimensions has applications to scalar dynamos [Bay2] and the theory of equilibrium of a confined plasma [LS1]. One can show that Theorem 2.8 holds in n dimensions. (Hint: adapt the proof of Theorem 2.2.)

REMARK 2.9. As discussed above, minimizers of energy (i.e., of the Dirichlet integral) among all smooth area-preserving changes of coordinates in a given function correspond to steady flows. The problem of existence of smooth minimizers is still open in any reasonable generality.

This problem admits a natural extension to a more general class of functions (for instance, from the L^p or Sobolev spaces), to all measure-preserving rearrangements of such functions on measure spaces, and to general variational functionals. There is vast literature on the existence of (usually, nonsmooth) extrema of variational problems in this setting and on their relation to 2D hydrodynamics, when one minimizes (or maximizes) the energy functional among the rearrangements (see [Bej, ATL, Bur]). In Section 2.D we discuss a different variational principle proposed in [Shn3] for two-dimensional flows, where one confines oneself to the same energy level, but constructs a partial order on functions. Minimal elements in this partial order correspond to steady flows.

A step towards the intrinsic characterization of the weak closure (in H_0^1) of the set of functions obtained from a given one by composing it with diffeomorphisms (not necessarily volume preserving) of the domain is obtained in [LS3]. It is done under the assumption that the function is *craterless*, i.e., in an appropriate weak sense it has no local minima in the interior of the domain. The authors define a subspace of this weak closure that captures robust (under weak limits) topological properties of the level sets.

2.C. Relation of two variational principles. We have observed that the smooth extremals of the energy functional among the vector fields diffeomorphic to a given one commute with their vorticities, and hence they coincide with the description of ideal steady flows (cf. Remark 1.4 and Theorem 2.2). This coincidence of the solutions in two problems is a manifestation of the duality of the two variational principles: in ideal hydrodynamics and in magnetohydrodynamics.

The steady solutions in ideal hydrodynamics correspond to critical points of the energy $\int (v,v)^2/2$ among all *isovorticed fields*, i.e., among the fields whose vorticities differ by the action of a volume-preserving diffeomorphism. In the Lie-algebraic language, steady flows correspond to stagnation points of the energy functional on the *coadjoint orbits* of the group of volume-preserving diffeomorphisms $S\mathrm{Diff}(M^n)$ (see Chapter I). On the other hand, in the above problem we are looking for an energy minimizer within the class of *diffeomorphic fields*, i.e., on the *adjoint orbits* of the same group of volume-preserving diffeomorphisms. Note that the latter principle of energy minimization among the diffeomorphic fields is encountered in the MHD theory (see Chapter III, or, e.g., [Arn9,20, Bej, Ser2, Mof2,4]).

Theorem 2.2 above can now be reformulated as follows: "Extrema for both variational principles coincide." This statement materializes in a very general phenomenon valid for any nondegenerate quadratic form E on an arbitrary Lie algebra $\mathfrak{g}$. Let E^* be the quadratic form on the dual space $\mathfrak{g}^*$ corresponding to the form E on $\mathfrak{g}$. If the form $E(x) = \frac{1}{2}\langle x, Ax\rangle$ is defined by means of an (invertible) inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$, then E^* is determined by $E^*(y)=\frac{1}{2}\langle A^{-1}y,y\rangle$ for any $y\in\mathfrak{g}^*$.

THEOREM 2.10. *Conditional extrema of the quadratic functional E on adjoint orbits in a Lie algebra $\mathfrak{g}$ are sent by the inertia operator $A:\mathfrak{g}\to\mathfrak{g}^*$ to the conditional extrema of the quadratic form E^* on the coadjoint orbits in $\mathfrak{g}^*$.*

PROOF. Let x_0 be a point of the Lie algebra $\mathfrak{g}$, and $\mathcal{O}$ the adjoint orbit of the point x_0. An arbitrary vector ζ of the tangent space $T_{x_0}\mathcal{O}$ can be written by definition as a variation of x_0, i.e., as $\zeta=\mathrm{ad}_\eta x_0$ for some element $\eta\in\mathfrak{g}$. Therefore, one has the following expression for the variation of the energy functional $E(v)=\frac{1}{2}\langle x,Ax\rangle$ along the vector ζ:

$$dE(\zeta)=\langle\zeta,Ax_0\rangle=\langle\mathrm{ad}_\eta x_0,Ax_0\rangle=\langle x_0,\mathrm{ad}^*_\eta(Ax_0)\rangle=\langle A^{-1}y_0,\mathrm{ad}^*_\eta y_0\rangle=dE^*(\zeta^*),$$

where $y_0\in\mathfrak{g}^*$ denotes the image of x_0 under the inertia operator ($y_0=Ax_0$), and the vector $\zeta^*=\mathrm{ad}^*_\eta y_0$ represents an arbitrary vector tangent to the coadjoint orbit $\mathcal{O}^*$ of the point y_0.

Now assume that $x_0\in\mathfrak{g}$ is a critical point of the function $E(x)$ restricted to the adjoint orbit $\mathcal{O}$ of x_0. Then the differential of E vanishes on the tangent space $T_{x_0}\mathcal{O}$ and so does the differential of E^* restricted to the tangent space to $\mathcal{O}^*$ at y_0. Hence y_0 is a critical point of E^* restricted to the coadjoint orbit $\mathcal{O}^*$. □

2.D. Semigroup variational principle for two-dimensional steady flows. In [Shn3], Shnirelman proposed a different variational principle in two dimensions that recovers some of the steady solutions of the Euler equation. Roughly speaking, instead of the energy minimization among all isovorticed fields, one can stay among the fields with the same energy and construct a partial order on their vorticities. In a sense, the extremal fields obtained by this method have the most mixed vorticity functions.

Consider a bounded connected two-dimensional domain $M\subset\mathbb{R}^2$ with a measure μ and boundary $\Gamma=\partial M$. We wish to describe generalized area-preserving mappings of M into itself that are not necessarily one-to-one. It is natural to define them in terms of their actions on functions on M.

DEFINITION 2.11. A *polymorphism* is a bounded operator $\widetilde{K}$ in $L^2(M,\mathbb{R})$ of the form

$$\widetilde{K}u(x)=\int_M K(x,y)u(y)\ \mu_y,$$

where the (distributional) kernel $K(x,y)$ obeys the following conditions:

i) $K(x,y)\geq 0$, i.e., $K(x,y)$ is a nonnegative measure on $M\times M$;

ii) $\int_M K(x,y)\ \mu_x\equiv 1$ for every $y\in M$; and

iii) $\int_M K(x,y)\ \mu_y\equiv 1$ for every $x\in M$.

EXAMPLES 2.12. Two obvious, yet important, examples of such operators are:

A) Let $\varphi \in S\mathrm{Diff}(M)$ be an area-preserving diffeomorphism of M. Set $K_\varphi(x,y) = \delta(y - \varphi^{-1}(x))$, where $\delta(*)$ is the 2-dimensional δ-function. Then the operator $\widetilde{K}_\varphi$ whose kernel is $K_\varphi(x,y)$ sends a function $u(x)$ to the function $u(\varphi^{-1}(x))$ and is unitary in $L^2(M)$.

B) If $K_0(x,y) \equiv 1/\mu(M)$ where $\mu(M)$ is the total measure of M, the operator $\widetilde{K}_0$ maps a function $u(x)$ to the constant that is the mean value of $u(x)$.

In a sense, an arbitrary operator $\widetilde{K}$ interpolates between those two extreme cases.

Conditions ii) and iii) generalize the volume-preserving property of diffeomorphisms: They demand that the probabilistic measure of the "image" of the element dy and the "inverse image" of the element dx under an operator $\widetilde{K}$ be equal to the measures of the elements dy and dx, respectively.

All polymorphisms form a (weakly compact) semigroup $\mathcal{P}$ of (contractive, or more precisely, nonexpanding) operators in $L^2(M)$. The operators $\widetilde{K}_\varphi$ corresponding to diffeomorphisms constitute a weakly dense subset of $\mathcal{P}$. Representations of the group of diffeomorphisms can be extended to the semigroup of polymorphisms [Ner2].

DEFINITION 2.13. The *partial ordering* in $L^2(M)$ is dictated by the action of $\mathcal{P}$: $f \prec g$ if there exists an operator $\widetilde{K} \in \mathcal{P}$ such that $f = \widetilde{K}g$. If $f \prec g$ and $g \prec f$, we say that f and g are equivalent: $f \sim g$.

The following property of the relation $\prec$ will be useful in the sequel.

PROPOSITION 2.14 [Shn3]. *If $f, g \in L^2(M)$ and $f \prec g$, then $\|f\|_{L^2} \leq \|g\|_{L^2}$. For $f \prec g$ the equality of the norms $\|f\|_{L^2} = \|g\|_{L^2}$ is possible if and only if $g \prec f$.*

Let $L^{2,2}(M)$ be the Sobolev space that consists of functions φ obeying

$$\sum_{|k|\leq 2} \|D^k\varphi\|^2_{L^2(M)} < \infty, \qquad \varphi\,|_{\partial M} = \text{ const.}$$

DEFINITION 2.15. Given a function $\varphi \in L^{2,2}(M)$, denote by $\bar{\Omega}_\varphi$ the set of such functions $\psi \in L^{2,2}(M)$ that

$$\Delta\psi \prec \Delta\varphi. \tag{2.2a}$$

If φ is regarded as a stream function for a fluid flow, then the set $\bar{\Omega}_\varphi$ contains the fields isovorticed with φ, i.e., the fields with the stream functions ψ for which there exists a diffeomorphism $g : M \to M$ such that $\Delta\psi(x) = \Delta\varphi(g(x))$. These fields constitute the coadjoint orbit $\mathcal{O}_\varphi$ of φ.

Let $\Omega_\varphi \subset \bar{\Omega}_\varphi$ be the set of stream functions ψ obeying one extra condition of the conservation of energy:

$$E(\psi) = E(\varphi), \tag{2.2b}$$

where $E(\psi) = \frac{1}{2}\|\nabla\psi\|^2_{L^2}$ is the kinetic energy of the flow with the stream function ψ.

An element $\nu \in \Omega_\varphi$ is *minimal* relative to the partial ordering on Ω_φ if $\Delta\nu' \sim \Delta\nu$ whenever $\nu' \in \Omega_\varphi$ and $\Delta\nu' \prec \Delta\nu$.

THEOREM 2.16 [Shn3]. *For each function $\varphi \in L^{2,2}(M)$ there exists a minimal element $\nu \in \Omega_\varphi$ in the set Ω_φ.*

A minimal element is not necessarily unique. The proof is essentially a combination of the *Zorn lemma* (claiming that if for each linearly ordered decreasing chain of elements of a partially ordered set there is a lower bound, then there exists a minimal element in the set) with the relative weak compactness of the set of measures $\{K(x,y)\}$.

THEOREM 2.17 [Shn3]. *Let u be a minimal element of Ω_φ. Then u is the stream function of a stationary flow, and moreover, there exists a single-valued monotone function F such that $\Delta u = F(u)$ almost everywhere in M.*

The equivalent statement is that if u is a minimal element of Ω_φ, then, for almost all points $x, y \in M$, the products $(u(x) - u(y))(\omega(x) - \omega(y))$, where $\omega := \Delta u$, all have the same sign. We refer to [Shn3] for the proof and all the details.

REMARK 2.18. Though a classical solution of the Euler equation is a trajectory on the coadjoint orbit $\mathcal{O}_\varphi$ for some function φ, for large times the flow transformations become similar to the mixing described by polymorphisms. These are the heuristics lying behind the relation between the minimal elements and the stationary solutions of the Euler equation.

REMARK 2.19. For a non-simply connected M, the boundary conditions for functions in the space $L^{2,2}(M)$ are $\varphi\,|_{\Gamma_i} \equiv \text{const}_i$, where Γ_i is a connected component of ∂M. In the latter case the set $\bar{\Omega}_\varphi$ consists of the functions ψ that, in addition to the condition (2.2a) satisfy the property

$$\int_{\Gamma_i} \frac{\partial\psi}{\partial n}\, ds = \int_{\Gamma_i} \frac{\partial\varphi}{\partial n}\, ds \qquad \text{for all } i. \tag{2.2c}$$

Property (2.2c) follows from (2.2a) for a simply connected M.

One can classify minimal elements of the "orbit" Ω_φ by comparing their energy to other points of the set $\bar{\Omega}_\varphi \supset \Omega_\varphi$ consisting of the stream functions obeying conditions (2.2a) and (2.2c), but without the requirement (2.2b) on the energy.

THEOREM 2.20 [Shn3]. *Each minimal element* $u \in \Omega_\varphi$ *is one of the following three types:*

a) *energy-excessive, i.e.,* $E(u) \geq E(\psi)$,

b) *energy-deficient, i.e.,* $E(u) \leq E(\psi)$, *or*

c) *neutral, i.e.,* $E(u) = E(\psi)$

for all $\psi \in \bar{\Omega}_\varphi$. *All the minimal elements of* Ω_φ *are of the same type.*

PROBLEM 2.21. It would be interesting to relate these types of minimal elements and the above variational principle to various types of energy relaxation discussed in Section 2.B (cf. numerical simulations in [Mof4, Baj]).

This variational principle might be a basis for formulating for *semigroups* an analogue of the (geodesic) variational principle for groups (Chapter I). In Section IV.7.G, we discuss a natural passage from the geodesics on the group of volume-preserving diffeomorphisms of a manifold to the extremals of the least action principle for the so-called generalized flows (which are similar to the semigroup of polymorphisms), i.e., the passage from classical fluid motions to generalized solutions of the Euler equation; see [Bre1, Shn5].

§3. Stability of stationary points on Lie algebras

In order to study the stability of stationary fluid flows in the next section, we obtain below a stability criterion for the Euler equation on an arbitrary Lie algebra.

Consider a system of ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{3.1}$$

DEFINITION 3.1. A point x_0 at which $f(x_0) = 0$ is *(Lyapunov) stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(t) - x_0| < \varepsilon$ for all $t > 0$, provided that $|x(0) - x_0| < \delta$.

