APM 236H1F term test 1

9 October, 2013

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

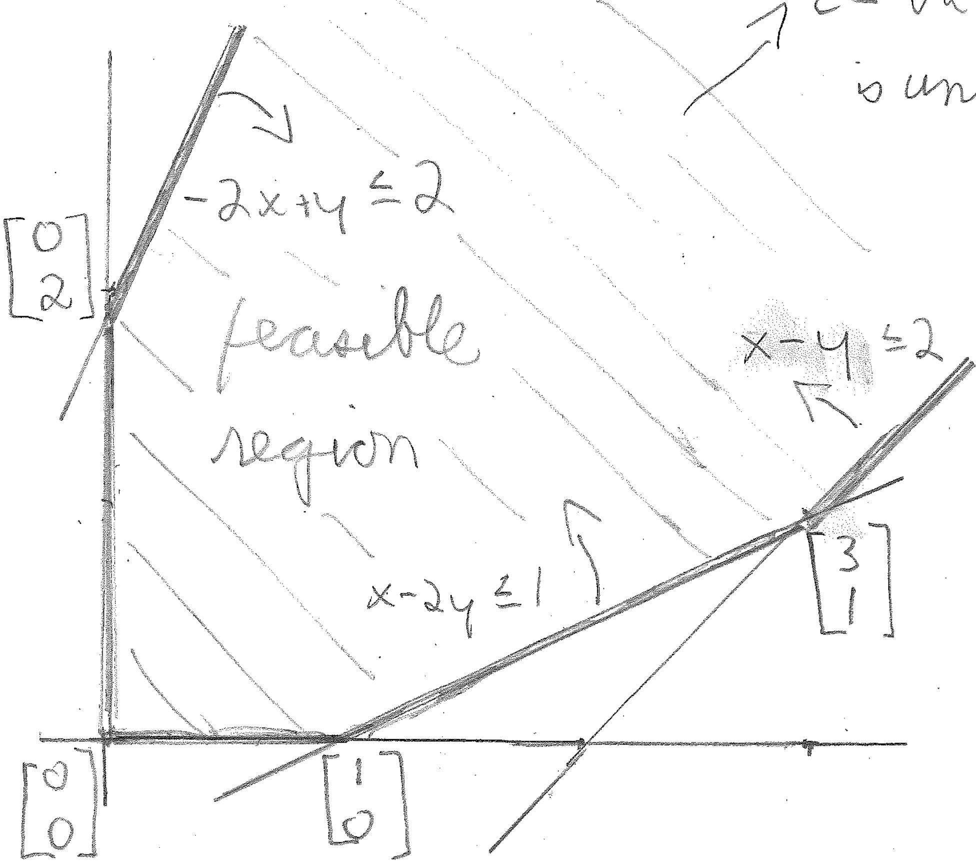
1. (13 marks) Consider the problem: Maximize $z = 3x + 5y$ subject to the constraints

$$\begin{array}{rcrcl} x & - & 2y & \leq & 1 \\ x & - & y & \leq & 2 \\ -2x & + & y & \leq & 2 \end{array}$$

$$x \geq 0, y \geq 0$$

1.(a) (9 marks) **Solve** the problem **graphically**.

$z = 3x + 5y$ is unbounded



To find $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$:

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

This problem is unbounded above.

1.(b) (4 marks) List the extreme points of the feasible region of the problem.

$\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

2.(a) (7 marks) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } 0 \le x \le 1 \text{ **or** } 0 \le y \le 1 \right\}$. **Prove** that S is **not convex.**

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in S$ because $0 \le x_1 \le 1$

$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \in S$ because $0 \le y_2 \le 1$

Yet $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin S$

because both $x \not\le 1$ and $y \not\le 1$.

2.(b) (7 marks) Let S denote the solution set of the set of inequalities:

$$\begin{array}{rcrcl} 2x & + & y & \le & 6 \\ x & + & 2y & \le & 6 \end{array}$$

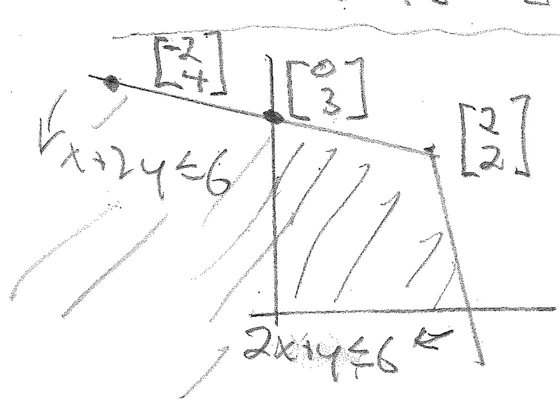
x unrestricted, y unrestricted

Prove that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ is **not an extreme point** of S.

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \in S$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \in S$

Since $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ but $\begin{bmatrix} x \\ y \end{bmatrix} \ne \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

and $\begin{bmatrix} x \\ y \end{bmatrix} \ne \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix}$ is not extreme. Q.E.D.



(This diagram is not part of the proof completed above. It was used to motivate the proof.)

3. (14 marks) Let S denote the solution set (in $\mathbb{R}^4$) of the system of equalities and inequalities:

$$\begin{array}{ccccccccc} x_1 & & & & & + & 2x_4 & \geq & 1 \\ & & x_2 & - & 2x_3 & & & = & 2 \\ x_1 & - & 2x_2 & + & 4x_3 & + & 2x_4 & = & 3 \end{array}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

List the extreme points of S. Note that points in S <u>belong to $\mathbb{R}^4$</u>. **Show your work.**

After putting the constraints in canonical form with slack variable $x_5 \geq 0$, the resulting system of equations has augmented matrix

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 & 0 & 2 \\ 1 & -2 & 4 & 2 & 0 & 3 \end{array}\right] \end{array}$$

. (The x_1- and x_4-columns are linearly dependent, as are the x_2- and x_3-columns)

Basic solutions are $\begin{bmatrix} 7 \\ 2 \\ 0 \\ 0 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 0 \\ -1 \\ 0 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 0 \\ \frac{7}{2} \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ -1 \\ \frac{7}{2} \\ 6 \end{bmatrix}$.

(In each case the basic variables are the non-zero variables). Omitting the infeasible solutions and truncating the slack variable, S has extreme points $\begin{bmatrix} 7 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 0 \\ \frac{7}{2} \end{bmatrix}$.