APM 236H1F term test 1

8 October, 2014

FAMILY NAME ____________________

GIVEN NAME(S) ____________________

STUDENT NUMBER ____________________

SIGNATURE ____________________

Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) Write **one linear programming problem** which satisfies **all** of the following:

(i) It has **two decision variables**, x and y.

(ii) It is in **standard form.**

(iii) It has **no optimal objective value.**

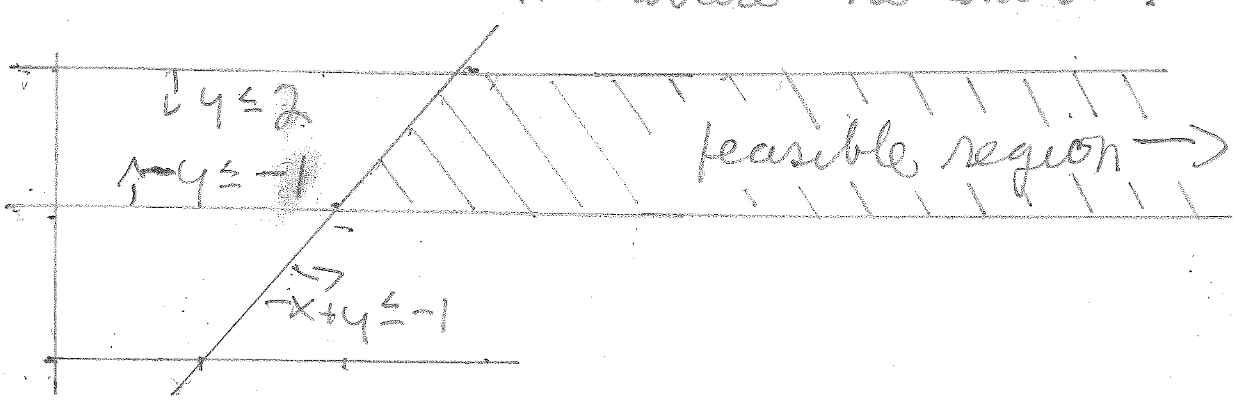
(iv) Its feasible region has $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as **extreme points** and these are its **only** extreme points.

((iv) implies the feasible region is non-empty. Hence (iii) implies the problem must be unbounded. Then the extreme point theorem implies the feasible region itself is unbounded.)

This problem has infinitely many correct answers. One example is:

$$\text{Maximize } Z = x \text{ s.t. } \begin{array}{r} -x + y \le -1 \\ y \le 2 \\ -y \le -1 \end{array}, \; x \ge 0, y \ge 0$$

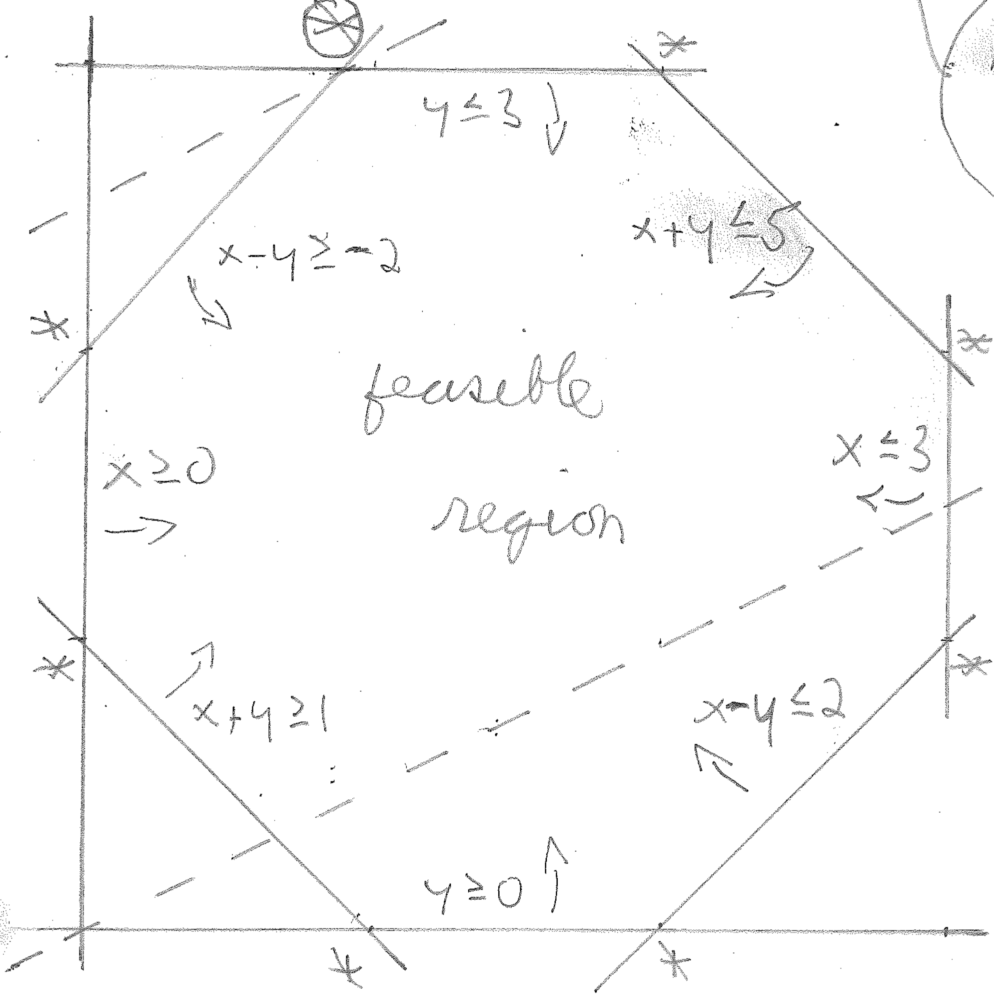
This sketch is not part of the answer above. It was used to motivate the answer.



