APM 236H1F term test 1

18 October, 2006

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

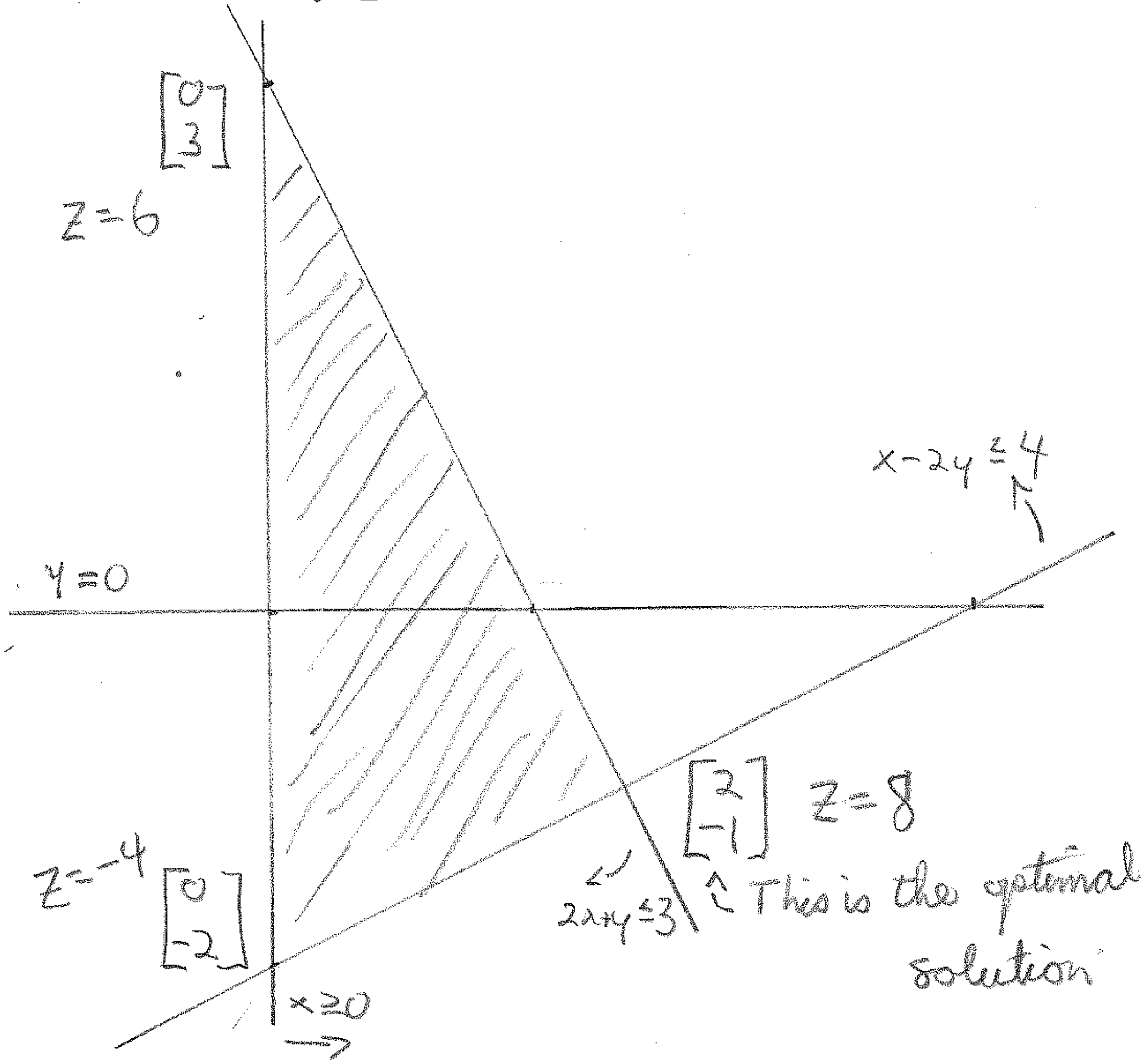
Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1.(a) (10 marks) **Solve** the following problem **graphically**: Maximize $z = 5x + 2y$ subject to the constraints $\begin{array}{rcrcl} 2x & + & y & \leq & 3 \\ x & - & 2y & \leq & 4 \end{array}$, $x \geq 0, y$ unrestricted.