Assume that we are also given a foliation in the space $\mathbb{R}^n$. A point x_0 is called *regular* for the foliation if the partition of a neighborhood of x_0 into the leaves of the foliation is diffeomorphic to a partition of the Euclidean space into parallel planes (in particular, all leaves near the point x_0 have the same dimension).

EXAMPLE 3.2. In the case of the Lie algebra $\mathfrak{so}(3)$ the orbits form a partition of three-dimensional space $\mathfrak{so}(3) \simeq \mathbb{R}^3$ into spheres centered at 0 and the point 0 itself. Then all points of the space $\mathbb{R}^3$, except the origin, are regular for the partition into orbits.

Suppose now that the system (3.1) leaves the foliation invariant, and E is a first integral of the system such that

i) x_0 is a critical point of E restricted to the leaf containing x_0;

ii) x_0 is a regular point of the foliation; and

iii) the second differential of E restricted to the leaf of x_0 is a nondegenerate quadratic form.

The following statement is essentially a reformulation of Lagrange's theorem.

THEOREM 3.3. *A point x_0 obeying conditions i)–iii) is a stationary point of the system (3.1). If, in addition, the second differential of E restricted to the leaf of x_0 is positively or negatively defined, then the point x_0 is (Lyapunov) stable.*

PROOF. If y is a coordinate on the leaf such that $y(x_0) = 0$, then the function E restricted to the leaf can be written as $E(y) = E_0 + \frac{1}{2}(E_2 y, y) + \mathcal{O}(y^3)$ as $y \to 0$, where the matrix E_2 is symmetric: $(E_2 y, z) = (y, E_2 z)$. Hence the time derivative along the trajectories of our system is

$$\dot{E} = (E_2 y, \dot{y}) + \mathcal{O}(y^2)\dot{y} \quad \text{ as } y \to 0.$$

If $\dot{y} \neq 0$ at the origin $y = 0$, then one can choose a point y arbitrarily close to the origin such that $(E_2 y, \dot{y}) \neq 0$. The latter contradicts the invariance of E. Therefore, $\dot{y} = 0$, and x_0 is a stationary point.

The regularity of the leaves near x_0 implies that on every neighboring leaf there exists near x_0 a point that is a conditional maximum or minimum of E. The stability part of the statement is evident (Lagrange, Dirichlet, etc.): The definiteness of E ensures that in every leaf near x_0 the E-levels form a family of ellipsoid-like hypersurfaces. Every trajectory of the system (3.1) that begins inside such an ellipsoid will never leave it, due to the invariance of E and of the foliation (see Fig.14). □

Let ν be a stationary point of the Euler equation on a Lie algebra $\mathfrak{g}$ (see Chapter I). The space $\mathfrak{g}$ is foliated by the images of the coadjoint orbits in the algebra, and we suppose that ν is a regular point of the foliation.

THEOREM 3.4 [Arn4,16]. *The second differential of the kinetic energy restricted to the image of an orbit of the coadjoint representation in the algebra $\mathfrak{g}$ is given at a critical point $\nu \in \mathfrak{g}$ by the formula*

$$2\delta^2 E\mid_\nu(\xi) = \langle B(\nu, f), B(\nu, f)\rangle + \langle [f, \nu], B(\nu, f)\rangle, \tag{3.2}$$

where ξ is a tangent vector to this image expressed in terms of $f \in \mathfrak{g}$ by the formula $\xi = B(\nu, f)$, and $B(\cdot, \cdot)$ is the operation on $\mathfrak{g}$ defined by (I.4.3).

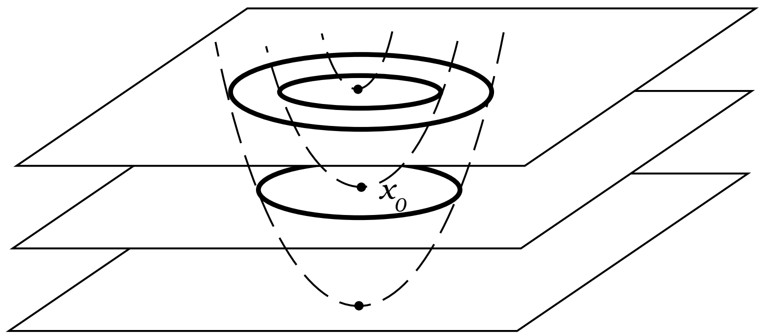

FIGURE 14. Trajectories enclosed in ellipsoid-like intersections of foliation leaves (here, horizontal planes) and energy levels (paraboloids) will never leave a vicinity of the stationary point.

COROLLARY 3.5. *If the quadratic form above is positive or negative definite, then the stationary point* ν *is a stable solution of the Euler equation.*

EXAMPLE 3.6. In the case of the rigid body ($\mathfrak{g} = \mathfrak{so}(3)$), the coadjoint orbits are spheres centered at zero, while the levels of the kinetic energy form a family of ellipsoids. The energy restricted to every orbit has 6 critical points (being points of tangency of the sphere with the ellipsoids): 2 maxima, 2 minima, and 2 saddles (Fig.15). The maxima and minima correspond to the stable rotations of the rigid body about the shortest and the longest axes of the inertia ellipsoid. The saddles correspond to the unstable rotations about its middle axis.

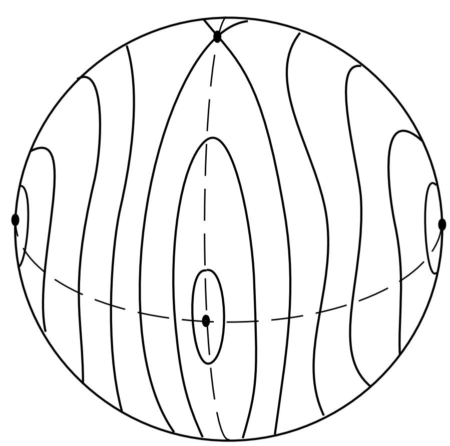

FIGURE 15. Energy levels on a coadjoint orbit of the Lie algebra $\mathfrak{so}(3, \mathbb{R})$ of a rigid body.

We emphasize that the question under discussion is not stability "in a linear approximation," but the actual Lyapunov stability (i.e., with respect to finite perturbations in the nonlinear problem). The difference between these two forms of stability is substantial in this case, since our problem has a Hamiltonian character. For Hamiltonian systems asymptotic stability is impossible, so stability in a linear approximation is always neutral and inconclusive on the stability of an equilibrium position of the nonlinear problem.

REMARK 3.7. In general, an indefinite quadratic form $\delta^2 E$ does not imply instability of the corresponding point. An equilibrium position of a Hamiltonian system can be stable even if the Hamiltonian function at this position is neither a maximum nor a minimum. The quadratic Hamiltonian

$$E = \omega_1 \frac{p_1^2 + q_1^2}{2} - \omega_2 \frac{p_2^2 + q_2^2}{2}$$

is the simplest example of this kind. Note that the behavior of the corresponding eigenvalues under the introduction of a small viscosity is different: $\pm i\omega_1$ are moving into the left (stable) hyperplane, while $\pm i\omega_2$ are moving into the right (unstable) one.

PROOF OF THEOREM 3.4. The action of an element $\varepsilon \cdot f \in \mathfrak{g}$ on a point ν is given by the Taylor expansion for motion along a coadjoint orbit; cf. formula (2.1):

$$\nu \mapsto \bar{\nu} = \nu + \varepsilon \cdot \xi + \frac{\varepsilon^2}{2} \cdot \zeta + O(\varepsilon^3), \qquad \varepsilon \to 0,$$

where $\xi = B(\nu, f)$, $\zeta = B(B(\nu, f), f)$. Substitute $\bar{\nu}$ into the expression for the energy $E(\bar{\nu}) = \frac{1}{2}\langle \bar{\nu}, \bar{\nu} \rangle$:

$$E(\bar{\nu}) = E(\nu) + \varepsilon \cdot \delta E + \varepsilon^2 \cdot \delta^2 E + O(\varepsilon^3), \qquad \varepsilon \to 0,$$

where $\delta E = \langle \nu, \xi \rangle$ and $2\delta^2 E = \langle \xi, \xi \rangle + \langle \nu, \zeta \rangle$.

The first variation of the energy vanishes at ν:

$$\delta E = \langle \nu, B(\nu, f) \rangle = -\langle B(\nu, \nu), f \rangle = 0,$$

since ν is stationary, and therefore $B(\nu, \nu) = 0$.

The required expression (3.2) for $\delta^2 E$ follows due to the identity

$$\langle \nu, B(B(\nu, f), f) \rangle = \langle [f, \nu], B(\nu, f) \rangle.$$

Now we would like to show that the quadratic form $\delta^2 E$ depends on $\xi = B(\nu, f)$ rather than on f, so it is indeed a form on the tangent space in $\mathfrak{g}$.

First verify that the auxiliary bilinear form $C(x, y) := \langle [x, \nu], B(\nu, y)\rangle$ is symmetric: $C(x, y) = C(y, x)$. It readily follows from the definition of B, the Jacobi identity in $\mathfrak{g}$, and from the stationarity condition $B(\nu, \nu) = 0$ that

$$\begin{aligned}\langle [x, \nu], B(\nu, y)\rangle &= \langle B(\nu, [\nu, x]),\ y\rangle = \langle [[\nu, x], y],\ \nu\rangle = \langle [\nu, [x, y]],\ \nu\rangle + \langle [x, [y, \nu]],\ \nu\rangle \\ &= \langle B(\nu, \nu),\ [x, y]\rangle + \langle B(\nu, x),\ [y, \nu]\rangle = \langle [y, \nu], B(\nu, x)\rangle.\end{aligned}$$

Finally, assume that $B(\nu, f_1) = B(\nu, f_2)$ and show that the corresponding values of $\delta^2 E$ coincide. Set $x = f_1 - f_2, y = f_1$, and notice that $B(\nu, x) = 0$. The expression (3.2) for $\delta^2 E$, combined with the symmetry of $C(x, y)$, gives the desired identity:

$$2(\delta^2 E(f_1) - \delta^2 E(f_2)) = \langle [x, \nu], B(\nu, y)\rangle = \langle [y, \nu], B(\nu, x)\rangle = 0.$$

Thus, the quadratic form $\delta^2 E$ indeed depends on $\xi = B(\nu, f)$, and Theorem 3.4 is proved. □

REMARK 3.8. For the Euler equation on a Lie algebra $\mathfrak{g}$ consider the *equation in variations* at a stationary point ν:

$$\dot{\xi} = B(\nu, \xi) + B(\xi, \nu). \tag{3.3}$$

PROPOSITION 3.9. *The quadratic form $d^2 E$ is the first integral of the equation in variations (3.3).*

PROOF. The proposition can be verified by the following straightforward calculation. From (3.2), it follows that

$$\frac{d}{dt}\delta^2 E = \langle \xi, \dot{\xi}\rangle + \langle [f, \nu], \dot{\xi}\rangle.$$

Therefore the substitution of $\dot{\xi}$ from the equation in variations (3.3) leads to

$$\begin{aligned}\frac{d}{dt}\delta^2 E &= \langle \xi, B(\nu, \xi)\rangle + \langle \xi, B(\xi, \nu)\rangle + \langle [f, \nu], B(\nu, \xi)\rangle + \langle [f, \nu], B(\xi, \nu)\rangle \\ &= \langle \xi, B(\xi, \nu)\rangle + \langle [\xi, \nu], \xi\rangle + \langle [\nu, [f, \nu]], \xi\rangle \\ &= \langle [\nu, [f, \nu]], B(\nu, f)\rangle = -\langle [f, \nu], B(\nu, [f, \nu])\rangle = 0.\end{aligned}$$

□

§4. Stability of planar fluid flows

The analogy between the equations of a rigid body and of an incompressible fluid enables one to study stability of steady flows by considering critical points of the energy function on the sets of isovorticed vector fields (i.e., on the coadjoint orbits of the diffeomorphism group).

This approach was initiated in [Arn4], and we refer to Fjortoft [Fj] as a predecessor, and to [HMRW] for further applications manifesting the fruitfulness of this method for a variety of dynamical systems. In this section we touch on a few selected facts.

In Section 3 we saw that the variational approach to the study of the stationary solutions of the Euler equation of an incompressible fluid suggests that:

i) A steady fluid flow is distinguished from all flows isovorticed to it by the fact that it is a (conditional) critical point of the kinetic energy.

ii) If the indicated critical point is actually an extremum, i.e., a local conditional maximum or minimum, and this extremum is nondegenerate (the second differential d^2E is positive or negative definite), then (under some regularity condition) the stationary flow is Lyapunov stable.

Though these assertions do not formally follow from the theorems of Section 3 because of the infinite-dimensionality of our consideration here, one can justify the final conclusion about stability without justifying the intermediate constructions.

4.A. Stability criteria for steady flows. Let M be a two-dimensional domain, say, an annulus with a steady flow in it (Fig.16). In what follows we show, in particular, that the steady flow in M is stable if its stream function ψ satisfies the following condition on the velocity profile:

$$0 < c \leq \frac{\nabla\psi}{\nabla\Delta\psi} \leq C < \infty \tag{4.1}$$

for some constants c and C.

For an arbitrary stationary flow in two dimensions the gradient vectors of the stream function and of its Laplacian are collinear. Therefore the ratio $\nabla\psi/\nabla\Delta\psi$ makes sense. Furthermore, in a neighborhood of every point that is not critical for the vorticity function $\Delta\psi$, the stream function ψ is a function of the vorticity.

We begin the study of the two-dimensional case by obtaining the following explicit expression for the second variation of the energy.