2. (13 marks) Consider the problem: Minimize $z = x - 2y$ subject to the constraints

$$\begin{array}{rcrcr} x & + & y & \geq & 1 \\ x & + & y & \leq & 5 \\ x & - & y & \leq & 2 \\ x & - & y & \geq & -2 \\ x & & & \leq & 3 \\ & & y & \leq & 3 \end{array}, \quad x \geq 0,\ y \geq 0.$$

2.(a) (9 marks) **Solve** the problem **graphically**.



x decreases
y increases
$x - 2y$ decreases

The solution is at ⊛, so satisfies $x - y = -2$, $y = 3$. So the solution is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

2.(b) (4 marks) List the **extreme points** of the feasible region of the problem.

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

Alternate solution to 2.(a), using 2.(b) and the extreme point theorem (the graph shows the feasible region is non-empty and bounded): z-values are -5, -4, -2, 1, 2, 1, -1, -4.

3.(a) (7 marks) Let $n = 1, 2, 3, \ldots$, let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ denote vectors in $\mathbb{R}^n$, and let $C = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \in \mathbb{R}^n \text{ s.t. } a_1 \geq 0, a_2 \geq 0, a_3 \geq 0, \text{ and } a_1 + a_2 + a_3 = 1\}$.
Prove that C is convex.

Pick $\lambda \in [0,1]$ and 2 points in C: $a_1 v_1 + a_2 v_2 + a_3 v_3$ and $b_1 v_1 + b_2 v_2 + b_3 v_3$ so that ($\star$) $\lambda \geq 0$, $1-\lambda \geq 0$,
($\star$ cont'd) $a_i \geq 0$, $b_i \geq 0$ for $i = 1, 2, 3$ and ($\ddagger$) $a_1 + a_2 + a_3 = 1$, $b_1 + b_2 + b_3 = 1$.
Then $\lambda(a_1 v_1 + a_2 v_2 + a_3 v_3) + (1-\lambda)(b_1 v_1 + b_2 v_2 + b_3 v_3)$
$= (\lambda a_1 + (1-\lambda) b_1) v_1 + (\lambda a_2 + (1-\lambda) b_2) v_2 + (\lambda a_3 + (1-\lambda) b_3) v_3$,
in which the coefficients of v_1, v_2, v_3 are ≥ 0 by ($\star$)
and add up to 1: $(\lambda a_1 + (1-\lambda) b_1) + (\lambda a_2 + (1-\lambda) b_2) + (\lambda a_3 + (1-\lambda) b_3)$
$= \lambda(a_1 + a_2 + a_3) + (1-\lambda)(b_1 + b_2 + b_3) = \lambda + (1-\lambda)$ by ($\ddagger$).

3.(b) (7 marks) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } x + y^2 \geq 0 \right\}$. **Prove** that S is **not convex.**

$\begin{bmatrix} -1 \\ -1 \end{bmatrix} \in S$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \in S$ because they satisfy $x + y^2 \geq 0$.
But the convex combination, $\frac{1}{2}\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$
does not belong to S because $-1 + 0^2 \not\geq 0$.

This sketch is not part of the proof of 3.(b).
It was used to motivate the proof.