TOPOLOGICAL METHODS IN HYDRODYNAMICS

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Introduction

"...ad alcuno, dico, di quelli, che troppo laconicamente vorrebbero vedere, nei piu' angusti spazii che possibil fusse, ristretti i filosofici insegnamenti, si' che sempre si usasse quella rigida e concisa maniera, spogliata di qualsivoglia vaghezza ed ornamento, che e' propria dei puri geometri, li quali ne' pure una parola proferiscono che dalla assoluta necessita' non sia loro suggerita.

Ma io, all'incontro, non ascrivo a difetto in un trattato, ancorche' indirizzato ad un solo scopo, interserire altre varie notizie, purche' non siano totalmente separate e senza veruna coerenza annesse al principale instituto."[1]

Galileo Galilei
"Lettera al Principe Leopoldo di Toscana" (1623)

Hydrodynamics is one of those fundamental areas in mathematics where progress at any moment may be regarded as a standard to measure the real success of mathematical science. Many important achievements in this field are based on profound theories rather than on experiments. In turn, those hydrodynamical theories stimulated developments in the domains of pure mathematics, such as complex analysis, topology, stability theory, bifurcation theory, and completely integrable dynamical systems. In spite of all this acknowledged success, hydrodynamics with its spectacular empirical laws remains a challenge for mathematicians. For instance, the phenomenon of turbulence has not yet acquired a rigorous mathematical theory. Furthermore, the existence problems for the smooth solutions of hydrodynamic equations of a three-dimensional fluid are still open.

The simplest but already very substantial mathematical model for fluid dynamics is the hydrodynamics of an ideal (i.e., of an incompressible and inviscid) homogeneous fluid. From the mathematical point of view, a theory of such a fluid filling a certain domain is

[1] *"... Some prefer to see the scientific teachings condensed too laconically into the smallest possible volume, so as always to use a rigid and concise manner that whatsoever lacks beauty and embellishment, and that is so common among pure geometers who do not pronounce a single word which is not of absolute necessity.*
I, on the contrary, do not consider it a defect to insert in a treatise, albeit devoted to a single aim, other various remarks, as long as they are not out of place and without coherency with the main purpose," see [Gal].

nothing but a study of geodesics on the group of diffeomorphisms of the domain that preserve volume elements. The geodesics on this (infinite-dimensional) group are considered with respect to the right-invariant Riemannian metric given by the kinetic energy.

In 1765, L. Euler [Eul] published the equations of motion of a rigid body. Eulerian motions are described as geodesics in the group of rotations of three-dimensional Euclidean space, where the group is provided with a left-invariant metric. In essence, the Euler theory of a rigid body is fully described by this invariance. The Euler equations can be extended in the same way to an arbitrary group. As a result, one obtains, for instance, the equations of a rigid body motion in a high-dimensional space and, especially interesting, the Euler equations of the hydrodynamics of an ideal fluid.

Euler's theorems on the stability of rotations about the longest and shortest axes of the inertia ellipsoid have counterparts for an arbitrary group as well. In the case of hydrodynamics, these counterparts deliver nonlinear generalizations of Rayleigh's theorem on the stability of two-dimensional flows without inflection points of the velocity profile.

The description of ideal fluid flows by means of geodesics of the right-invariant metric allows one to apply the methods of Riemannian geometry to the study of flows. It does not immediately imply that one has to start by constructing a consistent theory of infinite-dimensional Riemannian manifolds. The latter encounters serious analytical difficulties, related in particular to the absence of existence theorems for smooth solutions of the corresponding differential equations.

On the other hand, the strategy of applying geometric methods to the infinite-dimensional problems is as follows. Having established certain facts in the finite-dimensional situation (of geodesics for invariant metrics on finite-dimensional Lie groups), one uses the results to *formulate* the corresponding facts for the infinite-dimensional case of the diffeomorphism groups. These final results often can be proved directly, leaving aside the difficult questions of foundations for the intermediate steps (such as the existence of solutions on a given time interval). The results obtained in this way have an *a priori* character: the derived identities or inequalities take place for any reasonable meaning of "solutions," provided that such solutions exist. The actual existence of the solutions remains an open question.

For example, we deduce the formulas for the Riemannian curvature of a group endowed with an invariant Riemannian metric. Applying these formulas to the case of the infinite-dimensional manifold whose geodesics are motions of the ideal fluid, we find that the curvature is negative in many directions. Negativeness of the curvature implies instability of motion along the geodesics (which is well-known in Riemannian geometry of finite-dimensional manifolds). In the context of the (infinite-dimensional) case of the diffeomorphism group, we conclude that the ideal flow is unstable (in the sense that a small variation of the initial data implies large changes of the particle positions at a later time).

Moreover, the curvature formulas allow one to estimate the increment of the exponential deviation of fluid particles with close initial positions and hence to predict the time period when the motion of fluid masses becomes essentially unpredictable.

For instance, in the simplest and utmost idealized model of the earth's atmosphere (regarded as two-dimensional ideal fluid on a torus surface), the deviations grow by the factor of 10^5 in 2 months. This circumstance ensures that a dynamical weather forecast for such a period is practically impossible (however powerful the computers and however dense the grid of data used for this purpose).

The table of contents is essentially self explanatory. We have tried to make the chapters as independent of each other as possible. Cross references within the same chapter do not contain the chapter number.

For a first acquaintance with the subject, we address the reader to the following sections in each chapter: Sections I.1-5 and I.12, Sections II.1 and II.3-4, Sections III.1-2 and III.4, Section IV.1, Sections V.1-2, Sections VI.1 and VI.4.

Some statements in this book may be new even for the experts. We mention the classification of the local conservation laws in ideal hydrodynamics (Theorem I.9.9), M. Freedman's solution of the A. Sakharov–Ya. Zeldovich problem on the energy minimization of the unknotted magnetic field (Theorem III.3), a discussion of the construction of manifold invariants from the energy bounds (Remark III.2.6), a discussion of a complex version of the Vassiliev knot invariants (in Section III.7.E), a nice remark of B. Zeldovich on the Lobachevsky triangle medians (Problem IV.1.4), the relation of the covariant derivative of a vector field and the inertia operator in hydrodynamics (Section IV.1.D), a digression on the Fokker–Planck equation (Section V.3.C), and the dynamo construction from the geodesic flow on surfaces of constant negative curvature (Section V.4.D).

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