THEOREM 4.1 [Arn6,16]. *The second variation of the energy E on the set of fields*

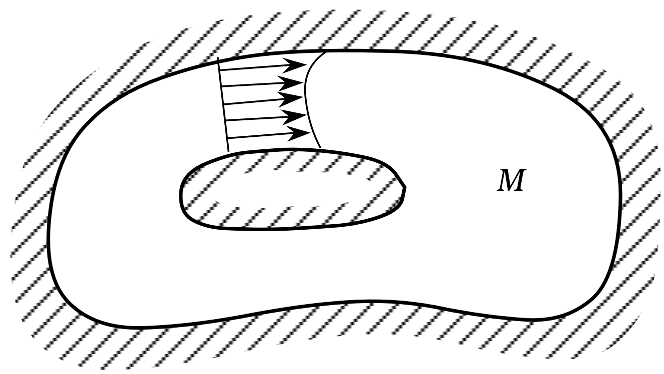

FIGURE 16. A profile of a stable steady flow in an annulus.

isovorticed to a given steady field v with the stream function ψ is

$$\delta^2 E\big|_v = \frac{1}{2}\iint_M \left((\delta v)^2 + \frac{\nabla\psi}{\nabla\Delta\psi}(\delta\omega)^2\right)\ dxdy,$$

where δv is a variation of the velocity field, $\delta\omega$ is the corresponding variation of the vorticity function $\omega = curl\ v = \Delta\psi$, and $dxdy$ is the area form in M.

REMARK 4.2. The condition (4.1) on the ratio $\nabla\psi/\nabla\Delta\psi$ implies that the quadratic form $\delta^2 E$, with respect to δv, is positively defined.

In the case of the negative ratio $\nabla\psi/\nabla\Delta\psi$ satisfying

$$0 < c \le -\frac{\nabla\psi}{\nabla\Delta\psi} \le C < \infty,$$

the form $\delta^2 E$ is negatively defined, provided that the inequality $\|\nabla\varphi\|^2_{L^2} \le \alpha\|\Delta\varphi\|^2_{L^2}$ holds for all $\varphi \in C^2(M)$ with $0 < \alpha < c$. The latter inequality is essentially an estimate on the first eigenvalue of the Laplace operator in the domain M, and it relies on the shape and size of the domain.

PROOF. Formula (3.2) for the second variation of the energy $E = \frac{1}{2}\iint_M (v,v)dxdy$ gives

$$2\delta^2 E = \iint_M \left((\delta v)^2 + (\delta v,\ [f,v])\right)\ dxdy, \tag{4.2}$$

where $\delta v = B(v,f)$.

Integrating by parts the second term, we come to

$$\iint_M (\delta v,\ [f,v])\ dxdy = \iint_M (\delta v,\ \text{curl}\ (f\times v))\ dxdy = \iint_M (\delta\omega)\cdot(f\times v)\ dxdy \tag{4.3}$$

with evident notations: $f \times v$ is a function on M whose value at any point is the oriented area of the parallelogram spanned by f and v, and curl $(f \times v) =$ sgrad $(f \times v)$. The formula $v = \operatorname{sgrad} \psi = (-\psi_y, \psi_x)$ implies that

$$f \times v = f \times (\operatorname{sgrad} \psi) = (f, \nabla \psi).$$

On the other hand, for $\omega = \Delta\psi$, the variation $\delta\omega$ is the derivative of ω along the field f:

$$\delta\omega = L_f \omega = (f, \nabla\Delta\psi).$$

The comparison of the two formulas above immediately gives

$$f \times v = \frac{\nabla\psi}{\nabla\Delta\psi}\ \delta\omega,$$

which, along with (4.2-4.3), implies the statement of the theorem. □

The above heuristic consideration of stability, based on the definiteness of the quadratic differential of the kinetic energy $\delta^2 E$, can be justified to obtain the actual stability with the following *a priori* bound.

THEOREM 4.3 (STABILITY THEOREM, [Arn6,16]). *Suppose that the stream function of a stationary flow, $\psi = \psi(x, y)$, in a region M is a function of the vorticity function (i.e., of the function $\Delta\psi$) not only locally but globally. Suppose that the derivative of the stream function with respect to the vorticity satisfies the inequality*

$$c \le \frac{\nabla\psi}{\nabla\Delta\psi} \le C, \ \textit{where}\ 0 < c \le C < \infty.$$

Let $\psi + \varphi(x, y, t)$ be the stream function of another flow, not necessarily stationary. Assume that at the initial moment, the circulation of the velocity field of the perturbed flow (with the stream function $\psi + \varphi$) around every boundary component of the region M is equal to the circulation of the original flow (with the stream function ψ). Then the perturbation $\varphi = \varphi(x, y, t)$ at every moment of time is bounded in terms of the initial perturbation $\varphi_0 = \varphi(x, y, 0)$ by the inequality

$$\iint_M (\nabla\varphi)^2 + c(\Delta\varphi)^2 dxdy \le \iint_M (\nabla\varphi_0)^2 + C(\Delta\varphi_0)^2 dxdy.$$

THEOREM 4.3′ (SECOND STABILITY THEOREM, [Arn6,16]). *If the stationary flow satisfies the condition*

$$c \leq -\frac{\nabla\psi}{\nabla\Delta\psi} \leq C \quad \textit{with} \quad 0 < c \leq C < \infty$$

(as well as other assumptions of the preceding theorem), then the perturbation φ is bounded in terms of φ_0 by the inequality

$$\iint_M c(\Delta\varphi)^2 - (\nabla\varphi)^2 dxdy \leq \iint_M C(\Delta\varphi_0)^2 - (\nabla\varphi_0)^2 dxdy. \tag{4.4}$$

REMARK 4.4. If for a certain α satisfying $0 < \alpha < c$ the inequality $\|\nabla\varphi\|^2_{L^2} \leq \alpha\|\Delta\varphi\|^2_{L^2}$ holds for all $\varphi \in C^2(M)$, then the quadratic form $\iint_M c(\Delta\varphi)^2 - (\nabla\varphi)^2 dxdy$ is positive definite:

$$\iint_M c(\Delta\varphi)^2 - (\nabla\varphi)^2 dxdy \geq (c-\alpha)\iint_M (\Delta\varphi)^2 dxdy.$$

Therefore it follows from (4.4) that

$$\iint_M (\Delta\varphi)^2 dxdy \leq \frac{C}{c-\alpha}\iint_M (\Delta\varphi_0)^2 dxdy,$$

which manifests the stability of the stationary flow ψ.

The underlying heuristic idea of the proof of the Stability Theorem is as follows. A first integral $H(\varphi)$ having a nondegenerate minimum or maximum at the stationary point ψ can be regarded as a squared "norm" (setting $H(\psi) = 0$). It gives us control of the trajectory φ_t in the norm that is positive in a punctured neighborhood of ψ on the set of isovorticed fields.

EXAMPLE 4.5. Consider a circular motion with the stream function $\psi = \psi(\rho)$, $\rho = \sqrt{x^2+y^2}$, in the annulus $M = \{\ R_1 \leq \rho \leq R_2\ \}$. Rewriting the Laplace operator in polar coordinates, we get the following sufficient condition for stability: If the ratio $\psi'/(\psi''+\frac{1}{\rho}\psi')'$ does not change sign, then the flow is stable (see [Arn16]).

EXAMPLE 4.6. Consider a planar shear flow in the strip $0 \leq y \leq 2\pi$ in the (x,y)-plane with a velocity profile $v(y)$ (i.e., with a velocity field $(v(y),0)$, Fig.17). Such a flow is stationary for every velocity profile.

The form $\delta^2 E$ is positively or negatively defined if the velocity profile has no zeroes and no points of inflection (i.e., $v \neq 0$ and $v_{yy} \neq 0$). The conclusion, that the planar parallel flows are stable, provided that there are no inflection points in the velocity profile, is a

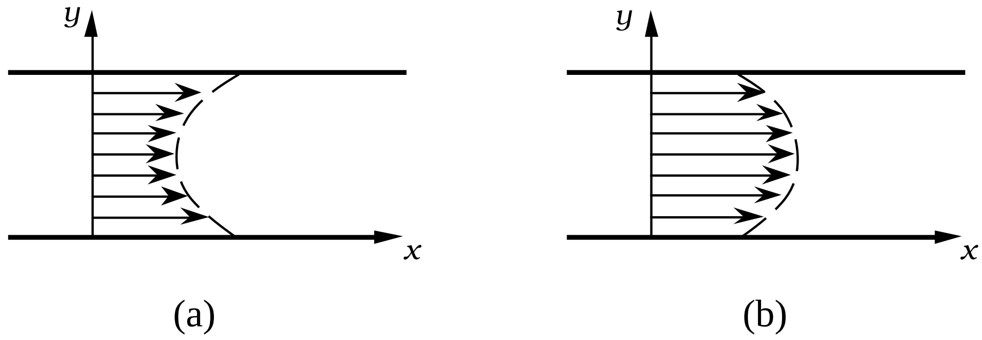

FIGURE 17. Lyapunov stable fluid flows in a strip. Profiles with the ratio (a) $v/v_{yy} > 0$ and (b) $v/v_{yy} < 0$.

nonlinear analogue of the so-called Rayleigh theorem. Profiles with the ratio $v/v_{yy} > 0$ and $v/v_{yy} < 0$ are sketched in Figs.17a and 17b, respectively.

To make the region of the flow compact, we impose the periodicity condition $x \pmod X$ along the x-coordinate and obtain the torus $\{(x, y) \mid\; x \pmod X),\; y \pmod{2\pi}\}$. Fix the velocity field $v = (\sin y, 0)$ determined by the stream function $\psi = -\cos y$. Its vorticity is $\omega = -\cos y$. The velocity profile has two inflection points, but the stream function can be expressed as a function of the vorticity. The ratio $\nabla\psi/\nabla\Delta\psi$ is equal to minus one. By applying the Second Stability Theorem, we have obtained the stability of our stationary flow in the case when

$$\int_0^{2\pi}\int_0^X (\Delta\varphi)^2 dxdy \geq \int_0^{2\pi}\int_0^X (\nabla\varphi)^2 dxdy$$

for all functions φ of period X in x and 2π in y. It is easy to calculate that the last inequality is satisfied for $X \leq 2\pi$ and is violated for $X > 2\pi$.

Thus the Second Stability Theorem implies the stability of a sinusoidal stationary flow on a short torus when the period in the direction of the basic flow (X) is less than the width of the flow (2π). On the other hand, one can directly verify that on a long torus (for $X > 2\pi$) our sinusoidal flow is unstable [MSi]. Hence, in this example, the sufficient condition for stability from the Second Stability Theorem turns out to be necessary as well.

Stability of certain plane-parallel and spherical two-dimensional flows was considered in [Dik].

PROOF OF STABILITY THEOREM. Assume that the stream function ψ and the vorticity function $\omega = \Delta\psi$ are related by means of $\psi = \Psi(\Delta\psi)$, and set $\Phi(\tau) := \int^\tau \Psi(\theta)\, d\theta$ to be the primitive of $\Psi(\theta)$. Then the second derivative Φ'' evaluated at the function $\Delta\psi$ is $\Phi''(\Delta\psi) = \nabla\psi / \nabla\Delta\psi$, and hence for τ within the limits $\min \Delta\psi \le \tau \le \max \Delta\psi$, we have

$$c \le \Phi''(\tau) \le C. \tag{4.5}$$

We extend the definition of $\Phi(\tau)$ to cover the whole τ-axis subject to this inequality, and in what follows Φ denotes the function extended in this way.

Form the functional

$$H_2(\varphi) = \iint_M \left(\frac{(\nabla\varphi)^2}{2} + [\Phi(\Delta\psi + \Delta\varphi) - \Phi(\Delta\psi) - \Phi'(\Delta\psi)\Delta\varphi] \right) \, dxdy.$$

LEMMA 4.7. *The functional H_2 is the first integral of the Euler equation,*

$$H_2(\varphi(x, y, t)) \equiv H_2(\varphi(x, y, 0)),$$

for the stream function $\varphi(x, y, t)$ of any velocity field evolving according to the Euler equation.

PROOF OF LEMMA. Consider the functional

$$H(u) = \iint_M \left(\frac{(\nabla u)^2}{2} + \Phi(\Delta u) \right) \, dxdy.$$

It is preserved along every solution of the Euler equation by virtue of the laws of energy and vortex conservation. Therefore, $\hat{H}(\varphi) := H(\psi + \varphi) - H(\psi)$ is also a conserved functional for a given steady flow ψ:

$$\hat{H}(\varphi(x, y, t)) \equiv \hat{H}(\varphi(x, y, 0)). \tag{4.6}$$

Decompose $\hat{H}(\phi)$ into the sum $\hat{H}(\varphi) = H_1(\varphi) + H_2(\varphi)$, where

$$H_1(\varphi) = \iint_M ((\nabla\varphi, \ \nabla\psi) + \Phi'(\Delta\psi)\Delta\varphi) \ dxdy,$$
$$H_2(\varphi) = \iint_M \left(\frac{(\nabla\varphi)^2}{2} + [\Phi(\Delta\psi + \Delta\varphi) - \Phi(\Delta\psi) - \Phi'(\Delta\psi)\Delta\varphi] \right) \, dxdy.$$

The term $H_1(\varphi)$ vanishes, since it is the first variation of the invariant functional $H(u)$ at the stationary flow ψ. Explicitly, after integration by parts we have

$$H_1(\varphi) = \iint_M (-\psi\Delta\varphi + \Phi'(\Delta\psi)\Delta\varphi) \, dxdy + \oint_{\partial M} \psi \frac{\partial\varphi}{\partial n} \, d\ell.$$

Recall that $\Phi' = \Psi$ and $\Psi(\Delta\psi) = \psi$. Furthermore, by assumption the stream function ψ is constant on the boundary components Γ_i ($\partial M = \bigcup_i^n \Gamma_i$), and the perturbed fields have the same circulation around every boundary component: $\oint_{\Gamma_i} \partial\varphi/\partial n \, d\ell = 0$. Hence $H_1(\varphi) \equiv 0$. Therefore $\hat{H}(\varphi) = H_2(\varphi)$, and in accordance with (4.6), the functional $H_2(\varphi)$ is preserved. This proves Lemma 4.7. □

Returning to the proof of the theorem, we note that it follows from (4.5) that for any h,

$$c\frac{h^2}{2} \leq \Phi(\tau + h) - \Phi(\tau) - \Phi'(\tau)h \leq C\frac{h^2}{2}.$$

Hence,

$$H_2(\varphi(t)) \geq \iint_M \left(\frac{(\nabla\varphi)^2}{2} + c\frac{(\Delta\varphi)^2}{2}\right) \, dxdy,$$
$$H_2(\varphi(0)) \leq \iint_M \left(\frac{(\nabla\varphi_0)^2}{2} + C\frac{(\Delta\varphi_0)^2}{2}\right) \, dxdy.$$

By combining these inequalities with the invariance of $H_2(\varphi)$ we complete the proof of the Stability Theorem. □

We leave to the reader to complete the proof of stability for the negative ratio (the Second Stability Theorem)

$$c \leq -\frac{\nabla\psi}{\nabla\Delta\psi} \leq C, \qquad 0 < c \leq C < \infty.$$

Remark 4.8 [M-P]. Notice that the condition $0 < c \leq \frac{\nabla\psi}{\nabla\Delta\psi} \leq C$ cannot be obeyed in domains without boundary. Indeed, the existence of a function Ψ obeying the condition $0 < c \leq \Psi'(\tau) \leq C$ and such that $\psi = \Psi(\Delta\psi)$ implies the existence of the inverse function F for which $\Delta\psi = F(\psi)$, and moreover, $0 < c' \leq F'(\psi) \leq C'$.

