APM 236H1F term test 2

6 November, 2013

FAMILY NAME ____________________

GIVEN NAME(S) ____________________

STUDENT NUMBER ____________________

SIGNATURE ____________________

Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1.(a) (7 marks) **Find an optimal solution** of the problem: Maximize $z = 2x_1 + 4x_2 + 5x_3$

subject to the constraints $\begin{array}{rcrcrcr} 2x_1 & - & 2x_2 & - & x_3 & \geq & -4 \\ 4x_1 & - & x_2 & + & x_3 & \leq & 1 \end{array}$, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

1.(b) (6 marks) **Find all optimal solutions** of the problem of question 1.(a).

1.(a) An equivalent canonical problem with slacks x_4, x_5 is:

Maximize $z = 2x_1 + 4x_2 + 5x_3$ s.t. $-2x_1 + 2x_2 + x_3 + x_4 = 4$

$4x_1 - x_2 + x_3 + x_5 = 1$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$

Tableau 1

	x_1	x_2	x_3	x_4	x_5	
x_4	-2	2	1	1	0	4
x_5	4	-1	1	0	1	1
	-2	-4	-5	0	0	0

Tableau 2

	x_1	x_2	x_3	x_4	x_5	
x_4	-6	3	0	1	-1	3
x_3	4	-1	1	0	1	1
	18	-9	0	0	5	5

Tableau 3

	x_1	x_2	x_3	x_4	x_5	
x_2	-2	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	1
x_3	2	0	1	$\frac{1}{3}$	$\frac{2}{3}$	2
	0	0	0	3	2	14

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	1	$\frac{2}{3}$	$\frac{1}{3}$	3
x_1	1	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	1
	0	0	0	3	2	14

1.(b) Noting that if x_1 enters, the objective row will not change, and exiting x_3, we get Tableau 4. From Tableaux 3 and 4 (and expressing solutions in $\mathbb{R}^3$), the optimal solutions consist of the line segment joining $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$. That is, $\left\{ \begin{bmatrix} \lambda \\ 1+2\lambda \\ 2-2\lambda \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } 0 \leq \lambda \leq 1 \right\}$.

2. (13 marks) Suppose that the simplex solution of a linear programming problem includes the following non-optimal tableau.

	x_1	$\cdots$	x_n	
x_1	a_{11}	$\cdots$	a_{1n}	b_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_m	a_{m1}	$\cdots$	a_{mn}	b_m
	r_1	$\cdots$	r_n	s

Suppose also, that by following the rules of the simplex method, we enter x_n and exit x_1, but in doing so, the objective value does not change. **Prove** that the basic solution given by the above tableau is degenerate.

Upon entering x_n and exiting x_1, the objective row will become:

$$[r_1 \cdots r_n \mid s] - r_n a_{1n}^{-1} [a_{11} \cdots a_{1n} \mid b_1]$$

($a_{1n} \neq 0$ because we have been following the simplex method) In particular the objective value changes by the amount $-r_n a_{1n}^{-1} b_1$, so by hypotheses, $r_n a_{1n}^{-1} b_1 = 0$. Since the tableau is not optimal and we are following the simplex method, $r_n \neq 0$. (Actually, $r_n < 0$) Therefore, the factor b_1 in the product $r_n a_{1n}^{-1} b_1 = 0$ must be 0.

3. (14 marks) **Solve** the problem: Maximize $z = x_1 - 4x_2 - x_3$

subject to the constraints $\begin{array}{rcrcrcl} -2x_1 & + & 2x_2 & + & x_3 & = & 2 \\ x_1 & + & 2x_2 & + & x_3 & \geq & 5 \end{array}$, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

Phase 1: x_4 is slack and y_1, y_2 are artificial.

Tableau 1

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	-2	2	1	0	1	0	2
y_2	1	2	1	-1	0	1	5
	1	-4	-2	1	0	0	-7

Tableau 2

	x_1	x_2	x_3	x_4	y_1	y_2	
x_2	-1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1
y_2	3	0	0	-1	-1	1	3
	-3	0	0	1	2	0	-3

Tableau 3

	x_1	x_2	x_3	x_4	y_1	y_2	
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	2
x_1	1	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1
	0	0	0	0	1	1	0

Phase 2, Tableau 1

	x_1	x_2	x_3	x_4	
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{3}$	2
x_1	1	0	0	$-\frac{1}{3}$	1
	0	0	-1	1	-7

$$\begin{array}{l} [-1 \;\; 4 \;\; 1 \;\; 0 \mid 0] \\ +[1 \;\; 0 \;\; 0 \;\; -\tfrac{1}{3} \mid 1] \\ -4[0 \;\; 1 \;\; \tfrac{1}{2} \;\; -\tfrac{1}{3} \mid 2] \end{array}$$

Phase 2, Tableau 2

	x_1	x_2	x_3	x_4	
x_3	0	2	1	$-\frac{2}{3}$	4
x_1	1	0	0	$-\frac{1}{3}$	1
	0	2	0	$\frac{1}{3}$	-3