APM 236H1F term test 2

9 November, 2011

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

Instructions: **No calculators or other aids allowed.**

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) **Solve** the problem: Maximize $z = 2x_1 + 3x_2 + x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 2x_3 & \leq & 6 \\ 2x_1 & + & x_2 & + & 3x_3 & \leq & 5 \\ x_1 & + & 2x_2 & + & 4x_3 & \leq & 3 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Tableau 1

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1	2	1	0	0	6
x_5	2	1	3	0	1	0	5
x_6	1	2	4	0	0	1	3
	-2	-3	-1	0	0	0	0

Tableau 2

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	$\frac{1}{2}$	0	0	1	0	$-\frac{1}{2}$	$\frac{9}{2}$
x_5	$\frac{3}{2}$	0	1	0	1	$-\frac{1}{2}$	$\frac{7}{2}$
x_2	$\frac{1}{2}$	1	2	0	0	$\frac{1}{2}$	$\frac{3}{2}$
	$-\frac{1}{2}$	0	5	0	0	$\frac{3}{2}$	$\frac{9}{2}$

Tableau 3

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{10}{3}$
x_1	1	0	$\frac{2}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{7}{3}$
x_2	0	1	$\frac{5}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
	0	0	$\frac{16}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{17}{3}$

2.(a) (7 marks) The following tableau is the final tableau in the simplex solution of a certain linear programming problem (problem $\mathcal{P}$, say):

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	5	9	0	−4	1	−7	6
x_3	2	−5	1	−8	0	2	0
	1	9	0	5	0	3	7

Prove that $\mathcal{P}$ has at most one optimal solution.

The tableau represents the problem

Maximize $Z = 7 - x_1 - 9x_2 - 5x_4 - 3x_6$ subject to the constraints $5x_1 + 9x_2 - 4x_4 - 7x_6 \leq 6$

$$2x_1 - 5x_2 - 8x_4 + 2x_6 \leq 0$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_4 \geq 0,\ x_6 \geq 0$$

This problem has the same feasible values for x_1, x_2, x_4, and x_6 as $\mathcal{P}$ and the same objective function for feasible x_1, x_2, x_4, and x_6 as $\mathcal{P}$; hence the same optimal solution(s).

But $x_1 = 0$, $x_2 = 0$, $x_4 = 0$, $x_6 = 0$ is the <u>only</u> optimal solution of the above problem. From the ≥ 0 constraints, any feasible change from this solution would be a change to positive value(s), which would cause Z (having <u>all</u> coefficients negative) to <u>decrease</u>.

2.(b) (7 marks) The following tableau is the final tableau in the simplex solution of a certain linear programming problem (problem Q, say):

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	-7	1	5	0	3	-4	8
x_4	-4	0	6	1	-2	8	5
	6	0	1	0	0	3	-9

Find **all optimal solutions** of problem Q.

The non-basic variable x_5 has "0" in the objective row so entering x_5, exiting x_2 yields

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	$-\frac{7}{3}$	$\frac{1}{3}$	$\frac{5}{3}$	0	1	$-\frac{4}{3}$	$\frac{8}{3}$
x_4	$-\frac{26}{3}$	$\frac{2}{3}$	$\frac{28}{3}$	1	0	$\frac{16}{3}$	$\frac{31}{3}$
	6	0	1	0	0	3	-9

These two tableaux represent the extreme points of the optimal region, which is the line segment having endpoints $\begin{bmatrix} 0 \\ 8 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{31}{3} \\ \frac{8}{3} \\ 0 \end{bmatrix}$.

That is, $\left\{ \begin{bmatrix} 0 \\ 8\lambda \\ 0 \\ \frac{31}{3} - \frac{16}{3}\lambda \\ \frac{8}{3} - \frac{8}{3}\lambda \\ 0 \end{bmatrix} \in \mathbb{R}^6 \text{ s.t. } 0 \le \lambda \le 1 \right\}$.

3. (13 marks) **Solve** the problem: Maximize $z = 2x_1 + x_2 + x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} 2x_1 & - & x_2 & & & = & -1 \\ 4x_1 & - & 3x_2 & - & x_3 & \leq & -5 \\ -2x_1 & + & 3x_2 & + & 2x_3 & \geq & 7 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Phase 1, Tableau 1

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	-2	1	0	0	0	1	0	0	1
y_2	-4	3	1	-1	0	0	1	0	5
y_3	-2	3	2	0	-1	0	0	1	7
	8	-7	-3	1	1	0	0	0	-13

Phase 1, Tableau 2

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_2	-2	1	0	0	0	1	0	0	1
y_2	2	0	1	-1	0	-3	1	0	2
y_3	4	0	2	0	-1	-3	0	1	4
	-6	0	-3	1	1	7	0	0	-6

Phase 1, Tableau 3 (optimal since $z=0$)

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_2	0	1	1	-1	0	-2	1	0	3
x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{3}{2}$	$\frac{1}{2}$	0	1
y_3	0	0	0	2	-1	3	-2	1	0
	0	0	0	-2	1	-2	3	0	0

Phase 2, Tableau 1

	x_1	x_2	x_3	x_4	x_5	y_3	
x_2	0	1	1	-1	0	0	3
x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1
y_3	0	0	0	2	-1	1	0
	0	0	1	-2	0	0	5

Phase 2, Tableau 2

	x_1	x_2	x_3	x_4	x_5	y_3	
x_2	0	1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	3
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1
x_4	0	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0
	0	0	1	0	-1	1	5

This problem is unbounded above.