On the other hand, from $\Delta\psi = F(\psi)$ one gets $\partial_{x_1}\Delta\psi = F'(\psi)\partial_{x_1}\psi$, and therefore

$$\iint_M \partial_{x_1}\psi \ (\Delta\partial_{x_1}\psi) \ dxdy = \iint_M F'(\psi)(\partial_{x_1}\psi)^2 \ dxdy.$$

Integrating by parts we come to the following:

$$\int_{\partial M} \partial_{x_1}\psi \frac{\partial(\partial_{x_1}\psi)}{\partial n} \, d\ell - \iint_M (\nabla\partial_{x_1}\psi)^2 \ dxdy = \iint_M F'(\psi)(\partial_{x_1}\psi)^2 \ dxdy.$$

Now one can see that the absence of the boundary term leads to a contradiction: The left- and the right-hand sides of the equality are of different signs unless ψ is constant (the trivial case of $\partial_{x_1}\psi \equiv 0$ is treated by replacing ∂_{x_1} with ∂_{x_2}). In particular, it excludes unbounded domains (such as $M = \mathbb{R}^2$, important for meteorological and oceanographic simulations) from the scope of applicability of the Stability Theorem. A way to overcome this difficulty is to exploit the symmetry properties of the domains accompanied by the stability analysis outlined above.

THEOREM 4.9 [M-P]. *In the hypotheses of the Stability Theorem, the stability result is achieved if the condition* $c \leq \frac{\nabla\psi}{\nabla\Delta\psi} \leq C$ *holds with* $c \geq 0$.

The proof is based on the use of a family of Lyapunov functions $H^{\epsilon}(\varphi)$ for which the first variation at the stationary flow ψ is given by $H_1^{\epsilon}(\varphi) = \epsilon \iint (\nabla\varphi,\ \nabla\Delta\psi)\, dxdy$.

REMARK 4.10. It turns out that the stability test based on the second variation of steady flows is inconclusive in dimensions greater than two: The second variation of the kinetic energy is never sign definite in that case (see Section 5.G).

Invariants of isovorticed fields (i.e., Casimir functions of the group of area-preserving diffeomorphisms) play the role of Lagrange multipliers in the above study of the conditional extremum. We refer to the survey [HMRW] for a study of stability by combining the energy function with Casimir functions for a number of physically interesting infinite-dimensional systems. Various modifications and extensions of the *Routh* (or *Casimir-momentum*) method outlined above can be found in, e.g., [MaR, MaS, Vla1,2, W-G].

REMARK 4.11 (J. MARSDEN). ABBREVIATED GUIDE TO THE ENERGY–MOMENTUM METHOD. For a more complete guide to the literature, see http://www.cds.caltech.edu/~marsden/

The energy–momentum (em) method extends the Arnold (or the energy–Casimir) method, which was developed for Lie–Poisson systems on duals of Lie algebras, especially those of fluid dynamical type. The motivation for this extension is threefold. First, it can deal with Lie–Poisson systems for which there are not sufficient Casimir functions available, such as 3D ideal flow and certain problems in elasticity. In fact, [A-H] use (with hindsight) the em-method to show that 3D equilibria for ideal flow are always formally unstable due to vortex stretching. Other fluid and plasma situations, such as ABC flows and certain multiple hump situations in plasma dynamics, provided additional motivation in the Lie–Poisson setting. Second, it extends the method to systems that need not be Lie–Poisson. Examples such as rigid bodies with vibrating antennas (see [KrM]) motivate this need. Finally, it gives sharper stability conclusions in material representation (stability is modulo a *subgroup* of the symmetry group) as well as giving links with geometric phases (Berry phases); see [Pat, MMR]. This is seen already in rigid body problems.

The setting of the energy–momentum method is that of a mechanical system with symmetry with a configuration space Q and phase space T^*Q and a symmetry group G acting, with a standard momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the Lie algebra of G. One gets the Lie–Poisson case when $Q = G$.

The rough idea is first to formulate the problem on the unreduced space T^*Q. Here, relative equilibria associated with a Lie algebra element ξ are critical points of the *aug-*

mented Hamiltonian $H_\xi := H - \langle \mathbf{J}, \xi \rangle$. One now computes the second variation $\delta^2 H_\xi(z_e)$ at a relative equilibrium z_e with the momentum value μ_e subject to the constraint $\mathbf{J} = \mu_e$ and on a space transverse to the action of G_{μ_e}. Although the augmented Hamiltonian H_ξ plays the role of $E + Casimir$ in the Arnold method, Casimir functions are not explicitly needed.

In explicit splittings based on the mechanical connection, the second variation $\delta^2 H_\xi(z_e)$ is block diagonal. In the *same* coordinates the symplectic structure has a simple block structure, so the linearized equations also have a canonical form. Even in the Lie–Poisson setting, this often leads to simpler second variations. This block diagonal structure is what gives the method its computational power. The theory for the em-method can be found in [MaS, SPM, SLM] (see also the exposition in [Mar]). For Lagrangian versions, see [Lew]. There is also a converse, building on classical work of Thompson and Tait, Chetayev, and others, which states that when one has a saddle point for $\delta^2 H_\xi(z_e)$, the addition of dissipation linearly (and hence nonlinearly) destabilizes the relative equilibrium; see [BKMR].

The energy–momentum method is effective in many examples. For instance, [LeS] dealt with the stability problem for pseudo-rigid bodies, which was thought to be analytically intractable. For the heavy top, see [LRSM]; for underwater vehicle dynamics, see [LMa]; and for ABC flows, see [CMa]. The em-method has also been used in the context of free boundary and Hamiltonian bifurcation problems [LMMR, LMR]. Finally, the method also extends to nonholonomic systems (systems with rolling constraints), as shown in [ZBM].

4.B. Wandering solutions of the Euler equation. *Poincaré's recurrence theorem* claims that for any volume-preserving continuous mapping of a bounded region into itself, almost every moving point returns repeatedly to the vicinity of its initial position.

In particular, the phase flow of the Euler equation on any *finite-dimensional* Lie algebra acquires this property. Indeed, level surfaces of the kinetic energy (i.e., of a positively definite quadratic form) E are compact. Every trajectory of the Euler equation belongs to the intersection of some energy level with a certain coadjoint orbit of the Lie algebra.

PROPOSITION 4.12. *The intersections of the coadjoint orbits with the noncritical energy levels can be equipped with a natural volume form conserved by the Euler equation.*

PROOF. If ω is the symplectic structure on a $2m$-dimensional coadjoint orbit $\mathcal{O}$, then the symplectic volume form $\mu = \omega^m$ is preserved by any Hamiltonian flow on the orbit. The flow with the Hamiltonian function E preserves the differential $(2m-1)$-form $\mu_E := \omega^m / dE$ on the intersections of the orbit $\mathcal{O}$ with the E-levels. These intersections are compact, due to the positive-definiteness of the form E. □

COROLLARY 4.13. *The Poincaré recurrence theorem is applicable in this case: almost every trajectory of the Euler equation returns at times to a neighborhood of the initial point.*

REMARK 4.14. The Euler equation with a nondegenerate inertia operator has an invariant C^1-measure on the whole dual Lie algebra $\mathfrak{g}^*$ (not only on the coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ of the group) if and only if the group G is *unimodular*, i.e., the operators ad_η are traceless for all $\eta \in \mathfrak{g}$ [Ko2].

However, the Euler equation of an ideal fluid does not enjoy the recurrence property: The passage to the infinite-dimensional case is not harmless (see [Shn6] for other peculiar features of 2D fluid dynamics). Fix, for instance, the region $M = \{1 \le |x| \le 2 \mid x \in \mathbb{R}^2\}$ and consider the space $\mathcal{V}$ of C^1-smooth divergence-free vector fields in M tangent to the boundary $\partial M = \Gamma_1 \cup \Gamma_2$, Fig.18.

THEOREM 4.15 [Nad]. *There exists a smooth divergence-free vector field ξ on M (tangent to the boundary ∂M) such that for any initial condition C^1-close to ξ the corresponding solution of the Euler equation in M does not return to a vicinity of the point ξ after a certain moment of time (i.e., there exist $\varepsilon, T > 0$ such that for any initial condition $v(0) \in \mathcal{V}$ satisfying $\|v(0) - \xi\|_{C^1} < \varepsilon$, the corresponding solution $v(t)$ satisfies the inequality $\|v(t) - \xi\|_{C^1} > \varepsilon$, whereas $t > T$).*

PROOF. Consider the steady flow v^* with the stream function $\psi(x) = \ln |x|$: $v^* = \mathrm{sgrad}(\ln |x|)$. Let $v^* + h$ be a C^1-small (divergence-free) perturbation of the field v^* : $\|h\|_{C^1} < \delta$.

LEMMA 4.16. *There exists $\delta > 0$ such that for any perturbation h with $\|h\|_{C^1} < \delta$, the solution $v(t)$ with the initial condition $v(0) = v^* + h$ obeys the inequality $\|v(t) - v^*\|_{C^0} < \frac{1}{4}$ for all $t \ge 0$.*

PROOF OF LEMMA. The vorticity function curl $v(t)$ of the solution $v(t)$ is transported by the flow, and so is the function curl $(v(t) - v^*) =$ curl $v(t)$, since curl $v^* \equiv 0$. Therefore, the C^0-norm of the function curl $(v(t) - v^*)$ is conserved as well as the circulation of the field $v(t) - v^*$ along the circumferences Γ_1 and Γ_2. Therefore, the statement of the lemma is essentially the maximum principle for the stream function $\psi(t)$ of the field $v(t)$, which obeys the equation $\Delta\psi(t) = -$ curl $(v(t) - v^*)$. □

Denote by $M_- = \{x \in M, x_1 < 0\}$ and $\ell = \{x \in M, x_2 = 0, x_1 > 0\}$ the semiannulus and the segment, respectively (Fig.18). Choose some smooth divergence-free field u satisfying the following conditions:

$$\|u\|_{C^1} < \frac{\delta}{2}, \qquad u\big|_{M_-} \equiv 0, \qquad \mathrm{curl}\ u\big|_\ell > \frac{\delta}{4}.$$

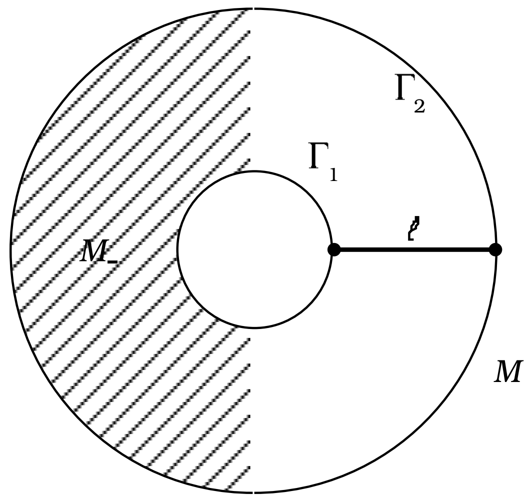

FIGURE 18. Pick a smooth field on the annulus M vanishing on the left semiannulus M_- and whose vorticity is greater than $\delta/4$ on the segment ℓ.

Finally, set $\xi = v^* + u$, and notice that curl $\xi \,|_{M_-} \equiv 0$.

Now let $v(0) \in \mathcal{V}$ be the initial condition close enough to $\xi : \|v(0) - \xi\|_{C^1} < \varepsilon$, and $v(t)$ the corresponding solution of the Euler equation on M. Such a solution defines for each $t \in \mathbb{R}$ an area-preserving diffeomorphism g^t of the annulus M. The circumferences Γ_1 and Γ_2 are mapped by g^t into themselves.

Moreover, by choosing ε to be $\varepsilon = \delta/4$, we ensure that the solution $v(t)$ is close enough to v^*. According to the lemma, the linear velocity of every point on the inner circumference Γ_1 is greater than $3/4$, while that on the outer circumference Γ_2 is smaller than $3/4$. The corresponding angular velocities are greater than $3/4$ on Γ_1 and smaller than $3/8$ on Γ_2, respectively.

The image $\ell_t := g^t(\ell)$ of the segment ℓ under the action of transformation g^t joins the points on different circumferences. The angular coordinates of the connected points diverge from each other at the rate $3t/8$. It follows that for $t > 8\pi/3$, the curve ℓ_t definitely hits $M_- : \ell_t \cap M_- \neq \emptyset$. On the other hand, curl $v(t)$ is carried over by the flow g^t and is greater than $\delta/4 = \varepsilon$ when restricted to ℓ_t. Hence, for $t > 8\pi/3$, we have $\|\xi - v(t)\|_{C^1} > \varepsilon$. $\square$

§5. Linear and exponential stretching of particles and rapidly oscillating perturbations

In this section we study the short-wave asymptotics of the perturbations of a stationary motion of an ideal fluid (following [Arn8]).

5.A. The linearized and shortened Euler equations.