1.(b) (3 marks) List the **extreme points** of the feasible region of the problem in question 1.(a).

$\begin{bmatrix} 0 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

2. (13 marks) The Nutandbolt Company has factories in Ajax and Guelph which produce metal nuts and bolts. At full capacity, the Ajax factory can produce 400 nuts **or** 200 bolts per day. By using the fraction x_1 of its total capacity to produce nuts and the fraction y_1 of its total capacity to produce bolts, the Ajax factory can produce $400x_1$ nuts **and** $200y_1$ bolts per day. Similarly, if the Guelph factory uses the fraction x_2 of its total capacity to produce nuts and the fraction y_2 of its total capacity to produce bolts, it can produce $300x_2$ nuts and $500y_2$ bolts per day. At the end of each day, the outputs of the two factories are shipped to Oakville, where they are paired and assembled, one nut for each bolt. The Company wishes to maximize its daily output of nut-and-bolt assemblies. Write **one linear programming problem** in **canonical form** whose solution will tell the Company how to do this. Hints: the daily output of nut-and-bolt assemblies is maximized only if the total daily production of nuts equals the total daily production of bolts and both factories are working at full capacity. After you have written a linear programming problem, **do not solve it**.

There exist infinitely many correct answers to this question. The following requires the minimum number of decision variables.

Maximize $z = 400x_1 + 300x_2$ s.t.

$$\begin{aligned} 400x_1 + 300x_2 - 200y_1 - 500y_2 &= 0 \\ x_1 + y_1 &= 1 \\ x_2 + y_2 &= 1 \\ x_1 \geq 0,\ x_2 \geq 0,\ y_1 \geq 0,\ y_2 &\geq 0. \end{aligned}$$

A second example is

Maximize $z = 200y_1 + 500y_2$ subject to the same constraints as the first example.

3.(a) (7 marks) In $\mathbb{R}^2$, determine whether $\begin{bmatrix}3\\2\end{bmatrix}$ is a convex combination of $\begin{bmatrix}2\\1\end{bmatrix}$, $\begin{bmatrix}4\\1\end{bmatrix}$, and $\begin{bmatrix}2\\4\end{bmatrix}$. That is, determine whether there exist $c_1 \geq 0$, $c_2 \geq 0$, and $c_3 \geq 0$ such that $c_1 + c_2 + c_3 = 1$ and $\begin{bmatrix}3\\2\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}4\\1\end{bmatrix} + c_3\begin{bmatrix}2\\4\end{bmatrix}$.

Any such c_i must satisfy the system
$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ 2c_1 + 4c_2 + 2c_3 &= 3. \\ c_1 + c_2 + 4c_3 &= 2 \end{aligned}$$

By row reduction,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 3 \\ 1 & 1 & 4 & 2 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 3 & 1 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right].$$

Since each $c_i \geq 0$, $\begin{bmatrix}3\\2\end{bmatrix}$ <u>is</u> a convex combination of $\begin{bmatrix}2\\1\end{bmatrix}$, $\begin{bmatrix}4\\1\end{bmatrix}$, and $\begin{bmatrix}2\\4\end{bmatrix}$.

3.(b) (7 marks) **Prove** that $\left\{\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } y \leq x^2\right\}$ is **not convex.**

$\begin{bmatrix}x_1\\y_1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}x_2\\y_2\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$ both belong to the set, as in each case $y_i \leq x_i^2$.

But their midpoint, $\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$, does not (since $1 \not\leq 0^2$).