DEFINITIONS 5.1. The 3D Euler equation in the *vortex* (or *Helmholtz*) form

$$\frac{\partial w}{\partial t} = [v, w], \quad \text{where } w = \text{ curl } v,$$

can be *linearized* in a neighborhood of a steady flow v:

$$\frac{\partial s}{\partial t} = [v, s] + [\text{curl}^{-1} s, w]. \tag{5.1}$$

Here $[\ ,\ ] = -\{\ ,\ \}$ is the Lie bracket (i.e., minus the Poisson bracket) of two vector fields, and s is a perturbation of the vorticity field: curl $(v + u) = w + s$, where u is a small perturbation of the steady flow v. The operator curl^{-1} is understood as the reconstruction of the divergence-free vector field from its vorticity (and from the circulations over the boundary components if $\partial M \neq \emptyset$).

We will examine the behavior of solutions of this equation linear in s. Note that the first term on the right-hand side of (5.1) is a more powerful linear operator on functions s than the second. This means that the value of $[v, s]$ on the rapidly oscillating s of the type $s = e^{ikx}$ will contain a higher degree of the wave number k than those occurring in $[\text{curl}^{-1} s, w]$. Hence, for the rapidly oscillating perturbing field s, the second term in (5.1) may be considered as a perturbation of the first. In this way we obtain the *shortened equation*

$$\frac{\partial s}{\partial t} = [v, s]. \tag{5.2}$$

If the stationary flow is potential ($w = 0$), the second term in Equation (5.1) vanishes, and in that case the shortened Equation (5.2) is the same as the linearized Euler equation (5.1). In accordance with perturbation theory [Fad], it is reasonable to assume that the shortened equation defines the continuous part of the spectrum of the linearized equation (5.1).

The shortened Equation (5.2) implies that vector s is carried by the steady flow. If the geometry of the steady flow v is known, this equation can be solved explicitly. Let g^t be a one-parameter group of diffeomorphisms generated by the field v. Then the solution of the shortened equation is expressed in terms of its initial conditions by the formula

$$s(t, x) = g^t_* s(0, g^{-t}(x)), \tag{5.3}$$

where g^t_* is the derivative of the image of g^t.

5.B. The action–angle variables. Below we present two lines of reasoning for the following statement.

PROPOSITION 5.2. *For a non-Beltrami steady field (i.e., for a steady field that is not collinear with its vorticity in any region) on a closed three-dimensional manifold M, almost all solutions of the shortened equation are linearly unstable.*

PROOF. If the fields v and w are not identically collinear in any region, then the manifold without boundary splits into cells in each of which the stream and vorticity lines lie on two-dimensional tori (see Theorems 1.2 and 1.10 in Section 1, or [Arn3,4]). One can introduce the angular coordinates $\varphi = (\varphi_1, \varphi_2) \mod 2\pi$ along the tori and the "action variable" z, which provides the numbering for the tori, such that the volume element is defined by $d\varphi_1 d\varphi_2 dz$, and the fields v and w are given by

$$v(\varphi, z) = v_1(z)\frac{\partial}{\partial \varphi_1} + v_2(z)\frac{\partial}{\partial \varphi_2}, \quad w(\varphi, z) = w_1(z)\frac{\partial}{\partial \varphi_1} + w_2(z)\frac{\partial}{\partial \varphi_2}.$$

These equations are integrable in the system of coordinates $(\varphi_1, \varphi_2, z)$. For the components of the field

$$s(t; \varphi, z) = s_1\frac{\partial}{\partial \varphi_1} + s_2\frac{\partial}{\partial \varphi_2} + s_3\frac{\partial}{\partial z},$$

by using (5.3) we obtain the expressions

$$\begin{aligned} s_k(t; \varphi, z) &= s_k(0; \varphi_0, z) + t \cdot v_k'\, s_3(0; \varphi_0, z), \qquad k = 1, 2, \\ s_3(t; \varphi, z) &= s_3(0; \varphi_0, z), \end{aligned} \tag{5.4}$$

where $\varphi_0 = \varphi - vt$, and the prime denotes the derivative with respect to z. Formulas (5.4) imply that solutions of the shortened Equation (5.2) (for $v' \neq 0$) usually increase linearly with time. □

Hence the conventional (exponential) instability of the linearized Euler equation for non-Beltrami flows can be due only to the second term in formula (5.1). In accordance with perturbation theory, it is reasonable to expect the appearance of a finite number of unstable discrete eigenvalues.

The question of retention of the (detected above) slow instability, when passing from the shortened Equation (5.2) to the complete linearized equation (5.1), is discussed in Section 5.D below.

The other possibility of exponential instability is related to the collinearity of v and w, when the action–angle variables cannot be introduced and the geometry of the steady flow differs from the one described above (cf. [Hen]). This form of instability is examined in Section 5.E.

REMARK 5.3. An integrable (non-Beltrami) steady flow can be thought of as a Hamiltonian system with two degrees of freedom that is restricted to a three-dimensional energy level. The KAM theory for volume-preserving flows on three-dimensional manifolds guarantees that under certain nondegeneracy conditions, all flows sufficiently close to the integrable ones preserve a large set of two-dimensional invariant tori (see, e.g., the survey on the KAM theory of Hamiltonian systems [AKN] or the volume-preserving case in [C-S, D-L, B-L]).

The above implies that for nonstationary Euler solutions that get close enough to a steady non-Beltrami field, the *vorticity* fields of the solutions have plenty of invariant tori. Indeed, those vorticity fields of the solutions approach the integrable vorticity field of the steady flow. (The vortex form of the Euler equation is more suitable for this consideration, since the vorticity, unlike the velocity, is frozen into the flow.) Similarly, for the Navier–Stokes equation the steady flows close to the Beltrami ones have many invariant tori.

5.C. Spectrum of the shortened equation. For a more detailed analysis of solutions of Equation (5.2) (and another viewpoint at Proposition 5.2), we expand s into a Fourier series in terms of φ, using the following notation. Let m, which we shall call the *wave vector,* be a pair of integers m_1 and m_2. We denote $m_1\varphi_1 + m_2\varphi_2$ by (m, φ), the number $\sqrt{m_1^2 + m_2^2}$ by $|m|$, and the pair $n_1 = -m_2$ and $n_2 = m_1$ by n.

For each wave vector we determine the "longitudinal," "transverse," and "normal" vector fields

$$e_m = \frac{m_1}{|m|}\frac{\partial}{\partial\varphi_1} + \frac{m_2}{|m|}\frac{\partial}{\partial\varphi_2}, \quad e_n = -\frac{m_2}{|m|}\frac{\partial}{\partial\varphi_1} + \frac{m_1}{|m|}\frac{\partial}{\partial\varphi_2}, \quad e_z = \frac{\partial}{\partial z}.$$

(For $m = 0$ we assume, e.g., $e_m = \partial/\partial\varphi_1$ and $e_n = \partial/\partial\varphi_2$.)

The Fourier expansion of a field s can now be written as

$$s = \sum_m (A_m e_m + B_m e_n + C_m e_z) e^{i(m,\varphi)},$$

where A_m, B_m, and C_m are functions of z.

It can be readily verified that the vector fields e_m, e_n, and e_z have zero divergence with respect to the volume element $d\varphi_1 d\varphi_2 dz$. Hence,

$$\operatorname{div} s = \sum_m (i|m|A_m + \partial_z C_m) e^{i(m,\varphi)} \quad \left(\partial_z := \frac{d}{dz}\right).$$

Consequently, the divergence-free fields are determined by the condition $i|m|A_m + \partial_z C_m = 0$ satisfied for all m.

By virtue of this condition, the set of functions B_m and C_m (for $m = 0$, we have $C_0 = \text{const}$, but A_0 is to be added) can be taken as the "coordinates" in the space of all fields. In this coordinate system Equation (5.2) decouples into a series of triangular systems

$$\begin{cases} \dot{B}_m = -i|m|v_m B_m + v_n' C_m, \\ \dot{C}_m = -i|m|v_m C_m, \end{cases} \tag{5.5}$$

where $v = v_m e_m + v_n e_n$ is the velocity field of the steady flow (for $m = 0$ we add the equation $\dot{A}_0 = v_0' C_0$); the prime and the dot denote differentiation with respect to z and t, respectively.

Formula (5.5) again implies the nonexponential instability of Equation (5.2) (and proves Proposition 5.2). Furthermore, it determines the spectrum of the latter equation: To each wave vector m one associates a segment of the continuous spectrum along the imaginary axis. The related "frequencies" $|m|v_m$ are equal to all kinds of frequencies (m, v) of the stationary flow on the tori, corresponding to various values of the z-coordinate. The multiplicity of each segment is not less than two (the B- and C-components have the same frequencies).

5.D. The Squire theorem for shear flows. Though the coordinates introduced above are suitable for analyzing the shortened Equation (5.2), analysis of the complete equation (5.1) is generally difficult, since in curvilinear coordinates the operator curl^{-1} is of a complicated form. A particular case in which the analysis can be reduced to a one-dimensional problem is that of a flow with straight streamlines. All plane rectilinear flows, as well as the more general ones in which the fluid particles move in parallel planes at constant velocity, which varies in magnitude and direction when passing from one plane to another, belong to this class. Study of the latter may be considered as an approximate analysis of a generic flow in the torus geometry, in which the torus curvature is neglected, while the shear (i.e., the variation of the direction of the streamlines from one torus to another) is taken into consideration.

Let φ_1, φ_2, and z be Cartesian coordinates and the length element $d\ell^2 = d\varphi_1^2 + d\varphi_2^2 + dz^2$. Let $v = v_m e_m + v_n e_n$ be the velocity field of a shear (rectilinear) flow in three-dimensional space (or in a three-torus, whose curvature is neglected).

PROPOSITION 5.4. *The rectilinear three-dimensional flow is exponentially unstable if and only if at least one of the two-dimensional flows of a perfect fluid obtained by the* *substitution for the velocity vector v of its longitudinal component v_m is exponentially unstable.*

Thus, the problem of exponential instability of the considered class of three-dimensional flows of a perfect fluid is reduced to a similar problem for a set of the two-dimensional flows corresponding to different values of the wave vector m. In the particular case of a nonshear flow (i.e., with a constant direction of the velocity v), all velocity profiles are proportional to each other, and the obtained result agrees with the Squire theorem for a perfect fluid [Squ].

PROOF. In this case it is expedient to consider periodic flows of not necessarily 2π-periodicity (e.g., we can assume the periods of φ_1 and φ_2 to be $2\pi X_1$ and $2\pi X_2$, respectively). The only alteration to be introduced in the formulas of Section 5.C is that now the wave vector m runs not through the lattice of integral points but through the lattice $\{(m_1/X_1, m_2/X_2)\}$.

Under these assumptions, the expansion of the vortex field w in terms of the unit vectors e_m, e_n, and e_z is of the form $w = -v_n' e_m + v_m' e_n$. The matrices of the operator curl in the coordinates B_m, C_m, and of the operator corresponding to the Poisson bracket containing w are, respectively,

$$i|m| \begin{pmatrix} 0 & -\operatorname{Id} + |m|^{-2}\ \partial_z^2 \\ \operatorname{Id} & 0 \end{pmatrix} \quad \text{and} \quad -\begin{pmatrix} i|m|v_n' & v_m'' \\ 0 & i|m|v_n' \end{pmatrix},$$

where Id is the identity transformation. Hence, in our coordinates the linearized Euler equation (5.1) decomposes into the systems of equations corresponding to various m. After some calculation, we obtain for $m \neq 0$ the triangular system

$$\text{(5.6)} \qquad \begin{cases} \dot{B}_m &= \left(i|m|v_m + \frac{v_m''}{i|m|}(id - |m|^{-2}\ \partial_z^2)^{-1}\right) B_m, \\ \dot{C}_m &= \quad i|m|v_m C_m + v_n'(id - |m|^{-2}\ \partial_z^2)^{-1} B_m, \end{cases}$$

and for $m = 0$, we have the system $A_0 = B_0 = C_0 = 0$. The first equation contains the B-component only. If the B-component does not have exponential instability, neither does the C-component (this is implied by the nonhomogeneous linear equation obtained for C_m). Finally, note that the equation for B_m contains only the longitudinal velocity component v_m. □

The Jordan form of system (5.6) indicates that in three-dimensional incompressible flows, unlike the two-dimensional ones, the linear increase of vortex perturbations with time is typical, even in the absence of exponential instability. Notice also that Equation (5.6) is the same as that derived in the analysis of the two-dimensional flow of a perfect fluid whose velocity profile is the component $v_m(z)$ of the velocity vector of a three-dimensional flow in the direction of the wave vector m.

5.E. Steady flows with exponential stretching of particles. In this section we will define a steady flow of an incompressible fluid for which the velocity field is Beltrami, i.e., it is proportional to its own vorticity, and the field does not have a family of invariant surfaces, as mentioned in Section 5.B. This simple example plays a key role in many other constructions of ideal hydrodynamics and of dynamo theory discussed in the sequel (see, e.g., Section V.4).

Imagine an ideal fluid filling a three-dimensional compact manifold M constructed in the following way. First consider the Euclidean three-dimensional space with coordinates x, y, z and define the following three diffeomorphisms of the space:

$$T_1(x,y,z) = (x+1,y,z), \qquad T_2(x,y,z) = (x,y+1,z),$$
$$T_3(x,y,z) = (2x+y, x+y, z+1).$$

Each of these transformations maps the integer lattice in the space x, y, z into itself. Identify all points of xyz-space that can be obtained from each other by the successive application of T_i and T_i^{-1} (in any order). The resulting compact analytic manifold M may be thought of as the product of a two-dimensional torus $\{(x,y) \mod 1\}$ by the segment $0 \le z \le 1$, whose end-tori are identified by means of the formula $(x,y,0) \equiv (2x+y, x+y, 1)$.

To equip the manifold M with a Riemannian metric, we define a metric in xyz-space invariant with respect to all T_i. We first examine the linear transformation of the xy-plane given by the matrix A ("cat map," Fig.19):

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{i.e.,} \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+y \end{pmatrix}.$$

The operator A has the eigenvalues $\chi_{1,2} = (3 \pm \sqrt{5})/2$. Note that $\chi_1 > 1 > \chi_2 > 0$, $\chi_1 \cdot \chi_2 = 1$, and the eigendirections are orthogonal to each other. Let (p,q) be a Cartesian system of coordinates in the xy-plane with the axes p and q directed along the eigenvectors with the eigenvalues $\chi_1 > 1$ and $\chi_2 < 1$, respectively.

Set the metric to be

$$d\ell^2 = e^{-2\beta z} dp^2 + e^{2\beta z} dq^2 + dz^2 \ , \quad \text{where } \beta = \ln \chi_1. \tag{5.7}$$

The metric $d\ell^2$ is invariant with respect to the transformations T_i, and therefore it defines an analytic Riemannian structure on the three-dimensional compact manifold M.

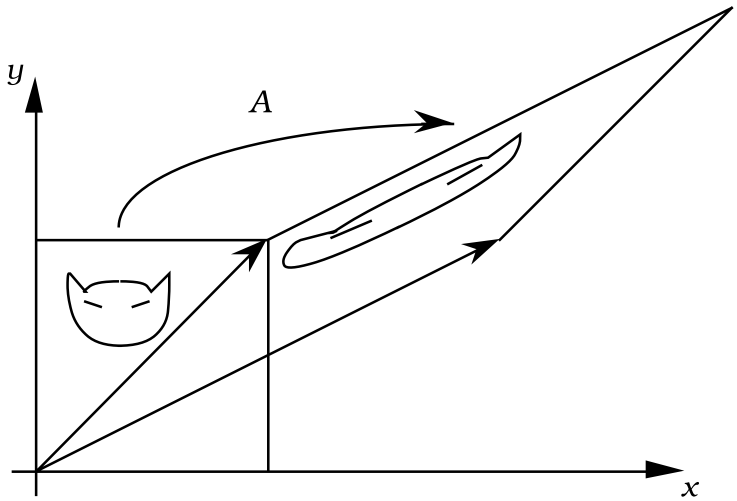

FIGURE 19. The cat map A of the torus onto itself.

Now consider the vector field grad $z = \partial/\partial z$ in xyz-space. Since it is invariant with respect to the transformations T_i, it descends to a vector field v on the Riemannian manifold M. The field v is harmonic on M : div $v = 0$, curl $v = 0$. Hence, v can be taken as the velocity field of a stationary potential flow of an ideal fluid. Every particle of the fluid moving along that field is stretched exponentially in the q-direction, and it is squeezed in the p-direction, as implied by formula (5.7).

5.F. Analysis of the linearized Euler equation. The Euler equation (5.1), linearized at v, is equivalent to the shortened equation (5.2), since the flow under consideration is potential. The simple geometry of the flow v allows one to solve the latter equation by using formula (5.3). It is convenient to express the solution in the following form. Consider the vector fields

$$e_p = e^{\beta z}\frac{\partial}{\partial p},\ e_q = e^{-\beta z}\frac{\partial}{\partial q},\ e_z = \frac{\partial}{\partial z}$$

in pqz-space. These fields are invariant with respect to all transformations T_i, and hence, they can be regarded as vector fields on the manifold M. The directions of the fields e_p, e_q, and e_z are invariant with respect to the phase flow g^t of the field e_z (in coordinate form $g^t(p,q,z) := (p,q,z+t)$), while the fields themselves are transformed as follows:

$$g_*^t e_p = e^{-\beta t} e_p,\ g_*^t e_q = e^{\beta t} e_q,\ g_*^t e_z = e_z$$

(this explains the names of the *stretching direction* e_q, the *compressing direction* e_p, and the *neutral direction* e_z). Every vector field u on M can be decomposed in these directions,

$$u = u_p e_p + u_q e_q + u_z e_z,$$

where u_p, u_q, and u_z are functions on the manifold M.

Formula (5.3) applied to the stationary flow $v = e_z$ has the form

$$s_p(t) = e^{-\beta t} U^t s_p(0),\ s_q(t) = e^{\beta t} U^t s_q(0),\ s_z(t) = U^t s_z(0), \tag{5.8}$$

where U^t is a linear operator acting on functions on the manifold M by the formula $(U^t f)(a) = f(g^{-t}a)$ for any point $a \in M$. Note that the operator U^t is unitary, since the flow g^t preserves the volume element.

Formulas (5.8) provide rather complete answers to all kinds of questions on the growth of perturbations of the steady flow v. First, they show that the q-component of any vortex perturbation exponentially increases with time, while the p-component decays exponentially.

Further, the spectrum of the operator U^t can be easily analyzed by the Fourier series expansion in terms of (x, y) with fixed z, and for functions independent of x and y by such expansion in terms of z. This spectrum has a countably multiple continuous (Lebesgue) component along the unit circle in $\mathbb{C}$, and also a discrete set of eigenvalues corresponding to the eigenfunctions $\varphi_k(z) = e^{2\pi i k z}$ (k are integers). This implies that the Euler equation (5.1) linearized at the stationary flow $v = e_z$ has a countable set of the (unstable) eigenvalues $\alpha - 2\pi i k$, related to the countable set of increasing perturbations of the vorticity $s = \varphi_k(z) e_q$ ($k = \pm 1, \pm 2, \dots$).

The difficulty of predicting solutions of the linearized Euler equation (5.1) for flows with the exponential stretching of particles is also indicated by formulas (5.8): To find an approximate solution, it is necessary to know, with considerable precision, a number of high-order harmonics of the initial perturbation $s(0)$, which rapidly increase with t. Formulas (5.8) and (5.4) show that the exponential particle stretching increases drastically the difficulty of predicting the perturbation growth, as compared to the flows defined by the "generic" stationary solutions of the Euler equation with the linear stretching of particles (see Sections 5.B-5.D).

Phenomena similar to those outlined in this example are also encountered in other flows with exponentially stretched particles, e.g., in the ABC flows

$$v_x = A \sin z + C \cos y,\ v_y = B \sin x + A \cos z,\ v_z = C \sin y + B \cos x$$

(see Sections II.1.A, V.4.B, and [Hen, Dom] for a study of symmetries and results of computer simulations) or in the geodesic flows on surfaces of negative curvature (see Section V.4.D).

5.G. Inconclusiveness of the stability test for space steady flows. In Section 4.A we gave a sufficient condition for stability of planar fluid flows. Unlike the two-dimensional case, the second variation of the kinetic energy of a stationary flow among isovorticed fields is never sign definite in higher dimensions. It implies that the sufficient stability criterion, based on the second variation, is inconclusive (see Remark 3.7): Quadratic Hamiltonians of a saddle type can govern both stable and unstable flows. This study is based on the consideration of rapidly oscillating perturbations of the steady flow.

THEOREM 5.5. *Let M be a three-dimensional closed manifold and v be a steady Euler flow. If curl v is not identically zero, then the spectrum of the quadratic form $\delta^2 E$ (i.e., of the corresponding self-adjoint operator) on the tangent space to the coadjoint orbit of v is neither bounded from below nor from above.*

REMARK 5.6. This theorem, along with its higher-dimensional version formulated below, has been proved in [S-V]. Indefiniteness of the second variation d^2E for the 3D case was earlier established in [Rou1] (and hinted at already in [Arn4]; see also [A-H], where the consideration was put forward for a generic equilibrium in the $3D$ case). The main idea underlying all the proofs is that the form $\delta^2 E$ is a sum of two terms, one of which is always positive, but of smaller order than the other. Picking the rapidly oscillating variation ξ, one can explicitly compute the asymptotic expression for $\delta^2 E$ and thus obtain an arbitrary sign for the second variation in the direction ξ.

The unboundedness of the spectrum of the second variation holds for the higher-dimensional generalization of the Euler equation as defined in Section I.7. Namely, let M be an n-dimensional smooth Riemannian manifold ($n \geq 3$) endowed with a volume form μ, and $v^\flat$ the one-form on M obtained from a μ-divergence-free vector field v by means of the identification $v^\flat(w) = (v,\ w)$ determined by the Riemannian metric $(\ ,\ )$. The kinetic energy is given by $E(v) = \frac{1}{2}\langle v,\ v\rangle = \frac{1}{2}\int_M (v,\ v)\ \mu$.

THEOREM 5.5′ [S-V]. *Let u be a smooth steady solution of the Euler equation in M. The second variation $\delta^2 E$ of the energy among the isovorticed vector fields is identically zero, whereas $v^\flat$ is locally "potential" in the sense that $d(v^\flat) \equiv 0$. Otherwise, the spectrum of the self-adjoint operator corresponding to the quadratic form $\delta^2 E$ on the space of isovorticed fields is neither bounded from below nor from above.*

REMARK 5.7. Actually, the Euler equation is defined on cosets of 1-forms on M : $[v^\flat] \in \Omega^1(M)/d\Omega^0(M)$ (see Chapter I). There are as many cosets furnishing the condition $d[v^\flat] = 0$ as elements in $H^1(M)$, i.e., a finite-dimensional space. Hence, among all stationary flows on the manifold M, there are exactly $b_1(M) := \dim H^1(M)$ linearly independent ones for which the second variation of the kinetic energy is zero. For all other steady flows this variation is indefinite.

LEMMA 5.8. *The second variation of the energy* $E(v) = \frac{1}{2}\langle v,\ v\rangle = \frac{1}{2}\langle v^\flat,\ v^\flat\rangle$ *on the (image in the Lie algebra of the coadjoint) orbit of the "isovorticed fields" is given by the quadratic form*

$$\delta^2 E(\xi) = \frac{1}{2}\langle i_\xi dv^\flat + dp, i_\xi dv^\flat + dp\rangle + \frac{1}{2}\langle i_\xi dv^\flat + dp, L_\xi(v^\flat)\rangle, \tag{5.9}$$

where the function p *is chosen to make the 1-form* $i_\xi dv^\flat + dp$ *correspond to a divergence-free field after the Riemannian identification.*

PROOF OF LEMMA is a straightforward application of formula (3.2) to the coadjoint operator $B(v,\xi) = i_\xi d(v^\flat) + dp$. All fields are supposed to be square-integrable. The formal tangent space to the coadjoint orbit of the 1-form $v^\flat$ is the image of the operator B. □

For a three-dimensional manifold M, this formula reads as

$$\begin{aligned}\delta^2 E(\xi) &= \frac{1}{2}\langle(\nabla\times v)\times\xi + \nabla p,\ (\nabla\times v)\times\xi + \nabla p\rangle \\ &+ \frac{1}{2}\langle\ (\nabla\times v)\times\xi + \nabla p,\ \nabla\times(\xi\times v)\rangle.\end{aligned}$$

The operator $B(v,\xi)$ in this case becomes $B(v,\xi) = (\nabla\times v)\times\xi + \nabla p$, where the pressure function p is chosen to make the vector field $(\nabla\times v)\times\xi + \nabla p$ divergence free.

PROOF OF THEOREM 5.5′. Certainly, $dv^\flat \equiv 0$ implies $dp = 0$, and hence, $\delta^2 E(\xi) \equiv 0$.

Assume now that the 2-form $dv^\flat$ and the vector field v are both nonzero at a point $x_0 \in M$. Fix some function $\varphi(x)$ for which $(v,\ \nabla\varphi)$ and $d\varphi \wedge dv^\flat$ are both nonzero in a neighborhood $\mathcal{U}$ of x_0. Pick smooth vector fields a_R and a_I that are orthogonal to $\nabla\varphi$ everywhere, vanish outside $\mathcal{U}$, and obey the inequalities $du^\flat(a_R, a_I) \geq 0$ everywhere, and $du^\flat(a_R, a_I) > 0$ in a smaller neighborhood $\mathcal{U}' \subset \mathcal{U}$. Finally, define a complex vector field $a = a_R + \sqrt{-1}\ a_I$ (where we use the notation $\sqrt{-1}$ for the imaginary unit to distinguish it from the operator i_v).

Our goal is to construct deformations ξ_ε (uniformly bounded in ε) for which $\delta^2 E(\xi_\varepsilon)$ is arbitrarily large positive or negative. Note that it is enough to choose ξ_ε to be a complex vector, if we extend the operator $\delta^2 E$, as well as the Hermitian inner product, to the complexification of the space of vector fields on the manifold M. Indeed, consider the Hermitian inner product $\langle\ ,\ \rangle_{\mathbb{C}}$, linear in the first argument and antilinear in the second, that extends the real inner product $\langle\ ,\ \rangle$ on the vector fields. Then boundedness of the spectrum of $\delta^2 E$ implies that the *real part* of the value $\langle(\delta^2 E)\xi_\varepsilon, \xi_\varepsilon\rangle_{\mathbb{C}}$ is bounded both from below and from above whenever ξ_ε belongs to some fixed ball in the Hilbert space of square-integrable complex vector fields.

To construct such deformations ξ_ε, consider for simplicity the case where μ is the Riemannian volume form on the manifold. Then a one-form $u^\flat$ corresponds to a divergence-free vector field u if and only if $d^*(u^\flat) \equiv 0$ (where the operator $d^* : \Omega^k(M, \mathbb{C}) \to \Omega^{k-1}(M, \mathbb{C})$ is dual to the exterior derivative operator $d : \Omega^k(M, \mathbb{C}) \to \Omega^{k+1}(M, \mathbb{C})$ by means of the identification of $\Omega^k(M^n, \mathbb{C})$ and $\Omega^{n-k}(M^n, \mathbb{C})$ provided by the metric).

Define the rapidly oscillating vector fields ξ_ε as the following $O(\varepsilon)$-correction of the field $a \cdot \exp(\sqrt{-1}\varphi/\varepsilon)$ to make it divergence free: ξ_ε is dual to the 1-form

$$\xi_\varepsilon^\flat = \varepsilon\sqrt{-1}\ d^*\left(\frac{d\varphi \wedge a^\flat}{\|d\varphi\|^2}\exp(\sqrt{-1}\varphi/\varepsilon)\right) = a^\flat \exp(\sqrt{-1}\varphi/\varepsilon) + O(\varepsilon).$$

Then the leading term of $\delta^2 E(\xi_\varepsilon)$ in the ε-expansion as $\varepsilon \to 0$ is

$$\begin{aligned}\delta^2 E(\xi_\varepsilon) &= \frac{1}{2\varepsilon}\langle i_{\xi_\varepsilon} du^\flat + dp, \sqrt{-1}(u, \nabla\varphi)\ a^\flat \exp(\sqrt{-1}\varphi/\varepsilon)\rangle_{\mathbb{C}} + O(1) \\ &= -\frac{\sqrt{-1}}{2\varepsilon}\int_M (u, \nabla\varphi)\ du^\flat(a, \bar{a})\ \mu + O(1) = -\frac{1}{\varepsilon}\int_M (u, \nabla\varphi)\ du^\flat(a_R, a_I)\ \mu + O(1),\end{aligned}$$

where $\langle\ ,\ \rangle_{\mathbb{C}}$ is the Hermitian inner product, extending the real inner product $\langle\ ,\ \rangle$.

By assumption, the inner product $(u, \nabla\varphi)$ is nonzero on $\mathcal{U}$, while $du^\flat(a_R, a_I)$ is positive in $\mathcal{U}'$ and nonnegative otherwise. Hence the integral is nonzero. Therefore, we can make the real part of $\delta^2 E(\xi_\varepsilon)$ arbitrarily large positive or negative by choosing ε to be of appropriate sign and sufficiently close to zero. Thus, $\delta^2 E$ is not a sign-definite form, and it has a spectrum unbounded in both directions. □

REMARK 5.9 [S-V]. For a manifold with boundary the same conclusion holds. One can take ξ_ε vanishing near the boundary and obtain arbitrarily large negative or positive values of $\delta^2 E(\xi_\varepsilon)$. The domain of the corresponding self-adjoint operator $\delta^2 E$ contains all smooth divergence-free vector fields with compact support in the interior of M.

REMARK 5.10. One can argue that indefiniteness of the second variation is indicative of instability (see, e.g., [A-H]). Though the sufficient criterion discussed above says nothing in this case, other methods can be applied to certain flows (see [Vla2] for the direct Lyapunov method and [FGV, FV1] for an instability criterion valid for some particular three-dimensional flows).

For instance, a fluid possessing surface tension and filling an upside-down cylindrical glass (with any cross section) is shown to be unstable [VlB, Vla2]. To the best of our knowledge, there is no proof of (actual nonlinear) instability if the shape of the container is not cylindrical.

The situation changes slightly for the system of MHD equations. In contrast with the purely hydrodynamical setting, it is possible to obtain three-dimensional examples of MHD equilibria for which the second variation of the total energy is definite [FV2]. The class of flows whose stability may be determined by the sufficient criterion discussed in this and preceding sections is very restricted. In particular, the second variation of energy turns out to be indefinite for the flows having a point where the vectors of the velocity v and of the vorticity curl v are nonzero and nonparallel to the vector of the magnetic field B. The same statement holds for fields with parallel magnetic and velocity fields if the magnetic field is weak enough: $\|v\| > \|B\|$ at some point [FV2]. Other applications of the stability analysis to MHD can be found in [VlM, VMI]. Stability of steady two- and three-dimensional flows of an ideal fluid with a free boundary was studied in [SYu]; for the stability analysis of stratified ideal, barotropic, and other fluids see [Dik, A-H, HMRW, Gri, Vla3]. We also refer to [Arn14, DoS, FV1, Lif, Shf] for various stability and asymptotic results for perturbations of steady solutions of the Euler and Navier–Stokes equations.

§6. Features of higher-dimensional steady flows

The existence of the Bernoulli function for a steady fluid flow is a general phenomenon valid for any dimension (see Section 1.A). In this section we discuss (following [GK1,2]) the consequences of the presence of this extra first integral for steady solutions of the Euler equation of an ideal fluid in higher dimensions.

6.A. Generalized Beltrami flows. Let v be an analytic divergence-free field of a steady flow on an odd-dimensional compact manifold M^{2n+1} equipped with a volume form μ.

DEFINITION 6.1. A trajectory of the field v is called *chaotic* if it is not contained in any analytic hypersurface in M^{2n+1}.

For instance, a generic trajectory of an ergodic flow is chaotic.

PROPOSITION 6.2 (=1.8′, [GK2]). *An analytic steady field v with at least one chaotic trajectory is proportional to its vorticity ξ; i.e., $\xi = C \cdot v$, where $C \in \mathbb{R}$.*

REMARK 6.3. Recall that in the odd-dimensional case the vorticity field is defined by the relation $i_\xi \mu = \omega^n$, where the two-form $\omega = du$ is the differential of the one-form u dual to the vector field v : $u(\cdot) = (v, \cdot)$; see Chapter I. Thus, by the proposition, the field v with a chaotic trajectory is an "eigenvector" of the operator curl: $v \mapsto \xi$, even though for $n > 1$ this operator is nonlinear! It is natural to call such a field v a *generalized Beltrami* flow. The theorem manifests that higher-dimensional Beltrami flows, as well as the three-dimensional ones, have quite a complicated structure. In particular, the mixing in a steady flow might occur only if at least one chaotic trajectory exists, i.e., only for the generalized Beltrami flows. On the contrary, a non-Beltrami steady flow is fibered by a family of hypersurfaces invariant under the flow, and therefore actual mixing for such a flow is impossible. The proof of the theorem closely follows the argument used for the three-dimensional case in [Arn3,4]; cf. Section 1.A.

PROOF. The vorticity field ξ commutes with the velocity field v for any steady flow (see Remark 1.4). The fields ξ and v are both tangent to the "Bernoulli surfaces," i.e., to the level hypersurfaces of the analytic Bernoulli function $\alpha = p + i_v u$, which is defined by the stationary Euler equation $i_v du = -d\alpha$.

If the Bernoulli function α is nonconstant, then trajectories of v lie on level hypersurfaces of α, which contradicts the assumption. (Note that similar to the three-dimensional case, the nonsingular Bernoulli surfaces ($d\alpha \neq 0$) have zero Euler characteristic, since the tangent field v has no singular points on them.) If the function α is constant, then the fields ξ and v are collinear (Remark 1.6). Consider the function $\kappa := v^2/\xi^2$ (or, alternatively, $(\xi, \xi) = \kappa \cdot (v, v)$). Owing to the commutativity of ξ and v, the function κ is invariant under the flow of v. Therefore, the field v is tangent to the level surfaces of κ. Since v has a chaotic trajectory, the only possibility remaining is that $\kappa \equiv const$. (Note that the Bernoulli function α is analytic, and the function κ is the ratio of two analytic functions.) Hence the functions α and κ are both constant, and the fields ξ and v are locally proportional: $\xi = C \cdot v$, where $C = \pm 1/\sqrt{\kappa} = const$. □

EXAMPLE 6.4. The Hopf vector field $(x_2, -x_1, x_4, -x_3, \ldots, x_{2n+2}, -x_{2n+1})$ is an example of an eigenvector field for the curl operator on $S^{2n+1} \subset \mathbb{R}^{2n+2}$ without chaotic trajectories. The theorem above claims that the existence of such a trajectory makes the vector field an "eigenvector" of curl. It would be interesting to find a nontrivial example of a higher-dimensional ABC flow and to compare its ergodic properties with those in the three-dimensional case (see, e.g., [Hen]). In particular, one wonders if there is an analogue for higher dimensions of the analytic nonintegrability of certain ABC flows, proved in [Zig2].

6.B. Structure of four-dimensional steady flows. The main result of this section shows that the steady flows of a four-dimensional fluid are very similar to integrable Hamiltonian systems with two degrees of freedom.

Here and below we deal with an even-dimensional orientable Riemannian manifold M^{2n} endowed with a volume form μ. In this case, a generic steady solution v gives rise to the closed 2-form $\omega = du$, which is symplectic (nondegenerate) almost everywhere on M. In particular, it allows one to define another (besides α) invariant function on the manifold: $\lambda(x) = \omega^n/\mu$, called the vorticity function (or "symplectic volume" element). The function λ is invariant, since $L_v\omega = 0$ and $L_v\mu = 0$. This means that the vorticity function λ and the Bernoulli function α are first integrals of the flow of v on M.

Let $\rho = (\alpha, \lambda) : M \to \mathbb{R}^2$ and Γ be the set formed by all $x \in M$ such that either $\lambda(x) = 0$ or $\rho(x)$ is a critical value of ρ. In other words, Γ is the union of the zero λ-level Λ and of the preimage of the set of critical values of ρ.

THEOREM 6.5 [GK1,2]. *Let M be a closed orientable four-dimensional manifold. Then*

(1) *the open set $U = M \setminus \Gamma$ is invariant under the flow of v;*
(2) *every connected component of U is fibered into two-dimensional tori invariant under the flow; and*
(3) *on each of these tori the flow lines are either all closed or all dense.*

PROOF. The form ω is symplectic on the complement to the set $\Lambda = \{\lambda = 0\}$. The vector field v is Hamiltonian (relative to this symplectic form) with the Hamiltonian function α: by definition $i_v\omega = -d\alpha$. Let ξ be the Hamiltonian vector field on $M \setminus \Lambda$ with the Hamiltonian λ. Observe that the Poisson bracket of the functions α and λ is identically zero on $M \setminus \Lambda$, since $\{\alpha, \lambda\} = L_v\lambda = 0$. Therefore, the fields v and ξ commute, and their flows together give rise to an $\mathbb{R}^2$-action on $M \setminus \Lambda$. The map ρ is, in fact, the momentum mapping for this action. The map ρ is invariant with respect to the action, and the orbits coincide with the connected components of ρ-levels. The projection $\rho\,|_U\colon U \to \rho(U)$ is a proper submersion, since defining U we have excluded from M all critical points of ρ. Hence each orbit in U is a smooth closed surface, and so it is either a torus or a Klein bottle. Furthermore, this surface is cooriented by $d\alpha \wedge d\lambda$. As a result, we see that the surface is orientable, i.e., a torus. Therefore, ρ fibers every connected component of U into tori.

On each orbit, the flow of ξ acts transitively on integral curves of v. Moreover, the field ξ does not have zeroes on U since its Hamiltonian function λ does not have critical points there. Thus the integral curves of v, on which ξ acts, are either all closed or all dense on each torus. □

Note that for a "generic" pair of α and λ the set U is open and dense in M. Thus the theorem gives an almost complete description of the flow of v.

The real-analytic version of the latter theorem for a manifold without boundary now looks as follows.

THEOREM 6.6 [GK2]. *Let M be as in the theorem above. Assume, in addition, that all the data (i.e., M, μ, and the metric), as well as ω, are real-analytic, and $d\alpha \wedge d\lambda \neq 0$ somewhere on M. Then Γ is a semianalytic subset nowhere dense in M, and $U = M \setminus \Gamma$ has a finite number of connected components. Every connected component is fibered into two-dimensional tori invariant under the flow. On each of these tori the flow lines are either all closed or all dense.*

A version of this theorem holds for a manifold M with boundary (see [GK2] for more detail).

REMARKS 6.7. A) For an arbitrary even-dimensional manifold M^{2n}, we can assert that M is a union of $(2n-2)$- (or less) dimensional submanifolds, such that the steady vector field v is tangent to them. These submanifolds are obtained as intersections of the levels $\alpha = \text{const}$ and $\lambda = \text{const}$ and have zero Euler characteristic.

B) For an arbitrary odd-dimensional M^{2n+1}, instead of the function $\lambda = \omega^n/\mu$ (and of the covector field $d\lambda$), we define the vorticity vector field ξ by $i_\xi \mu = \omega^n$. The fields ξ and v commute and thus give rise to an $\mathbb{R}^2$-action on M^{2n+1}. So in this case a steady flow gives rise to a foliation of *dimension* 2, unlike the foliation of *codimension* 2 in the even-dimensional case.

6.C. Topology of the vorticity function. Let ω be the two-form associated to a stationary divergence-free solution v on M^{2n} (i.e., $\omega = du$, where u is the differential 1-form $u(\cdot) = (v, \cdot)$ defined by the Riemannian metric $(\ ,\ )$ on M). In this section, we study the topology of the vorticity function $\lambda = \omega^n/\mu$ of the steady flow v. We describe some special features of such λ that the pair (λ, ω) (under a mild condition) does not admit "too many symmetries."

Let $\mathfrak{g}$ be the Lie algebra of all divergence-free vector fields on M. Steady flows are critical points of the energy on the coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ that consists of the 2-forms associated to the fields on M isovorticed with v. It is clear that topological invariants of λ, such as the number of its critical points and their indices, depend only on the orbit $\mathcal{O}$. This simple observation will enable us to find orbits with no stationary solutions at all (see Section 6.D).

DEFINITION 6.8. A function f on a compact symplectic manifold (P, ω) *does not admit extra symmetries* if an arbitrary function g satisfying $\{f, g\} = 0$ is constant on connected components of the level sets of f (i.e., $\{f, g\} = 0$ implies that the differential dg is proportional to df with coefficient depending on the point on P).

REMARK 6.9. On a two-dimensional symplectic manifold no functions admit extra symmetries. Conjecturally, a generic function on a compact symplectic manifold of any dimension does not admit extra symmetries. It is true for $\dim M = 4$ (cf. [MMe]). The question turns out to be closely related to some subtle problems in Hamiltonian dynamics. The general conjecture can be regarded as a Hamiltonian version of the following problem of generic nonintegrability.

REMARK 6.10: DIGRESSION ON NONINTEGRABILITY. From the time of Poincaré one usually has used the term "a nonintegrable dynamical system" in the sense of "a dynamical system having no analytic first integrals." However, there exists a number of other possibilities. For instance,

(1) the absence of invariant hypersurfaces (or of principal ideals),
(2) the absence of invariant closed 1-forms (or of multivalued first integrals),
(3) the absence of invariant distributions of tangent subspaces (or of invariant Pfaff modules), and
(4) the absence of invariant foliations (or of invariant completely integrable Pfaff systems).

Consider a dynamical system with discrete time (a diffeomorphism of a compact manifold) and an object of one of the above types (a function, an ideal, a closed 1-form, etc.) The images of this object under the iterations of the diffeomorphism may form a finite set (if they are repeated periodically) or an infinite sequence and may generate a finite-dimensional or infinite-dimensional space. These properties reflect the "degree of chaoticity" of the dynamical system.

PROBLEM 6.11. Do the nonintegrable systems (in the sense of each of the four definitions above) form an open set in the space of dynamical systems on manifolds of sufficiently high dimension? Conjecturally, this is the case in the space of Hamiltonian systems near an elliptic equilibrium point.

Even specific examples of systems that are nonintegrable in the strong sense ((1),(2),(3), or (4)) would be interesting. The following example of chaotic behavior is due to Kozlovsky [Koz1]. Consider a germ of an analytic mapping

$$z \mapsto e^{i\theta} z + z^2$$

of the complex line $z \in \mathbb{C}$ to itself in a neighborhood of the (elliptic) fixed point 0. Let an irrational θ be unusually well approximated by rational numbers. Then there are infinitely many periodic trajectories in any neighborhood of the origin. Such mappings are nonintegrable in the sense of (1)–(4).

One more extension of the integrability property has been suggested by Yudovich [Yu2]. He introduced the notion of cosymmetry of a vector field. A *cosymmetry* is a field of hyperplanes in the tangent spaces containing the given vector field (one might call them nonholonomic constraints). This field of hyperplanes is allowed to degenerate at some points of the manifold, and it is defined by a 1-form (possibly with zeroes) annihilated at every point by the given vector field.

Every nonzero vector field has locally some trivial cosymmetries. The existence of a global cosymmetry implies some restrictions on the topological properties of the field. Example: If a field with an equilibrium has a nontrivial cosymmetry, then the equilibrium is nonisolated (and generically belongs to a curve of equilibria). If a vector field admits two cosymmetries, it generically has a surface of equilibria, etc. This phenomenon is described by a "cosymmetric version" of the implicit function theorem [Yu2]. Furthermore, for dynamical systems with cosymmetries one observes generic bifurcations of an equilibrium point into a family of those points (the phenomenon of infinite codimension among all dynamical systems).

Yudovich has discovered nontrivial cosymmetries in some physical problems of hydrodynamical origin (fluid convection in porous media) and of Newtonian mechanics. For instance, if a vector field has a first integral ϕ, then the differential $d\phi$ is a (holonomic) cosymmetry. (Example: For Newton's second law $\ddot{x} = F(x)$ with a *potential* force $F(x)$, the sum of the kinetic and potential energy is the first integral of the equation.) The notion of cosymmetry provides a natural framework for the validity of the result of the Noether theorem on the existence of momentum-like first integrals for the Newton equation $\ddot{x} = F(x)$ with a *nonpotential* force $F(x)$ [Yu2]. The nonholonomic cosymmetries of this equation ensure (generically) the existence of continuous families of equilibria even for this classical situation.

Returning to steady fluid flows in even dimensions, we need the following

DEFINITION 6.12. A coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ *does not admit extra symmetries* if for any (or, equivalently, for some) 2-form $\omega \in \mathcal{O}$ the corresponding vorticity function λ does not admit extra symmetries on $\lambda^{-1}([a, b])$ for any pair of its regular values $0 < a < b$ or $a < b < 0$. (Note that the form ω is symplectic precisely on the complement to the zero level of $\lambda = \omega^n/\mu$.)

Definitions 6.8 and 6.12 are consistent: A function f on a compact symplectic manifold does not admit extra symmetries if and only if its restriction to the preimage of any segment with regular endpoints does not admit them.

DEFINITIONS 6.13. A function on a compact manifold is a *Morse function* if all its critical points are nondegenerate, i.e., the *Hessian matrix* of the second derivatives of the function is nondegenerate at every critical point. The number of negative eigenvalues of the Hessian matrix is called the *Morse index* of the critical point.

An orbit $\mathcal{O} \subset \mathfrak{g}^*$ has *Morse type* if for any (or, equivalently, for some) $\omega \in \mathcal{O}$ the function λ is a Morse function on M constant on every connected component of ∂M. The orbit is called *positive* if $\lambda(x)$ is positive for all $x \in M \setminus \partial M$.

THEOREM 6.14 [GK2]. *Let* $\dim M = 2n \geq 4$ *and* $\mathcal{O}$ *be a Morse-type orbit without extra symmetries. Assume that* $\mathcal{O}$ *contains a steady solution. Then, for every* $\omega \in \mathcal{O}$ *all the critical points of the vorticity function* λ *have indices either no less than* n *or no greater than* n *on every connected component of* $M \setminus \{\lambda = 0\}$.

EXAMPLE 6.15. If $\mathcal{O}$ is as above and $\lambda > 0$ on $M \setminus \partial M$, then λ cannot have both a local maximum (index $2n$) and a local minimum (index 0) on $M \setminus \partial M$.

PROOF OF THEOREM. For simplicity assume that $\mathcal{O}$ is a positive orbit, i.e., $\lambda > 0$ on M. Only a minor modification is required to prove the general case. Let $\omega \in \mathcal{O}$ be a stationary solution ($L_v\omega = 0$) and α the corresponding Bernoulli function such that $d\alpha = -i_v\omega$.

Since $\lambda = \omega^n/\mu$ does not admit extra symmetries and $\{\alpha, \lambda\} = 0$, the function α must be constant on the connected components of λ-levels.

LEMMA 6.16. *The functions* λ *and* α *have the same critical points. In particular, the critical points of* α *are isolated.*

PROOF OF LEMMA. Since λ does not admit extra symmetries, $d\lambda(x) = 0$ implies that $d\alpha(x) = 0$. The rest of the critical set of α may only be the union of some connected components of λ-levels. For a vector field v and the Riemannian dual 1-form $u(\cdot) = (v, \cdot)$ one has $u(v) = (v, v) \geq 0$.

Consider the vector field η on M defined by the formula $i_\eta\omega = u$. The field η is expanding for the 2-form $\omega = du$: $L_\eta\omega = \omega$. Furthermore, the field η is gradient-like for the Bernoulli function α:

$$L_\eta\alpha = i_\eta d\alpha = -i_\eta i_v\omega = i_v u = u(v) \geq 0.$$

Moreover,

$$L_\eta\alpha = 0 \Leftrightarrow u(v) = 0 \Leftrightarrow u = 0. \tag{6.1}$$

If the critical set of α contains a connected component Γ of a λ-level, then $L_{\eta_x}\alpha = 0$ for all $x \in \Gamma$, and as a consequence of (6.1), $u|_\Gamma = 0$. Hence, $\omega|_\Gamma = du|_\Gamma = 0$. This is impossible, because Γ is a hypersurface in the symplectic manifold (M, ω) and $2n = \dim M \geq 4$. The lemma is proved. □

Now observe that all zeroes of the vector field η are nondegenerate, as follows from $L_\eta \omega = \omega$. Therefore, the field η has smooth complementary dilating and contracting manifolds in a neighborhood of each of its stagnation points. Moreover, the dimension of the dilating manifold for each point must be at least n. Indeed, the restriction of the symplectic form ω to the contracting manifold of η must be zero by virtue of the expanding property of η, and hence all the contracting manifolds have dimension at most n.

Now we are ready to complete the proof of the theorem. The field η is gradient-like for the function α. Therefore, η is either gradient- or antigradient-like for λ on the whole of M, since the λ- and α-levels coincide in a neighborhood of every critical point and λ is a Morse function. Thus, at all critical points of the vorticity function λ the dimensions of all its dilating or of all its contracting manifolds are simultaneously bounded by n from below. This gives the desired inequality for the Morse indices of the λ-critical points. □

One may prove that all the critical points of α are nondegenerate except, possibly, for its maxima and minima.

THEOREM 6.17 [GK2]. *Let M be diffeomorphic to the two-dimensional disk B^2. If a Morse-type orbit $\mathcal{O} \subset \mathfrak{g}^*$ contains a stationary solution, then for any $\omega \in \mathcal{O}$ the vorticity function λ cannot simultaneously have a local maximum and a local minimum in M, provided that $\lambda > 0$ on $M \setminus \partial M$.*

Note that since $\dim M = 2$, the orbit $\mathcal{O}$ does not automatically admit extra symmetries.

The proof below is a formalization of the following argument, which is evident from a physical viewpoint. Minima and maxima of the vorticity function correspond to rotations of the fluid in opposite directions. On the other hand, the positivity of λ prescribes *a priori* a counterclockwise drift.

PROOF. First, recall that α cannot have maxima. Indeed, in a neighborhood of a maximum the gradient-like (for η) field η would shrink the area, which contradicts the equation $L_\eta \omega = \omega$. Let Γ be the critical set of α. Observe that since α is constant on ∂M, the set Γ either contains the boundary ∂M or does not meet it. We claim that $M \setminus \Gamma$ is connected. To prove this, assume the contrary. Then there exists an open set $U \subset M \setminus \Gamma$ such that $\partial U \subset \Gamma$. The set U is invariant under the flow of η, since $d\alpha$ (and thus η)

vanishes on Γ. On the other hand, as above, the existence of such a set U contradicts the area expansion.

Observe that the field η is gradient-like for λ in a neighborhood of every local minimum of λ: Indeed, every local minimum of λ is a local minimum of α, and the field η is gradient-like for α. Meanwhile, near a local maximum of λ, the field η must be antigradient-like for λ. Switching from being gradient-like to antigradient-like (and vica versa) may occur only on Γ. But Γ does not divide M. Hence η is either gradient-like or antigradient-like on all of M. The theorem follows. $\square$

6.D. Nonexistence of smooth steady flows and sharpness of the restrictions. Applying Theorems 6.14 and 6.17, one can easily find a coadjoint orbit that does not contain a steady solution.

The case of a two-dimensional M is particularly simple. Consider a disk $M = B^2 \subset \mathbb{R}^2_{x,y}$ with $\mu = dx \wedge dy$ and $\omega = \lambda \cdot \mu$, where λ is a positive Morse function on B such that $\lambda|_{\partial B} = \text{const}$. Assume also that λ has both a local maximum and a local minimum in the interior of B (see, e.g., Fig.20).

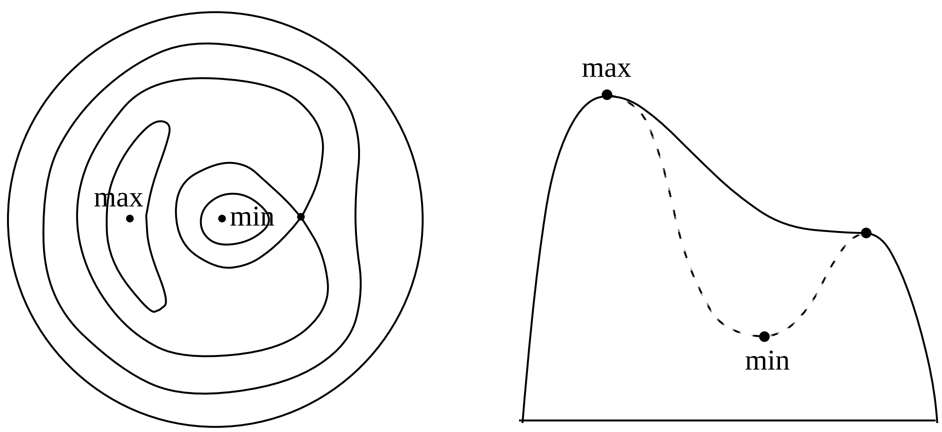

FIGURE 20. Level curves and a profile of the vorticity function having no smooth steady flow.

COROLLARY 6.18 (OF THEOREM 6.17). *There is no smooth steady solution on B^2 whose vorticity function is obtained from the function λ by an area-preserving diffeomorphism.*

Note that a "generalized steady solution" with a discontinuous vorticity function may still exist and be of certain interest for applications [Mof4].

Remark 6.19 [GK2]. It turns out that Theorems 6.8 and 6.10 are almost sharp as long as we are not concerned about the metric. Namely, there is no general restriction on the topology of the vorticity function except that given by the theorems.

In the two-dimensional case one can consider, for example, a positive smooth subharmonic function λ on $\mathbb{C} \approx \mathbb{R}^2$, constant on the unit circle. Then on the unit disk B^2 there exists a metric $(\cdot, \cdot)$ and an area form μ such that λ is the vorticity function of a steady solution. In particular, the vorticity function may have saddle critical points, at least for some metrics and volume forms.

A higher-dimensional version of Corollary 6.18 follows from Theorem 6.14. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a Morse-type orbit that is positive (i.e., $\lambda > 0$) and has no extra symmetries.

Corollary 6.20 (of Theorem 6.14). *Assume that for some $\omega \in \mathcal{O}$ the vorticity function λ has a critical point of index $k_1 < n$ and a critical point of index $k_2 > n$, where $2n = \dim M$. Then the coadjoint orbit $\mathcal{O}$ contains no steady solutions.* □

Corollary 6.21. *Assume that $H^{k_1}(M, \mathbb{R}) \neq 0$ and $H^{k_2}(M, \mathbb{R}) \neq 0$ for some $k_1 < n$ and $k_2 > n$. Then the coadjoint orbit $\mathcal{O}$ contains no steady solutions.*

Proof is the application of the Morse inequalities. □

Now the sharpness result reads as follows.

Theorem 6.22 [GK2]. *Let M be a compact manifold with boundary,* $\dim M = 2n \geq 6$, *and λ a smooth positive function on M such that f is constant on connected components of ∂M and all the critical points of λ have indices no greater than n. Assume, in addition, that M admits an almost complex structure. Then there exist a metric and a volume form on M such that λ is the vorticity function of a steady solution.*

The proof uses the result of Ya. Eliashberg [El2] that the manifold M admits a complex structure such that the closed 2-form $\omega = -2\,\mathrm{Im}\,\partial\bar{\partial}\lambda$ is a symplectic form on M.

Various connections between the steady solutions and complex structures, as well as further details and other subtle restrictions on the pairs (ω, λ) imposed by the existence of a steady solution, are discussed in [GK2].